An entropy preserving MOOD scheme for the Euler equations

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Abstract

The present work concerns the derivation of entropy stability properties to be satisfied by high-order accurate finite volume methods. Such a stability turns out to be crucial when approximating the weak solutions of hyperbolic systems of conservation laws. In fact, several recent works propose some kind of discrete entropy inequalities associated to high-order schemes. However, these entropy preserving schemes do not seem relevant to impose that the converged solution (in the sense of the Lax-Wendroff Theorem) satisfies the required entropy inequalities. We illustrate such a failure by exhibiting numerical schemes that, from one hand, satisfy *entropy stability* and, from the other hand, do not prevent numerical blow-up. Here, we recall the expected high-order discrete entropy inequalities to be certain that the approximate solution converges to an entropy solution. Equipped with these sufficient numerical entropy stability, we propose to extend the recently introduced high-order MOOD scheme to satisfy the required high-order entropy inequalities. In fact, the MOOD approach is based on an *a posteriori* estimation and it seems impossible to impose *a posteriori* the whole set of discrete entropy inequalities. We solve this difficulty by considering a finite volume scheme, which involves (at least one) discrete entropy inequalities with a *numerical transport* property. From one selected *numerical transport* discrete entropy inequality, we establish that all the needed discrete entropy inequalities are satisfied. Arguing this specific *numerical transport* entropy, we derive the expected *a posteriori* entropy condition to get an entropy preserving high-order MOOD scheme. Numerical experiments illustrate the relevance of the suggested numerical procedure.

1   Introduction

The present work concerns the derivation of entropy preserving high-order numerical schemes to approximate the weak solutions of the Euler equations given by

$$\begin{cases} \partial_t \rho + \partial_x \rho u = 0, \\ \partial_t \rho u + \partial_x (\rho u^2 + p) = 0, \\ \partial_t E + \partial_x (E + p) u = 0, \end{cases} \tag{1}$$

where the pressure is given by a perfect gas law:

$$p = (\gamma - 1) \left(E - \rho \frac{u^2}{2} \right),$$

for a given adiabatic coefficient $\gamma \in (1, 3]$.

To shorten the notations, let us introduce the conservative unknown state vector $w : \mathbb{R} \times \mathbb{R}^+ \to \Omega$ and the flux function $f : \Omega \to \mathbb{R}^3$ defined as follows:

$$w = {}^t(\rho, \rho u, E) \quad \text{and} \quad f(w) = {}^t(\rho u, \rho u^2 + p, (E + p)u), \tag{2}$$

with Ω the convex set of admissible states given by:

$$\Omega = \left\{ w \in \mathbb{R}^3; \ \rho > 0, e(w) = E - \rho \frac{u^2}{2} > 0 \right\}.$$

Here, the function $e : \Omega \to \mathbb{R}^+$ denotes the internal energy.

Because system (1) is well-known to be hyperbolic, the solutions may contains shock discontinuities (for instance, see [27, 39, 43, 20] and references therein). In order to rule out unphysical discontinuous solutions, the system under consideration must be endowed with entropy inequalities (see [37, 38, 43] for further details):

$$\partial_t \rho \mathcal{F}(\ln(s)) + \partial_x \rho \mathcal{F}(\ln(s)) u \leq 0 \quad \text{with} \quad s = \frac{p}{\rho^\gamma}, \tag{3}$$

where $\mathcal{F} : \mathbb{R} \to \mathbb{R}$ is a smooth function such that

$$w \mapsto S(w) = \rho \mathcal{F}(\ln(s)) \tag{4}$$

defines a convex map. To shorten the notation, we set

$$G(w) = \rho \mathcal{F}(\ln(s)) u. \tag{5}$$

After Tadmor [47] (see also [30, 43]), the function $\mathcal{F}$ must satisfy

$$\mathcal{F}'(y) < 0 \quad \text{and} \quad \mathcal{F}'(y) < \gamma \mathcal{F}''(y) \quad \text{for all } y \in \mathbb{R}. \tag{6}$$

Next, we consider the numerical approximation of the weak solutions of (1). Numerous numerical strategies can be found in the literature as soon as first-order finite volume methods are involved. For instance, the reader is referred to [28, 39, 48, 29, 9] where the usual numerical techniques are detailed. By denoting Δt a time step and $\Delta x = x_{i+1/2} - x_{i-1/2}$ a constant cell size, the approximation of $w(x_i, t^n + \Delta t)$ is given as follows:

$$w_i^{n+1} = w_i^n - \frac{\Delta t}{\Delta x} \left(f_{\Delta x}(w_i^n, w_{i+1}^n) - f_{\Delta x}(w_{i-1}^n, w_i^n) \right), \tag{7}$$

where $f_{\Delta x} : \Omega \times \Omega \to \mathbb{R}^3$ is a Lipschitz-continuous numerical flux function which is consistent:

$$f_{\Delta x}(w, w) = f(w).$$

Here, the time step is restricted according to a CFL condition:

$$\frac{\Delta t}{\Delta x} \max_{i \in \mathbb{Z}} |\lambda^\pm(w_i^n, w_{i+1}^n)| \leq \frac{1}{2}, \tag{8}$$

where $\lambda^\pm(w_i^n, w_{i+1}^n)$ represent some wave speeds associated to the considered numerical flux function $f_{\Delta x}(w_i^n, w_{i+1}^n)$.

The numerical flux function definition can be supplemented by additional robustness and stability properties. Concerning the robustness, the method must preserve the positiveness of both density and internal energy. Hence, as soon as the sequence $(w_i^n)_{i \in \mathbb{Z}}$ belongs to Ω, the adopted scheme must satisfy $w_i^{n+1} \in \Omega$.

In this work, the stability of the scheme is understood at the entropy level. Discrete entropy inequalities are imposed in order to exclude, at the discrete level, undesirable unphysical solutions. These reached discrete entropy inequalities read as follows:

$$\frac{1}{\Delta t} \left(S(w_i^{n+1}) - S(w_i^n) \right) + \frac{1}{\Delta x} \left(G_{\Delta x}(w_i^n, w_{i+1}^n) - G_{\Delta x}(w_{i-1}^n, w_i^n) \right) \leq 0, \tag{9}$$

where $S(w)$ is defined by (4) and $G_{\Delta x} : \Omega \times \Omega \to \mathbb{R}$ denotes the entropy numerical flux function which must be consistent:

$$G_{\Delta x}(w, w) = G(w).$$

The most common first-order finite volume methods (7) are proven to satisfy robustness and/or stability properties. For instance, we cite [31] for the HLL scheme, [26, 25, 14] for the extension to simple approximate Riemann solvers, [9, 15, 6] for relaxation schemes, [9, 49, 48, 2] for HLLC scheme, [42, 31, 23, 8, 13] for the Roe and the extension VFRoe schemes. Of course the previous list is not exhaustive.

Now, numerous strategies have been proposed to increase the order of accuracy. One of the most popular, and adopted in the present paper, is based on a suitable reconstruction of the state vector on each side of the interfaces located at $x_{i+1/2}$. Indeed, in (7), $f_{\Delta x}(w_i^n, w_{i+1}^n)$ is nothing but a first-order evaluation of the flux function at the interface $x_{i+1/2}$. The space second-order (or high-order) extension is obtained by involving a second-order (or high-order) evaluation of the flux now given by

$$f_{\Delta x} \left(w_{i+1/2}^-, w_{i+1/2}^+ \right),$$

where $w^{\pm}_{i+1/2}$ denote reconstructed states. Techniques to derive $w^{\pm}_{i+1/2}$ are widely studied in the literature and it is here impossible to refer all the papers devoted to such a topic. Let us just mention the MUSCL reconstruction [50, 39, 5, 10, 34, 40, 35, 16, 19], the kinetic second-order approaches [40, 35], the ENO/WENO reconstruction [41, 55, 54], the PPM reconstruction [52], the MOOD reconstruction [17, 21], and plenty of extensions...

