Developement of DDFV methods for the Euler equations

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Introduction

- Hyperbolic system of conservation laws in 2D

$$\partial_t W + \partial_x f(W) + \partial_y g(W) = 0 \qquad (1)$$

$W : \mathbb{R}^2 \times \mathbb{R}^+ \to \Omega \subset \mathbb{R}^d$: unknown state vector
$f, g : \Omega \to \mathbb{R}^d$: flux functions

- **Example:** the 2D Euler equations

$$W = \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ E \end{pmatrix}, \; f(W) = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho u v \\ u(E+p) \end{pmatrix}, \; g(W) = \begin{pmatrix} \rho v \\ \rho u v \\ \rho v^2 + p \\ v(E+p) \end{pmatrix},$$

where ρ is the density, (u, v) the velocity, E the total energy and p the pressure given by the perfect gas law

$$p = (\gamma - 1)\left(E - \frac{\rho}{2}\left(u^2 + v^2\right)\right) \qquad (2)$$

- Ω convex set of physical states. In the Euler case:

$$\Omega = \left\{ W \in \mathbb{R}^4; \rho > 0, (u, v) \in \mathbb{R}^2, E - \frac{\rho}{2}\left(u^2+v^2\right) > 0 \right\} \qquad (3)$$

- **Objectif:** derive a numerical scheme
  - → Second order accurate
  - → Ω-preserving
  - → Unstructured meshes
  - → CFL restriction

1. DDFV meshes

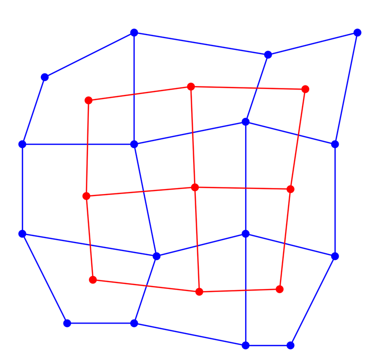

Fig. 1: The primal mesh and the dual mesh

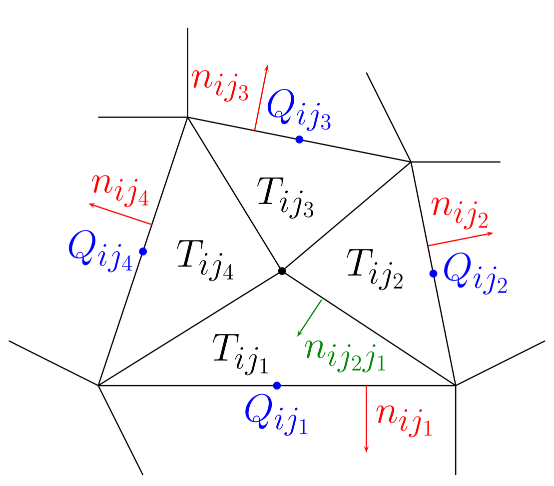

Fig. 2: A primal or dual cell K_i

Primal mesh: quadrilateral cells

Dual mesh: quadrilateral cells whose vertices are the mass centers of the primal cells

Global notations

For a primal or dual cell K_i, we define:

- $\nu(i)$ the index set of the cells K_j which share a common edge with K_i (neighbourhood of K_i);
- for $j \in \nu(i)$, ℓ_{ij} the edge between K_i and K_j and n_{ij} the outward unit normal to ℓ_{ij};
- Q_{ij} the midpoint of the edge ℓ_{ij};
- T_{ij} the triangle formed by the edge ℓ_{ij} and the mass center of K_i;
- $\nu(i,j) = \{k_1, k_2\}$ such that T_{ik_1} and T_{ik_2} are the triangles of K_i sharing an edge with T_{ij};
- for $k \in \nu(i,j)$, ℓ_{ijk} the edge between T_{ij} and T_{ik} and n_{ijk} the unit normal to ℓ_{ijk}, from T_{ij} to T_{ik}.

2. MUSCL scheme

We write a finite volume scheme on both the primal and dual meshes.

First-order scheme on the cell K_i

$$W_i^{n+1} = W_i^n - \frac{\Delta t}{|K_i|} \sum_{j \in \nu(i)} |\ell_{ij}| \phi\left(W_i^n, W_j^n, n_{ij}\right), \qquad (4)$$

where $\phi(W_L, W_R, n)$ is a numerical 2D flux function assumed to satisfy:

- consistency: $\phi(W, W, n) = \begin{pmatrix} f(W) \\ g(W) \end{pmatrix} \cdot n$,
- conservation: $\phi(W_L, W_R, n) = -\phi(W_R, W_L, -n)$.