In fact, these high-order finite volume methods, which now read as follows:

$$w_i^{n+1} = w_i^n - \frac{\Delta t}{\Delta x} \left(f_{\Delta x} \left(w^-_{i+1/2}, w^+_{i+1/2} \right) - f_{\Delta x} \left(w^-_{i-1/2}, w^+_{i-1/2} \right) \right), \qquad (10)$$

involve difficulties to derive robustness and stability properties. The Ω-preserving property to be satisfy by (10) is now well studied. It is obtained by introducing a suitable limitation procedure inside the reconstruction technique. We refer to [39, 9] where basic MUSCL reconstructions are considered, and to [5, 7] where robustness of more sophisticated approaches are studied. In [41], the required robustness is established within the WENO reconstruction framework.

Let us underline that these procedures to enforce the needed Ω-preserving property involve *a priori* limitation techniques. Put in other words, these limitations are global and, sometime, turn out to be too strong. As a consequence, such usual limitations may be too diffusive. To correct this loss of accuracy, the MOOD method has been recently presented in [17, 21]. It suggests to introduce an *a posteriori* limitation technique. Hence, the limitation is just local in space to reduce the numerical viscosity and to increase the accuracy of the method.

The difficulties turn out to be very distinct as soon as stability properties must be proven for high-order schemes given by (10). Several attempts are proposed in the literature. One proposed strategy is based on the Generalized Riemann Problem [3, 12, 11]. Unfortunately, the solutions of the GRP associated with (1) are very difficult to be exhibited, and this makes poorly attractive the resulting scheme. In [19, 18], the authors suggest to adopt new projection techniques but the obtained numerical methods are, in general, sophisticated and extensions to more complex problems seem delicate. In the same spirit, we cite the work by Bourdarias et al. [10] but, as specified by the authors, the derived scheme cannot be easily implemented. More recently, in [5], discrete entropy inequalities are obtained but for a specific entropy time derivative discrete operator (see also [10]). Moreover, these stability results are unluckily obtained by enforcing strong limitation procedures and thus by enforcing a lot of numerical viscosity. In addition, the relevance of the unusual entropy time derivative discrete operator, according to the well-known Lax-Wendroff Theorem, is not established. Put in other words, we are not able to prove (up to our knowledge) that the considered discrete entropy inequalities converge, in a sense to be prescribed, to the expected entropy inequalities (3). In [5] (see also [55, 54, 35]), an additional stability criterion is obtained by enforcing an entropy maximum principle [47]. However, this stability condition is weaker than the usual discrete entropy inequalities and, as a consequence, such a maximum principle is not considered in the present work.

In order to derive robust and entropy preserving high-order schemes, we here adopt the *a posteriori* MOOD technique [17, 21] (see also [33] for a related method).

In the next section, we give our main motivations by briefly studying the convergence behavior of the discrete entropy inequalities as stated in [5, 10]. These motivations are completed by numerical experiments performed with standard MUSCL schemes over very fine meshes. It turns out that these numerical approaches are not stable at all as soon as the mesh size is small enough. As a consequence, we suggest to modify the usual MUSCL schemes, or equivalently the usual high-order reconstructions, by introducing an *a posteriori* limitation according to only one discrete entropy inequality. Indeed, an *a posteriori* entropy evaluation cannot be performed by considering the whole space of convex entropy functions and we have to deal with one particular discrete entropy inequality. Hence, in Section 3, considering one specific discrete entropy inequality, we prove that all the reached discrete entropy inequalities can be restored. Equipped with such a result, Section 4 is devoted to the derivation of the e-MOOD scheme by introducing inside the adopted initial high-order scheme (here, MUSCL scheme to simplify), an *a posteriori* restriction given by the preservation of the relevant discrete entropy inequality. This procedure is illustrated by several numerical experiments in the last section.

2   Main motivations

The objective of the present paper is to derive high-order accurate entropy preserving schemes to approximate the weak solutions of (1). One of the main problem arising when dealing with high-order schemes concerns the derivation of suitable discrete entropy inequalities. Let us just recall that the discrete entropy inequalities are derived so that the converged solution is entropy preserving. Put in other words, the considered discrete entropy inequalities must converge, in the sense of the well-known Lax-Wendroff Theorem [36] (see also [39, 24]), to the expected continuous entropy inequalities (3).

In fact, several high-order (MUSCL) entropy inequalities have been derived in the recent literature (for instance, see [5, 10]). But, it is not convincing that these discrete inequalities satisfy the expected convergence behavior. The purpose of the present section is to briefly study the behavior of the usual high-order entropy inequalities inside the convergence regime. In fact, at the end of this section, we will present several numerical experiments to exhibit the failures of MUSCL schemes to restore (3), based on the usual reconstruction techniques and both first- and high-order time discretization. We do not rigorously justify these failures but some arguments are here given.

First, for the sake of completeness, we recall the Lax-Wendroff Theorem for high-order (space and time) accurate conservative schemes. It is the opportunity to report the expected high-order entropy inequalities to be satisfied so that the converged solution is entropy preserving according to (3). Next, we briefly review the usual discrete entropy inequalities coming from high-order space and time accurate schemes. We will show that these usual discrete entropy inequalities coincide with the required one within the Lax-Wendroff Theorem up to a positive measure. Put in other words, the usual discrete entropy inequalities seem insufficient to ensure that the converged solution is entropy preserving. We illustrate this negative result with

numerous numerical experiments.

2.1 Lax-Wendroff theorem for high-order schemes

We approximate the weak solutions of a hyperbolic system of conservation laws in the shortened form:

$$\begin{cases} \partial_t w + \partial_x f(w) = 0, \\ w(x, t = 0) = w_0(x), \end{cases} \tag{11}$$

where the state vector and the flux function are given by (2), and supplemented by the entropy inequalities (3).

We adopt a general m-step Runge-Kutta time scheme written as follows:

$$w_i^{n,(0)} = w_i^n,$$

$$w_i^{n,(\ell)} = w_i^n - \frac{\Delta t}{\Delta x} \sum_{j=0}^{\ell-1} c_{\ell,j} \left(f_{i+1/2}^{n,(j)} - f_{i-1/2}^{n,(j)} \right), \quad \ell = 1, \cdots, m, \tag{12}$$

$$w_i^{n+1} = w_i^{n,(m)}.$$

The coefficients $c_{\ell,j}$ are assumed to satisfy the following consistency conditions:

$$c_{\ell,j} \geq 0, \quad \sum_{j=0}^{m-1} c_{m,j} = 1. \tag{13}$$

We impose the scheme to be space high-order. To address such an issue, we consider a numerical flux function depending on a large stencil:

$$f_{i+1/2}^{n,(j)} = f_{\Delta x}^s \left(w_{i-s+1}^{n,(j)}, \cdots, w_{i+s}^{n,(j)} \right), \tag{14}$$

where $f_{\Delta x}^s : \Omega^{2s} \to \mathbb{R}^3$ is continuous and consistent:

$$f_{\Delta x}^s(w, \cdots, w) = f(w).$$

As usual, the initial data is here approximated as follows:

$$w_i^0 = \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} w_0(x) dx.$$

For the sake of simplicity in the forthcoming statements, we introduce the following piecewise constant functions:

$$w^\Delta(x, t) = w_i^n, \quad \text{for } (x, t) \in [x_{i-1/2}, x_{i+1/2}) \times [t^n, t^n + \Delta t),$$

$$w^{\Delta,(\ell)}(x, t) = w_i^{n,(\ell)}, \quad \text{for } (x, t) \in [x_{i-1/2}, x_{i+1/2}) \times [t^n, t^n + \Delta t).$$

THEOREM 2.1 (LAX-WENDROFF) Assume that the sequence Δx tends to zero with a constant positive ratio $\Delta t/\Delta x$, and assume

- there exists a compact $K \subset \Omega$ such that, for all $0 \leq \ell \leq m$, $w^{\Delta,(\ell)}$ belongs to K,

- the sequence w^Δ converges in $L^1_{\mathrm{loc}}(\mathbb{R} \times \mathbb{R}^+; \Omega)$ to a function w.

Then w is a weak solution of (11).

In addition, let us assume the existence of an entropy numerical flux $G^s_{\Delta x} : \Omega^{2s} \to \mathbb{R}$, which is Lipschitz-continuous and consistent:

$$G^s_{\Delta x}(w, \cdots, w) = G(w),$$

where G is defined by (5), such that we have the following discrete entropy inequality:

$$\frac{1}{\Delta t}\left(S(w_i^{n+1}) - S\left(w_i^n\right)\right) + \frac{1}{\Delta x}\sum_{j=0}^{m-1} c_{m,j}\left(G_{i+1/2}^{n,(j)} - G_{i-1/2}^{n,(j)}\right) \leq 0, \qquad (15)$$

with $G_{i+1/2}^{n,(j)} = G^s_{\Delta x}\left(w_{i-s+1}^{n,(j)}, \cdots, w_{i+s}^{n,(j)}\right)$.

Then w is an entropy solution of (11)-(3).

Here, we point out the difficulties coming from establishing the discrete entropy inequalities (15). In fact, several first-order schemes in the form (7), like Godunov scheme, HLL and HLLC schemes, relaxation schemes, Osher scheme [10, 6, 9, 4, 28] are proven to satisfy such required discrete entropy inequalities. Unfortunately, by extending these first-order entropy preserving schemes to get high-order numerical methods in the form (12), we do not recover high-order discrete entropy inequalities given by (15). Our purpose is now to exhibit high-order discrete entropy inequalities inheriting from time and space high-order extensions, and to consider their convergence behavior.

The proof of the Lax-Wendroff Theorem (2.1) is classical and several versions can be found in [36, 27, 39, 24]. However, because of the high-order discrete entropy inequalities (15), up to the authors knowledge, no complete proof can be found in the literature. Although the proof is standard, for the sake of completeness, we detail it in Appendix A.

2.2 Space high-order discrete entropy inequalities

From a given first-order scheme in the form (7) satisfying first-order discrete entropy inequalities (9), numerous methods have been introduced to increase the order of accuracy (for instance see [1, 3, 17, 51, 44]). In the present work, we restrict ourselves to MUSCL reconstruction techniques known to give second-order space accurate schemes. However, the reader is referred to [8, 53] where high-order MUSCL reconstructions are suggested.

We recall that the MUSCL approach is based on a vector state reconstruction on each side of the interface located at $x_{i+1/2}$ as follows:

$$w_{i+1/2}^- = w_i^n + \frac{1}{2}\mu_i^n \quad \text{and} \quad w_{i+1/2}^+ = w_{i+1}^n - \frac{1}{2}\mu_{i+1}^n. \qquad (16)$$

The increment $(\mu_i^n)_{i \in \mathbb{Z}}$ is defined by a limiter function to read:

$$\mu_i^n = L\left(w_i^n - w_{i-1}^n, w_{i+1}^n - w_i^n\right), \qquad (17)$$

where $L : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ is a Lipschitz-continuous function, which satisfies:

$$L(v, v) = v \quad \forall v \in \mathbb{R}^3, \tag{18}$$

$$\exists M > 0; \quad \|L(v_1, v_2)\| \leq M \max(\|v_1\|, \|v_2\|), \quad \forall v_1, v_2 \in \mathbb{R}^3. \tag{19}$$

Precise definitions of L are widely studied in the literature (for instance, see [39] and references therein). From now on, let us underline that the usual limiter functions (minmod, superbee, MC,...) satisfy the requirements (18)-(19).

Next, from the first-order scheme (7), we get a space second-order scheme in the form (10). Concerning the second-order discrete entropy inequalities associated with (10), several strategies have been recently proposed. For instance, in [5], independently from the limiter choice L, one get the following discrete entropy inequalities:

$$\frac{1}{\Delta t}\left(S\left(w_i^{n+1}\right) - \frac{1}{2}\left(S\left(w_{i+1/2}^+\right) + S\left(w_{i+1/2}^-\right)\right)\right) +$$
$$\frac{1}{\Delta x}\left(G_{\Delta x}\left(w_{i+1/2}^-, w_{i+1/2}^+\right) - G_{\Delta x}\left(w_{i-1/2}^-, w_{i-1/2}^+\right)\right) \leq 0. \tag{20}$$

A second example can be found in [10] where a specific MUSCL procedure is introduced to get

$$\frac{1}{\Delta t}\left(S\left(w_i^{n+1}\right) - \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} S\left(w_i^n + \frac{x - x_i}{\Delta x}\mu_i^n\right)dx\right) +$$
$$\frac{1}{\Delta x}\left(G_{\Delta x}\left(w_{i+1/2}^-, w_{i+1/2}^+\right) - G_{\Delta x}\left(w_{i-1/2}^-, w_{i-1/2}^+\right)\right) \leq 0. \tag{21}$$

Immediately, we notice that the discrete time derivative involved in both (20) and (21) does not coincide with the required one given by (15). Our purpose is now to illustrate that these variants of the discrete entropy inequalities are not efficient and are not relevant to get an entropy converged solution.

In the sequel, it will be useful to unify the notations. We rewrite (20) and (21) as follows:

$$\frac{1}{\Delta t}\left(S\left(w_i^{n+1}\right) - S\left(w_i^n\right)\right) + \frac{1}{\Delta x}\left(G_{i+1/2}^n - G_{i-1/2}^n\right) \leq \frac{1}{\Delta t}\left(P_i^n - S(w_i^n)\right), \tag{22}$$

where $P_i^n = P_S\left(w_i^n, \mu_i^n, \Delta x\right)$ finds an immediate definition with P_S being an operator associated with an entropy function S. Indeed, if we consider (20), we obtain:

$$P_S(w, \mu, \Delta x) = \frac{S(w - \mu/2) + S(w + \mu/2)}{2}. \tag{23}$$

Next, if we consider (21), we obtain:

$$P_S(w, \mu, \Delta x) = \frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2} S\left(w + \frac{x}{\Delta x}\mu\right)dx. \tag{24}$$

Let us emphasize that since S is a convex function, we have in both definitions (23) and (24) as long as P is well defined

$$P_S(w, \mu, \Delta x) - S(w) \geq 0, \quad \forall w \in \Omega.$$

In fact, P can be understood as a projection to approximate the entropy evaluated on w_i^n. We impose the existence of a positive constant C such that

$$0 \leq P_S(w, \mu, \Delta x) - S(w) \leq C \left\|\nabla^2 S\right\| \|\mu\|^2. \qquad (25)$$

This property is easily satisfied by both definition (23) and (24).

Now, we will see that $\frac{1}{\Delta t}(P_i^n - S(w_i^n))$ converges to a positive measure to make unsuitable the discrete entropy inequalities (22). In order to provide a complete illustration of the failure of (22), we propose to extend these space high-order discrete entropy inequalities by considering time high-order accurate schemes.

2.3 Time high-order discrete entropy inequalities

In order to increase the time order of accuracy, we here adopt the usual Runge-Kutta time scheme to consider a numerical approximation given by (12). To write the discrete entropy inequalities associated with (12), we adopt a reformulation of (12) introduced by Shu and Osher [45, 46]. It consists in writing (12) as a convex combination of time first-order schemes. We skip the computation details given in [45, 46], but we just recall that, for all positive parameters $(\alpha_{\ell,j})_{\substack{1 \leq \ell \leq m \\ 0 \leq j \leq \ell-1}}$ such that

$$\sum_{j=0}^{\ell-1} \alpha_{\ell,j} = 1, \quad \text{for all} \quad 1 \leq \ell \leq m,$$

the m-step Runge-Kutta time scheme (12) can be equivalently rewritten as follows:

$$\begin{aligned}
w_i^{n,(0)} &= w_i^n, \\
w_i^{n,(\ell)} &= \sum_{j=0}^{\ell-1} \alpha_{\ell,j} \left(w_i^{n,(j)} - \frac{\beta_{\ell,j}}{\alpha_{\ell,j}} \frac{\Delta t}{\Delta x} \left(f_{i+1/2}^{n,(j)} - f_{i-1/2}^{n,(j)} \right) \right), \qquad (26) \\
w_i^{n+1} &= w_i^{n,(m)},
\end{aligned}$$

where the coefficients $\beta_{\ell,j}$ are given by

$$\beta_{\ell,j} = c_{\ell,j} - \sum_{k=j+1}^{\ell-1} \alpha_{\ell,k} c_{k,j}.$$

The sequence $(\alpha_{\ell,j})_{\substack{1 \leq \ell \leq m \\ 0 \leq j \leq \ell-1}}$ is chosen in order to enforce the positiveness of the parameters $\beta_{\ell,j}$.