Second-order scheme on the cell K_i

$$W_i^{n+1} = W_i^n - \frac{\Delta t}{|K_i|} \sum_{j \in \nu(i)} |\ell_{ij}| \phi\left(W_{ij}, W_{ji}, n_{ij}\right), \qquad (5)$$

where W_{ij} and W_{ji} are second-order approximations of the solution at the point Q_{ij}, on each side of the edge ℓ_{ij}.

→ **How to compute W_{ij} ?**

3. Robustness

- We denote by $\lambda(W_L, W_R, n)$ the maximum absolute wave velocity associated to the numerical flux function $\phi(W_L, W_R, n)$.
- We assume that the numerical flux ϕ is first-order robust per direction, which means that for all $W_i \in \Omega$, $i \in \mathbb{Z}$, for all $n \in \mathbb{R}^2$, if the following first-order CFL condition is satisfied

$$\frac{\Delta t}{\Delta x} \max_{i \in \mathbb{Z}} \{\lambda(W_i, W_{i+1}, n)\} \le \frac{1}{2}, \qquad (6)$$

then for all $i \in \mathbb{Z}$, the 1D updated state
$W_i - \frac{\Delta t}{\Delta x}\left(\phi(W_i, W_{i+1}, n) - \phi(W_{i-1}, W_i, n)\right)$ remains in Ω.

- **Theorem 1.** Let us assume that all the states W_i^n and all the reconstructed states W_{ij} are in Ω. We suppose that the reconstruction satisfies the following conservation property

$$\sum_{j \in \nu(i)} \frac{|T_{ij}|}{|K_i|} W_{ij} = W_i^n. \qquad (7)$$

Consider the CFL condition

$$\Delta t \max_{\substack{i \\ j \in \nu(i)}} \left\{ \frac{|\ell_{ij}|}{|T_{ij}|} \lambda\left(W_{ij}, W_{ji}, n_{ij}\right) \right\} \le \frac{1}{6}, \qquad (8)$$

$$\Delta t \max_{\substack{i \\ j \in \nu(i) \\ k \in \nu(i,j)}} \left\{ \frac{|\ell_{ijk}|}{|T_{ij}|} \lambda\left(W_{ij}, W_{ik}, n_{ijk}\right) \right\} \le \frac{1}{6}. \qquad (9)$$

Then the updated states W_i^{n+1} given by (5) are in Ω.

4. Reconstruction procedure

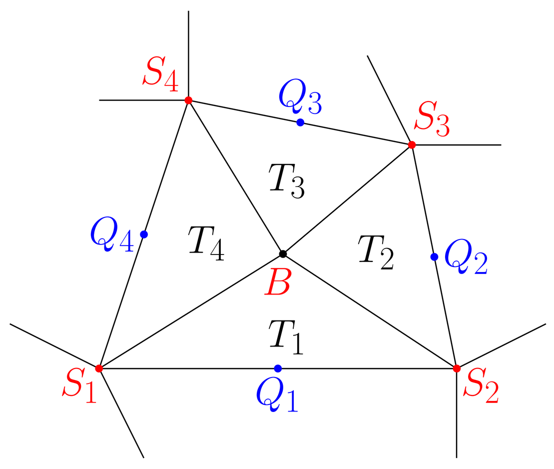

Fig. 3: Geometry of the cell K

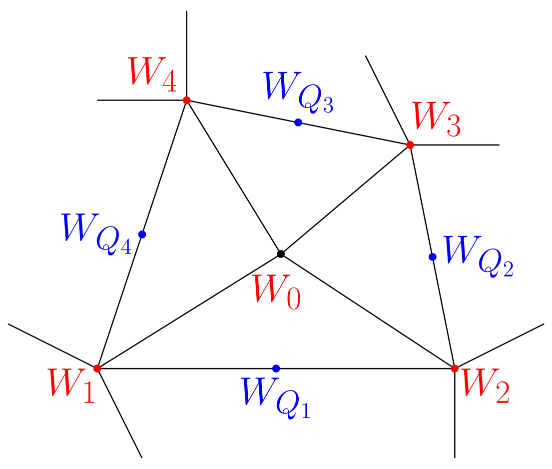

Fig. 4: Known states and reconstructed states

Local notations on cell K:

B, mass center	→ W_0, known state
S_j, vertex	→ W_j, known state
Q_j, midpoint of the edge	→ W_{Q_j}, to be reconstructed in Ω

Remark: The mass centers of the dual cells do not coincide with the vertices of the primal mesh ⇒ Consistency error on distorded meshes.

4.1 Gradient reconstruction

We define a function $\widehat{W} : K \to \mathbb{R}^d$ piecewise linear on each triangle T_j and such that $\widehat{W}(B) = W_0$ and $\widehat{W}(S_j) = W_j$, $1 \le j \le 4$.