Now, since the parameters $\alpha_{\ell,j}$ and $\beta_{\ell,j}$ are positive, we note that the intermediate states $w_i^{n,(\ell)}$ are nothing but a convex combination of first-order time schemes with time steps respectively given by $\frac{\beta_{\ell,j}}{\alpha_{\ell,j}} \Delta t$.

Next, we establish the discrete entropy inequalities satisfied by the time high-order scheme (12), or equivalently (26). Let us emphasize that the following result turns out to be independent from the adopted space order of accuracy.

LEMMA 2.2 Let us consider a time first-order scheme given by

$$w_i^{n+1} = w_i^n - \frac{\Delta t}{\Delta x} \left(f_{i+1/2}^n - f_{i-1/2}^n \right),$$
$$f_{i+1/2}^n = f_{\Delta x}^s(w_{i-s+1}^n, \cdots, w_{i+s}^n),$$

supplemented by discrete entropy inequalities as follows:

$$\frac{1}{\Delta t} \left(S(w_i^{n+1}) - S(w_i^n) \right) + \frac{1}{\Delta x} \left(G_{i+1/2}^n - G_{i-1/2}^n \right) \leq \delta \left(w_i^n \right), \qquad (27)$$

where $G_{i+1/2}^n = G_{\Delta x}^s(w_{i-s+1}^n, \cdots, w_{i+s}^n)$ and $\delta\left(w_i^n \right)$ is a positive perturbation.

Assume that the parameters $\alpha_{\ell,j} > 0$ are defined such that the parameters $\beta_{\ell,j}$ are nonnegative. Then the scheme (12) satisfies the following discrete entropy inequalities:

$$\frac{1}{\Delta t} \left(S(w_i^{n+1}) - S(w_i^n) \right) + \sum_{j=0}^{m-1} c_{m,j} \frac{1}{\Delta x} \left(G_{i+1/2}^{n,(j)} - G_{i-1/2}^{n,(j)} \right) \leq \sum_{j=0}^{m-1} \alpha_{m,j} \delta \left(w_i^{n,(j)} \right). \tag{28}$$

Before to prove this result, let us comment the role played by the perturbation $\delta(w_i^n)$ centered on w_i^n, but which may depends on other states. As soon as a standard space and time scheme, in the form (7), satisfies discrete entropy inequalities (9), we get a vanishing perturbation, $\delta(w) = 0$ for all w in Ω. As a consequence, inequalities (28) exactly coincide with the required discrete entropy inequalities (15). More generally, this means that, as long as the first-order scheme is entropy preserving (with $\delta(w_i^n) = 0$), then the time high-order Runge-Kutta scheme remains entropy preserving. Now, the situation turns out to be distinct whenever $\delta(w_i^n) \neq 0$, and the right hand side in (28) must be carefully studied.

Proof Let us introduce the intermediate states as follows:

$$\widetilde{w}_i^{n,j} = w_i^{n,(j)} - \frac{\beta_{\ell,j}}{\alpha_{\ell,j}} \frac{\Delta t}{\Delta x} \left(f_{i+1/2}^{n,(j)} - f_{i-1/2}^{n,(j)} \right).$$

Since $\frac{\beta_{\ell,j}}{\alpha_{\ell,j}} \geq 0$, the state $\widetilde{w}_i^{n,j}$ is nothing but the evolution state by a time first-order scheme. As a consequence of (27), the intermediate states $\widetilde{w}_i^{n,j}$ satisfies a discrete entropy inequality given by

$$\frac{1}{\Delta t} \left(S\left(\widetilde{w}_i^{n,j} \right) - S\left(w_i^{n,(j)} \right) \right) + \frac{\beta_{\ell,j}}{\alpha_{\ell,j} \Delta x} \left(G_{i+1/2}^{n,(j)} - G_{i-1/2}^{n,(j)} \right) \leq \delta \left(w_i^{n,(j)} \right).$$

From the equivalent formulation (26), let us notice that

$$w_i^{n,(\ell)} = \sum_{j=1}^{\ell-1} \alpha_{\ell,j} \widetilde{w}_i^{n,j}.$$

Next, since S is a convex function, we obtain

$$S\left(w_i^{n,(\ell)} \right) \leq \sum_{j=0}^{\ell-1} \alpha_{\ell,j} S\left(\widetilde{w}_i^{n,j} \right),$$

to immediately deduce

$$S\left(w_i^{n,(\ell)}\right) \leq \sum_{j=0}^{\ell-1} \left(\alpha_{\ell,j} S\left(w_i^{n,(j)}\right) - \beta_{\ell,j} \frac{\Delta t}{\Delta x} \left(G_{i+1/2}^{n,(j)} - G_{i-1/2}^{n,(j)}\right)\right) +$$

$$\Delta t \sum_{j=0}^{\ell-1} \alpha_{\ell,j} \delta\left(w_i^{n,(j)}\right). \quad (29)$$

Now, involving a standard proof by induction, we establish the following inequality:

$$S\left(w_i^{n,(\ell)}\right) \leq S\left(w_i^n\right) - \frac{\Delta t}{\Delta x} \sum_{j=0}^{\ell-1} c_{\ell,j} \left(G_{i+1/2}^{n,(j)} - G_{i-1/2}^{n,(j)}\right) +$$

$$\Delta t \sum_{j=0}^{\ell-1} \alpha_{\ell,j} \delta\left(w_i^{n,(j)}\right), \qquad 1 \leq \ell \leq m. \quad (30)$$

For $\ell = 1$, we immediately have $\alpha_{1,0} = 1$ and $c_{1,0} = \beta_{1,0}$. Then (30) is deduced from (29).

Next, let us assume that (30) holds true for all j such that $1 \leq j \leq \ell - 1$ and let us establish the equality for ℓ. From (29) and substituting $\left(S\left(w_i^{n,(j)}\right)\right)_{1 \leq j \leq \ell-1}$ by the estimation (30), we obtain

$$S\left(w_i^{n,(\ell)}\right) \leq \sum_{j=0}^{\ell-1} \left(\alpha_{\ell,j} \left(S\left(w_i^n\right) - \frac{\Delta t}{\Delta x} \sum_{k=0}^{j-1} c_{j,k} \left(G_{i+1/2}^{n,(k)} - G_{i-1/2}^{n,(k)}\right)\right) - \right.$$

$$\left. \beta_{\ell,j} \frac{\Delta t}{\Delta x} \left(G_{i+1/2}^{n,(j)} - G_{i-1/2}^{n,(j)}\right)\right) + \Delta t \sum_{j=0}^{\ell-1} \alpha_{\ell,j} \delta\left(w_i^{n,(j)}\right),$$

$$\leq S\left(w_i^n\right) - \frac{\Delta t}{\Delta x} \sum_{j=0}^{\ell-1} \left(\beta_{\ell,j} + \sum_{k=j+1}^{\ell-1} \alpha_{\ell,k} c_{k,j}\right) \left(G_{i+1/2}^{n,(j)} - G_{i-1/2}^{n,(j)}\right) +$$

$$\Delta t \sum_{j=0}^{\ell-1} \alpha_{\ell,j} \delta\left(w_i^{n,(j)}\right),$$

and (30) is stated. Since $w_i^{n+1} = w_i^{n,(m)}$, by involving (2.3), the proof is achieved. □

Equipped with the above result, we are now able to exhibit the discrete entropy inequalities associated with both space and time high-order accurate schemes (12)-(14). Indeed, since the associated time first-order scheme comes with discrete entropy inequalities given by (22), we easily get discrete entropy perturbations given by

$$\delta\left(w_i^n\right) = \frac{1}{\Delta t} \left(P_i^n - S\left(w_i^n\right)\right).$$

As a consequence, we are discussing about the following high-order discrete entropy inequalities:

$$\frac{1}{\Delta t}\left(S\left(w_i^{n+1}\right) - S\left(w_i^n\right)\right) + \sum_{j=0}^{m-1} \frac{c_{m,j}}{\Delta x}\left(G_{i+1/2}^{n,(j)} - G_{i-1/2}^{n,(j)}\right) \leq$$

$$\sum_{j=0}^{m-1} \alpha_{m,j}\frac{1}{\Delta t}\left(P_i^{n,(j)} - S\left(w_i^{n,(j)}\right)\right). \quad (31)$$

Under the assumptions stated in Theorem 2.1, we easily obtain the weak convergence of the left-hand side to $\partial_t S(w) + \partial_x G(w)$. Regarding the right-hand side, we set

$$a^\Delta(x,t) = \sum_{j=0}^{m-1} \alpha_{m,j}\frac{1}{\Delta t}\left(P_i^{n,(j)} - S\left(w_i^{n,(j)}\right)\right), \quad (x,t) \in [x_{i-1/2}, x_{i+1/2}) \times [t^n, t^{n+1}).$$

$$(32)$$

Now, let us introduce the nonnegative measure δ defined as the weak-star limit of the sequence a^Δ. Hence, in the limit of Δx and Δt to zero with a constant ratio $\Delta t/\Delta x$, the inequality (31) reads

$$\partial_t S(w) + \partial_x G(w) \leq \delta.$$

We suggest to compare the measure δ to the entropy dissipation measure β, which is defined as the weak-star limit of the following sequence:

$$b^\Delta(x,t) = \sum_{j=0}^{m-1} \alpha_{m,j}\frac{1}{\Delta x}\left\|w_i^{n,(j)} - w_{i-1}^{n,(j)}\right\|^2, \quad (x,t) \in [x_{i-1/2}, x_{i+1/2}) \times [t^n, t^{n+1}).$$

The entropy dissipation measure β was studied by Hou and LeFloch [32] (see also DiPerna [22]) in the scalar case and with a first-order time scheme. They conjectured that this measure is concentrated on the curves of discontinuity of w.

In the following statement, we establish that both measure δ and β have the same behavior.

THEOREM 2.3 The measure δ is absolutely continuous with respect to the entropy dissipation measure β.

Proof Let ϕ be a nonnegative test function with compact support K, and we set $\phi_i^n = \phi(x_i, t^n)$. Since P satisfies the property (25), we have

$$\sum_{i,n}\left(P_i^{n,(j)} - S\left(w_i^{n,(j)}\right)\right)\phi_i^n\Delta t \leq C\|\nabla^2 S\|_{L^\infty(K)}\sum_{i,n}\left\|\mu_i^{n,(j)}\right\|^2\phi_i^n\Delta t,$$

where $\nabla^2 S$ is bounded over K, and $\mu_i^{n,(j)}$ denotes the reconstructed increments defined by (17).

By involving (17) and (19), we get

$$\sum_{i,n} \left(P_i^{n,(j)} - S\left(w_i^{n,(j)} \right) \right) \phi_i^n \Delta t$$

$$\leq O(1) \sum_{i,n} \left(\left\| w_i^{n,(j)} - w_{i-1}^{n,(j)} \right\|^2 + \left\| w_{i+1}^{n,(j)} - w_i^{n,(j)} \right\|^2 \right) \phi_i^n \Delta t,$$

$$\leq O(1) \sum_{i,n} \left\| w_i^{n,(j)} - w_{i-1}^{n,(j)} \right\|^2 \left(\phi_i^n + \phi_{i-1}^n \right) \Delta t.$$

Since the ratio $\Delta t / \Delta x$ remains constant, we deduce

$$\sum_{i,n} \sum_{j=0}^{m-1} \alpha_{m,j} \left(P_i^{n,(j)} - S\left(w_i^{n,(j)} \right) \right) \phi_i^n \Delta t \leq$$

$$O(1) \sum_{i,n} \sum_{j=0}^{m-1} \alpha_{m,j} \left\| w_i^{n,(j)} - w_{i-1}^{n,(j)} \right\|^2 \left(\phi_i^n + \phi_{i-1}^n \right) \Delta x.$$

Passing to the limit, we get

$$\int \phi d\delta \leq O(1) \int \phi d\beta,$$

and the proof is completed. $\qquad\square$

To conclude this section, let us emphasize that we have established the absolute continuity of the measure δ with respect to β, while one may expect the equivalence between these two measures. In fact, the numerical results presented in Section 2.4 will confirm such an assumption. Nowadays we are not able to establish the absolute continuity of β with respect to δ. Moreover, the discrete entropy inequalities (31) cannot ensure the required entropy stability.

2.4 Numerical tests

We turn considering the numerical illustration of the above results. More precisely, our objective is here to numerically evaluate the measure δ introduced previously. According to the work by Hou and LeFloch [32], this measure must vanish as long as the solution is continuous. Reversely, whenever the solution admits shock discontinuities, the evaluation of δ must give $\delta > 0$.

All the presented numerical experiments are based on the same strategy. We adopt a space first-order numerical flux function $f_{\Delta x}(w_L, w_R)$ given by the well-known HLLC scheme [48, 49]. The benefit of such a numerical flux function is to exactly know the robustness and the discrete entropy inequalities [9, 6, 4, 15]. The space second-order accuracy is obtained by a MUSCL reconstruction (16) where the limiter function (17) is the minmod function, the van Albada 1 function, the van Leer function, the monotonized central-difference (MC) function or the Superbee function (see [39] where all the limiter functions are detailed). Concerning the time discretization, both first- and second-order accuracy are adopted.

According to [39, 9, 5], the time increment Δt is restricted by the following CFL condition:

$$\frac{\Delta t}{\Delta x} \max_{i \in \mathbb{Z}} \left(\left| \lambda^\pm \left(w_{i+1/2}^-, w_{i+1/2}^+ \right) \right|, \left| \lambda^\pm \left(w_{i-1/2}^+, w_{i+1/2}^- \right) \right| \right) \leq \frac{1}{4}.$$

After [5], this time restriction makes the considered scheme robust and preserves the discrete entropy inequalities (31).

The relevance of each compared scheme is evaluated by calculating the L^1-error:

$$E^\Delta = \sum_{i \in \mathbb{Z}} \left| \rho_i^N - \rho^{ex}(x_i, t^N) \right| \Delta x,$$

where $w^{ex} : \mathbb{R} \times \mathbb{R}^+ \to \Omega$ denotes the exact solution. In addition, we evaluate the measure δ by computing its total mass:

$$I^\Delta = \sum_{n=0}^{N} \sum_{i \in \mathbb{Z}} a^\Delta(x_i, t^n) \Delta x \Delta t.$$

Two numerical experiments are performed. Both are devoted to the approximation of the solution of Riemann problems. Hence, the initial data is made of two constant states separated by a discontinuity located at $x = 0$:

$$w_0(x) = \begin{cases} w_L & \text{if } x < 0, \\ w_R & \text{if } x > 0. \end{cases} \tag{33}$$

In the first test, left and right states are given by

$$\begin{aligned} \rho_L &= 1, & \rho_R &= 0.1989, \\ u_L &= -1, & u_R &= 1, \\ p_L &= 1.5, & p_R &= 0.1564, \end{aligned} \tag{34}$$

so that the exact solution is made of a continuous 1-rarefaction.