4.2 Projection

For $1 \le k \le d$, we define

$$E_k(\nu) = \int_K \left| \widehat{W}_k(X) - [(W_0)_k + \nu \cdot (X - B)] \right|^2 dX, \qquad (10)$$

where the subscript k denotes the k-th component.
Let $\mu \in \mathbb{R}^d$ be the vector whose k-th component is the solution of

$$E_k(\mu_k) = \min_{\nu \in \mathbb{R}^2} E_k(\nu). \qquad (11)$$

We define $\widetilde{W}_\mu(X) : K \to \mathbb{R}^d$ the linear function whose k-th component is $(W_0)_k + \mu_k \cdot (X - B)$.

4.3 Limitation of the slope μ

We restrict Ω to a close set Ω_ϵ. In the Euler case,

$$\Omega_\epsilon = \left\{ W \in \mathbb{R}^4; \rho \ge \epsilon, (u, v) \in \mathbb{R}^2, E - \frac{\rho}{2}\left(u^2+v^2\right) \ge \epsilon \right\}. \qquad (12)$$

We define the optimal slope limiter by

$$\theta = \max\left\{ t \in [0,1], \widetilde{W}_{t\mu}(Q_j) \in \Omega_\epsilon, \forall 1 \le j \le 4 \right\}. \qquad (13)$$

Finally, the reconstructed states are given by $W_{Q_j} = \widetilde{W}_{\theta\mu}(Q_j)$.

Limitation procedure	⇒ $W_{Q_j} \in \Omega$
$\widetilde{W}_{\theta\mu}$ linear and B mass center of K	⇒ Condition (7)