In Tables 1 and 2, we give respectively the evaluation of E^Δ and I^Δ obtained by considering a time first-order scheme with several limiter functions. First of all, we note that van Leer, MC and Superbee are not stable enough and a numerical blowup appears with very fine mesh. Concerning minmod and van Albada 1 limiter functions, the behavior is better because both schemes seem to converge since E^Δ goes to zero as Δx tends to zero. At this level, we may suspect that the blowups are consequences of some compression phenomena, while the minmod limiter and the van Albada 1 limiter seem diffusive enough to avoid such a failure. According to the work by Hou and LeFloch [32], since the converged solution is continuous, the entropy dissipation measure I^Δ goes to zero and thus the measure δ is equal to zero. Figure 1 illustrates the results stated in Tables 1 and 2.

Next, Tables 3 and 4 and Figure 2 are devoted to the results obtained with a time second-order scheme. Excepted with the superbee limiter, all considered schemes seem to converge like the measure δ, which tends to zero.

Nb cells	minmod	van Albada	van Leer	MC	superbee
125	7.54E-3	9.98E-3	1.09E-2	1.34E-2	3.42E-2
250	5.19E-3	6.64E-3	7.40E-3	1.06E-2	3.00E-2
500	3.12E-3	4.17E-3	5.00E-3	9.38E-3	2.35E-2
1000	1.84E-3	2.72E-3	3.70E-3	8.77E-3	1.88E-2
2000	1.07E-3	2.01E-3	2.82E-3	8.58E-3	2.48E-2
4000	6.13E-4	1.37E-3	2.35E-3	9.28E-3	9.98E-2
8000	3.63E-4	1.19E-3	2.44E-3	3.84E-2	1.21E-1
16000	2.41E-4	1.24E-3	3.51E-3	5.34E-2	
32000	1.80E-4	1.26E-3			

Table 1: L^1 error E^Δ for the 1-rarefaction using first-order time discretisation

Nb cells	minmod	van Albada	van Leer	MC	superbee
125	1.48E-2	1.79E-2	2.00E-2	2.27E-2	4.39E-2
250	9.36E-3	1.12E-2	1.25E-2	1.52E-2	3.22E-2
500	5.70E-3	6.81E-3	7.81E-3	1.01E-2	2.35E-2
1000	3.38E-3	4.10E-3	4.96E-3	8.53E-3	1.84E-2
2000	1.96E-3	2.51E-3	3.24E-3	7.12E-3	3.03E-2
4000	1.12E-3	1.55E-3	2.26E-3	6.38E-3	7.50E-1
8000	6.30E-4	1.02E-3	1.74E-3	8.04E-2	1.48E+1
16000	3.53E-4	7.54E-4	2.18E-3	1.42E-0	
32000	1.99E-4	6.18E-4			

Table 2: Total mass I^Δ of the right-hand side of the proven entropy inequality for the 1-rarefaction using first-order time discretisation
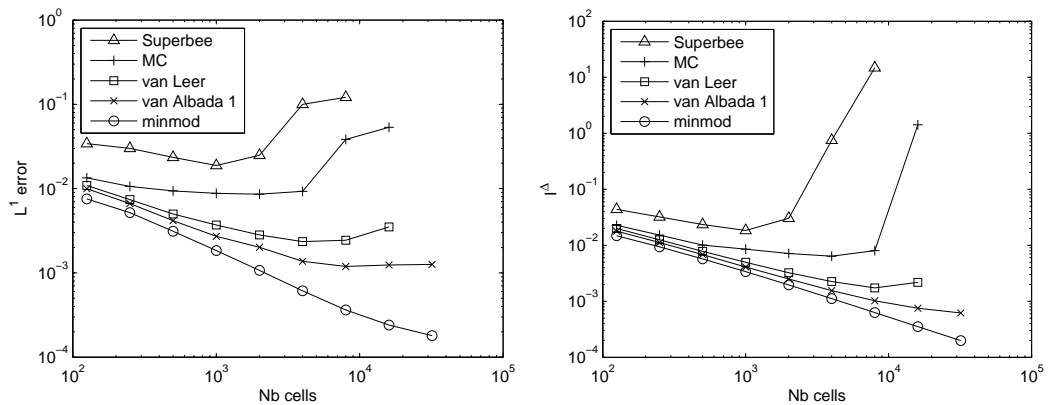

Figure 1: 1-rarefaction with first-order time scheme: L^1 error (left) and total mass I^Δ of the right-hand side of the proven entropy inequality (right)

Nb cells	minmod	van Albada	van Leer	MC	superbee
125	6.82E-3	5.41E-3	4.55E-3	3.87E-3	4.93E-3
250	5.38E-3	4.37E-3	3.86E-3	3.46E-3	2.37E-3
500	2.72E-3	2.19E-3	1.94E-3	1.74E-3	1.16E-3
1000	1.36E-3	1.10E-3	9.68E-4	8.69E-4	5.74E-4
2000	6.82E-4	5.49E-4	4.84E-4	4.35E-4	2.89E-4
4000	3.41E-4	2.74E-4	2.42E-4	2.17E-4	1.48E-4
8000	1.71E-4	1.37E-4	1.21E-4	1.09E-4	7.68E-5
16000	8.54E-5	6.86E-5	6.05E-5	5.43E-5	4.12E-5
32000	4.27E-5	3.43E-5	3.03E-5	2.72E-5	4.92E-5
64000				1.36E-5	8.39E-3

Table 3: L^1 error E^Δ for the 1-rarefaction using second-order time discretisation

Nb cells	minmod	van Albada	van Leer	MC	superbee
125	1.44E-2	1.67E-2	1.81E-2	1.93E-2	2.32E-2
250	9.08E-3	1.03E-2	1.10E-2	1.16E-2	1.38E-2
500	5.51E-3	6.17E-3	6.54E-3	6.85E-3	8.01E-3
1000	3.25E-3	3.60E-3	3.79E-3	3.96E-3	4.61E-3
2000	1.88E-3	2.06E-3	2.16E-3	2.25E-3	2.63E-3
4000	1.07E-3	1.16E-3	1.16E-3	1.27E-3	1.51E-3
8000	5.98E-4	6.50E-4	6.77E-4	7.03E-4	8.94E-4
16000	3.31E-4	3.59E-4	3.73E-4	3.88E-4	5.66E-4
32000	1.82E-4	1.97E-4	2.04E-4	2.12E-4	5.16E-4
64000				1.15E-4	1.03E-0

Table 4: Total mass I^Δ of the right-hand side of the proven entropy inequality for the 1-rarefaction using second-order time discretisation
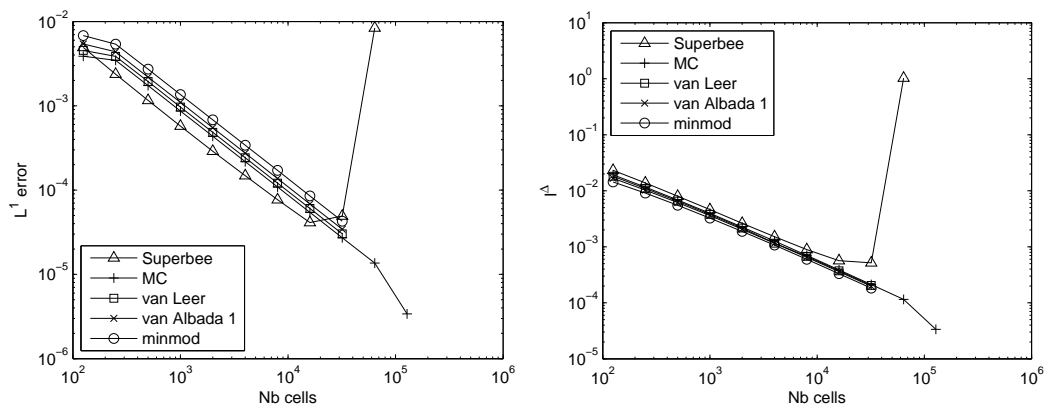

Figure 2: 1-rarefaction with second-order Runge-Kutta time scheme: L^1 error (left) and total mass I^Δ of the right-hand side of the proven entropy inequality (right)

The second proposed numerical experiment is devoted to approximate shock solutions. Once again, we consider a Riemann Problem where the initial left and right states are defined as follows:

$$
\begin{array}{ll}
\rho_L = 1, & \rho_R = 1, \\
u_L = 10, & u_R = -10, \\
p_L = 1, & p_R = 1,
\end{array}
\tag{35}
$$

to obtain an exact solution made of two shock discontinuities propagating with opposite velocities.

The results obtained with a time first-order discretisation are reported Tables 5 and 6 and Figure 3. We notice that van Leer, MC and Superbee limiter functions involve a numerical blowup. In fact, it seems that minmod and van Albada 1 limiters are also not stable but the blowup needs extremely fine meshes. Moreover, it is worth mentioning that the behavior of the measure δ, given by I^Δ, seems to coincide with a positive value (before a numerical blowup).

In Tables 7 and 8 and Figure 4, we present the convergence behavior of the L^1-error and the measure δ by considering time Runge-Kutta second-order schemes. Only superbee limiter involves a numerical blowup while the other schemes converge (or seem to converge). However, we remark that the measure δ does not converge to zero but to a positive value (according to [32]). As a consequence, the known discrete entropy inequalities (31) (for instance, given by [10, 5]) turn out to be not sufficient to ensure that the converged solution is entropy preserving in the sense of the Lax-Wendroff Theorem (Theorem 2.1).
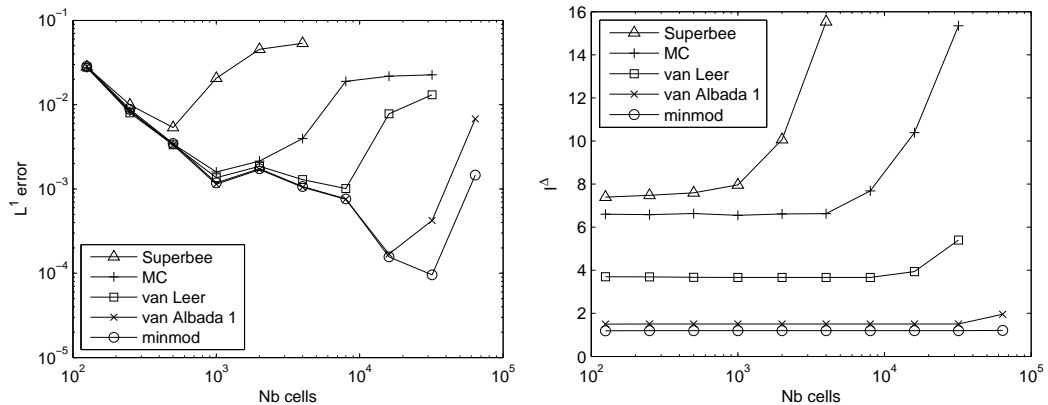

Figure 3: Shock-shock with first-order time scheme: L^1 error (left) and total mass I^Δ of the right-hand side of the proven entropy inequality (right)

To conclude these numerical illustrations, the discrete entropy inequalities (31) is clearly unsuitable since it does not prevent instabilities.

3 From one to all discrete entropy inequalities

From the above results, an entropy preserving high-order scheme must satisfy the entropy condition (15), while non-standard discrete formulation of the time deriva-

Nb cells	minmod	van Albada	van Leer	MC	superbee
125	2.85E-2	2.84E-2	2.78E-2	2.74E-2	2.81E-2
250	8.79E-3	8.42E-3	7.99E-3	8.36E-3	9.96E-3
500	3.47E-3	3.33E-3	3.34E-3	3.42E-3	5.35E-2
1000	1.18E-3	1.14E-3	1.36E-3	1.59E-3	2.06E-2
2000	1.74E-3	1.70E-3	1.86E-3	2.14E-3	4.54E-2
4000	1.07E-3	1.05E-3	1.29E-3	3.97E-3	
8000	7.62E-4	7.53E-4	1.04E-3	1.89E-2	
16000	1.56E-4	1.69E-4	7.78E-3	2.18E-2	
32000	9.59E-5	4.23E-4	1.31E-2		
64000	1.46E-3				

Table 5: L^1 error E^Δ for the shock-shock using first-order time discretisation

Nb cells	minmod	van Albada	van Leer	MC	superbee
125	1.19093	1.50166	3.69771	6.60340	7.39416
250	1.19348	1.50362	3.69027	6.57974	7.47961
500	1.19438	1.50505	3.67345	6.63387	7.59467
1000	1.19488	1.50596	3.66646	6.55043	7.95907
2000	1.19502	1.50636	3.66594	6.61186	10.0619
4000	1.19521	1.50650	3.66420	6.62561	
8000	1.19529	1.50648	3.66326	7.68269	
16000	1.19531	1.50625	3.93815	10.3855	
32000	1.19545	1.50600	5.40179		
64000	1.21136				

Table 6: Total mass I^Δ of the right-hand side of the proven entropy inequality for the shock-shock using first-order time discretisation

Nb cells	minmod	van Albada	van Leer	MC	superbee
125	2.87E-2	2.85E-2	2.80E-2	2.86E-2	2.96E-2
250	9.05E-3	8.63E-3	8.24E-3	8.22E-3	8.79E-3
500	3.60E-3	3.43E-3	3.55E-3	3.39E-3	4.20E-3
1000	1.23E-3	1.18E-3	1.57E-3	1.35E-3	1.97E-3
2000	1.77E-3	1.72E-3	2.06E-3	1.71E-3	2.85E-3
4000	1.08E-3	1.06E-3	1.47E-3	1.12E-3	3.21E-3
8000	7.59E-4	7.46E-4	1.21E-3	8.41E-4	1.92E-2
16000	1.50E-4	1.38E-4	6.19E-4	2.28E-4	2.36E-2
32000	6.52E-5	5.64E-5	5.45E-4	1.49E-4	3.22E-2
64000	2.84E-5		5.14E-4	1.13E-4	

Table 7: L^1 error E^Δ for the shock-shock using second-order time discretisation

Nb cells	minmod	van Albada	van Leer	MC	superbee
125	1.24970	1.52251	4.53453	6.40651	7.77453
250	1.25344	1.52491	4.56002	6.57310	7.95511
500	1.25475	1.52644	4.54521	6.63387	8.02181
1000	1.25538	1.52742	4.53574	6.68246	8.08829
2000	1.25556	1.52786	4.53815	6.69711	8.12208
4000	1.25580	1.52803	4.53633	6.75301	8.15356
8000	1.25594	1.52813	4.53547	6.87392	9.33707
16000	1.25600	1.52815	4.53571	6.93705	13.8100
32000	1.25602	1.52817	4.53550	6.99102	31.3636
64000	1.25603		4.53531	7.02895	

Table 8: Total mass I^Δ of the right-hand side of the proven entropy inequality for the shock-shock using second-order time discretisation
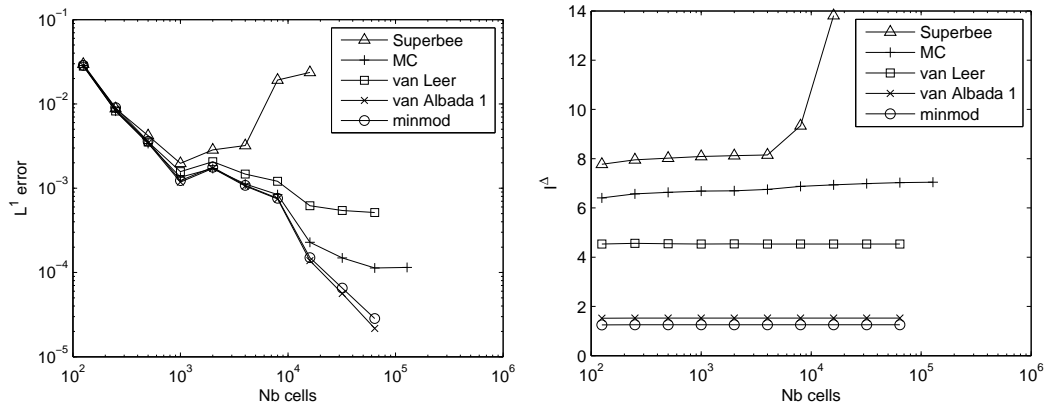

Figure 4: Shock-shock with second-order Runge-Kutta time scheme: L^1 error (left) and total mass I^Δ of the right-hand side of the proven entropy inequality (right)

tive may introduce unsuitable entropy inequalities including a positive measure. In order to derive a high-order scheme able to restore (15), we will adopt an *a posteriori* technique based on the discrete entropy inequalities satisfaction. Let us recall that the expected stability inequalities (15) must be satisfied by all entropy pairs $(\rho \mathcal{F}(\ln(s)), \rho \mathcal{F}(\ln(s))u)$ where $\mathcal{F}$ is a smooth function such that (6) holds true. *A posteriori* estimations are relevant whenever a finite number of estimations are considered, while we here have an infinite number of discrete entropy inequalities to be satisfied.