⇒ **Robustness by Theorem 1.**

5. Numerical tests

Meshes

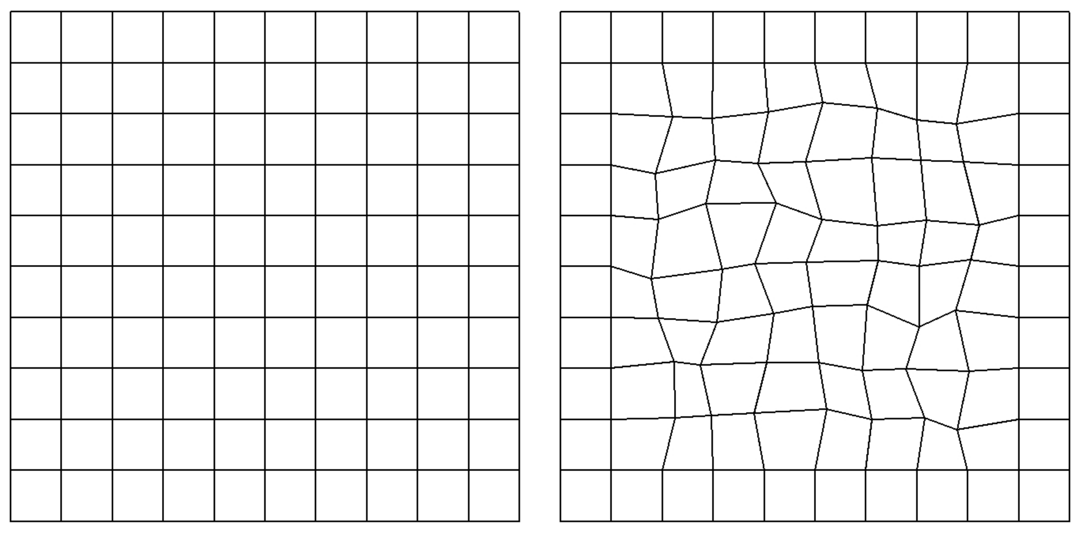

Fig. 5: *Square mesh 10 × 10 (left) and quadrilateral mesh 10 × 10 (right)*

Case 1 : four shocks

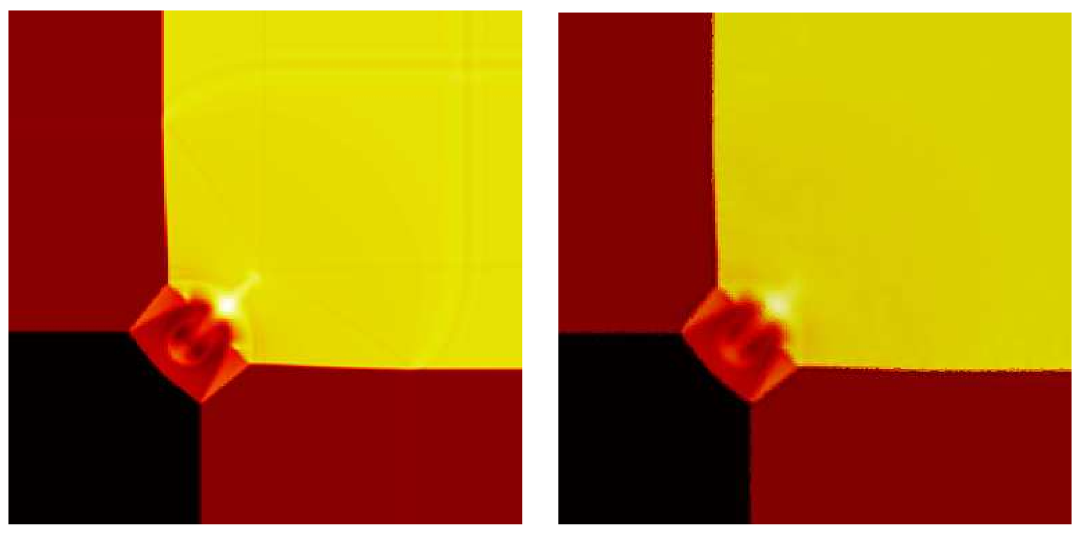

Fig. 6: *Square mesh 200 × 200 (left) and quadrilateral mesh 200 × 200 (right)*

Case 2 : four contact discontinuities

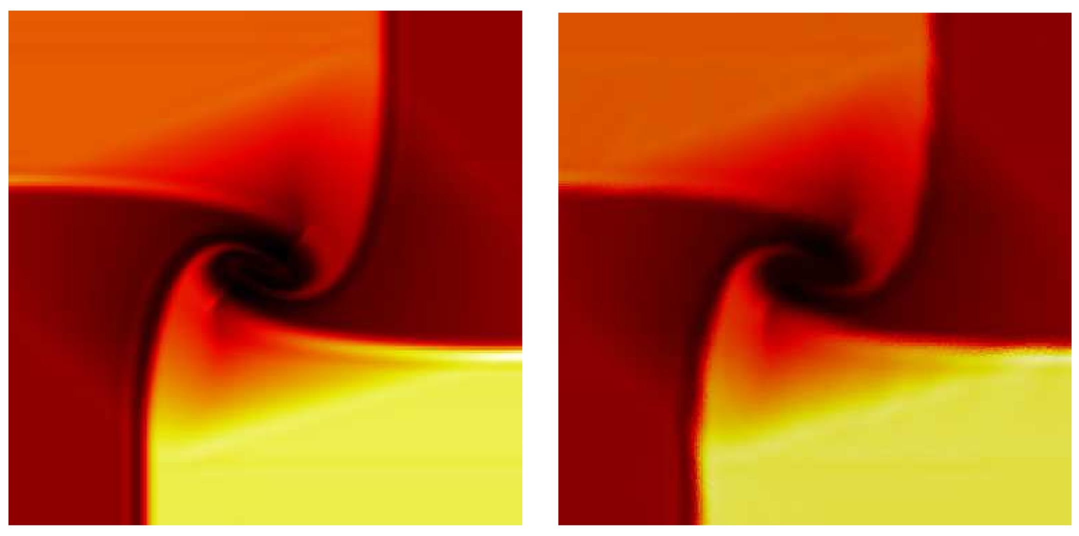

Fig. 7: *Square mesh 200 × 200 (left) and quadrilateral mesh 200 × 200 (right)*

Perspectives

- Extension to polygonal meshes;
- Allow non-conservative reconstructions, i.e. which don't satisfy (7);
- Optimization of the CFL conditions (8) and (9);
- Better approximation of the value at the vertices of the primal mesh, especially in the very distorded meshes (see Fig. 8).

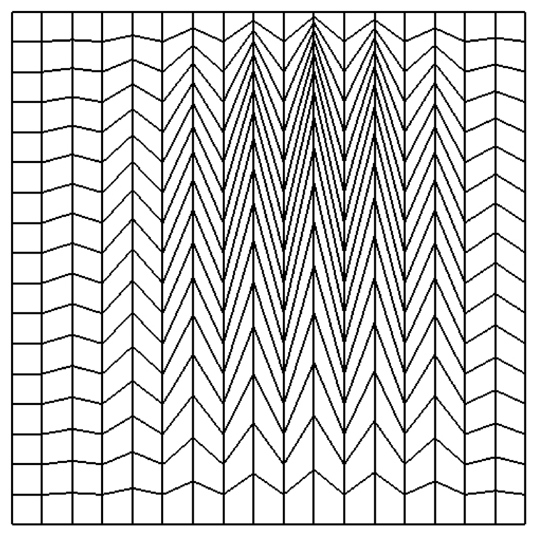

Fig. 8: *Distorded mesh*

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