The purpose of the present section is to detail arguments to derive all the required discrete entropy inequalities from just one. To address such an issue, we first reformulate the entropy pairs as follows:

LEMMA 3.1 The entropy pairs (S, G), defined by (4)-(5), rewrites

$$S(w) = \rho \psi(r), \qquad G(w) = \rho \psi(r)u,$$

where we have set

$$r = -\frac{p^{1/\gamma}}{\rho}, \tag{36}$$

and ψ denotes a smooth increasing convex function.

From now on, let us underline that this result is not essential in the sequel, but it makes easier several developments. Indeed, we will see that considering entropies $S(w)$ parameterized by a monotone convex function ψ will be more convenient than considering entropies parameterized by a function $\mathcal{F}$ with the property (6). However, we emphasize that all the following scheme derivations can be performed by adopting the usual entropy pairs given by (4)-(5).

Proof First, let us notice that the specific entropy, defined by (3), writes $r = -s^{1/\gamma}$. Now, let us consider two functions, $\tilde{S}$ and $\tilde{G}$ such that we have

$$\tilde{S}(w) = \rho \psi(r) \qquad \text{and} \qquad \tilde{G}(w) = \rho \psi(r)u,$$

where ψ is a smooth increasing convex function. By introducing

$$\mathcal{F}(\ln(s)) := \psi(-s^{1/\gamma}),$$

we get

$$\mathcal{F}(y) = \psi(-e^{y/\gamma}), \qquad \forall y \in \mathbb{R},$$

to write

$$\mathcal{F}'(y) = -\frac{1}{\gamma}\psi'\left(-e^{y/\gamma}\right) < 0$$

and

$$\mathcal{F}'(y) - \gamma \mathcal{F}''(y) = -\frac{1}{\gamma}\left(\psi'\left(-e^{y/\gamma}\right) + \psi''\left(-e^{y/\gamma}\right)\right) < 0.$$

As a consequence, the smooth function $\mathcal{F}$ satisfies (6) and the pair $(\tilde{S}, \tilde{G})$ is thus an entropy pair.

Conversely, let us consider an entropy pair $(S, G) = (\rho\mathcal{F}(\ln(s)), \rho\mathcal{F}(\ln(s))u)$, where $\mathcal{F}$ satisfies (6). Since we have

$$\mathcal{F}(\ln(s)) = \mathcal{F}(\gamma\ln(-r)),$$

we set

$$\psi(r) := \mathcal{F}(\gamma\ln(-r)),$$

to write the following relations:

$$S(w) = \rho\psi(r) \qquad \text{and} \qquad G(w) = \rho\psi(r)u.$$

Since (6) is satisfied, we easily obtain

$$\psi'(r) = \frac{\gamma}{r}\mathcal{F}'(\gamma\ln(-r)) > 0$$

and

$$\psi''(r) = -\frac{\gamma}{r^2}\left(\mathcal{F}'(\gamma\ln(-r)) - \gamma\mathcal{F}''(\gamma\ln(-r))\right) > 0.$$

As expected, ψ is an increasing convex function, and the proof is completed. $\square$

Arguing the above result, we now establish conditions so that a finite volume method is entropy preserving as soon as just one relevant discrete entropy inequality is satisfied. Let us consider a conservative scheme given by

$$w_i^{n+1} = w_i^n - \frac{\Delta t}{\Delta x}\left(f_{i+1/2}^n - f_{i+1/2}^n\right), \tag{37}$$

where $f_{i+1/2}^n = {}^t\left(f_{i+1/2}^\rho, f_{i+1/2}^{\rho u}, f_{i+1/2}^E\right)$ stands for the consistent numerical flux function, according to (7) or more generally to (14).

THEOREM 3.2 Under the CFL condition (8), assume the scheme (37) is Ω-preserving: for all $w_i^n \in \Omega$, we have $w_i^{n+1} \in \Omega$, for all $i \in \mathbb{Z}$. Assume the following specific discrete entropy inequality:

$$\rho_i^{n+1}r_i^{n+1} \le \rho_i^n r_i^n - \frac{\Delta t}{\Delta x}\left(f_{i+1/2}^\rho r_{i+1/2}^n - f_{i-1/2}^\rho r_{i-1/2}^n\right) \tag{38}$$

is satisfied, where we have set

$$r_i^n = -\frac{(p_i^n)^{-1/\gamma}}{\rho_i^n} \quad \text{and} \quad r_{i+1/2}^n = \begin{cases} r_{i+1}^n & \text{if } f_{i+1/2}^\rho < 0, \\ r_i^n & \text{if } f_{i+1/2}^\rho > 0. \end{cases} \tag{39}$$

Moreover, assume the following additional CFL like condition holds:

$$\frac{\Delta t}{\Delta x}\left(\max\left(0, f_{i+1/2}^\rho\right) - \min\left(0, f_{i-1/2}^\rho\right)\right) \le \rho_i^n. \tag{40}$$

Then the scheme (37) is entropy preserving: for all smooth increasing convex function ψ, we have

$$\rho_i^{n+1}\psi\left(r_i^{n+1}\right) \le \rho_i^n\psi\left(r_i^n\right) - \frac{\Delta t}{\Delta x}\left(f_{i+1/2}^\rho\psi_{i+1/2}^n - f_{i-1/2}^\rho\psi_{i-1/2}^n\right),$$

with $\psi_{i+1/2}^n$ defined as follows:

$$\psi_{i+1/2}^n = \begin{cases} \psi\left(r_{i+1}^n\right) & \text{if} \quad f_{i+1/2}^\rho < 0, \\ \psi\left(r_i^n\right) & \text{if} \quad f_{i+1/2}^\rho > 0. \end{cases} \tag{41}$$

From now on, let us emphasize the particular form of the numerical entropy flux function involved in (38). In fact, we impose to the entropy r to satisfy a *transport* like property. Such a condition is clearly more restrictive than usual. For instance, the HLL scheme is an entropy preserving scheme (see [31]) which does not satisfy (38). But there exists schemes preserving this restriction, like the Suliciu relaxation scheme [9, 15] or equivalently the HLLC scheme [49, 48], which are first-order entropy preserving schemes involving entropy numerical flux function given by $f_{i+1/2}^\rho \mathcal{F}\left(\ln\left(s_{i+1/2}^n\right)\right)$ where

$$s_{i+1/2}^n = \begin{cases} p_{i+1}^n/(\rho_{i+1}^n)^\gamma & \text{if} \quad f_{i+1/2}^\rho < 0, \\ p_i^n/(\rho_i^n)^\gamma & \text{if} \quad f_{i+1/2}^\rho > 0. \end{cases}$$

As a consequence, by introducing $r_i^n = -(s_i^n)^{1/\gamma}$, such schemes are able to preserve the inequalities (38)-(39).

Proof By definition of the numerical entropy flux function coming from (38), and arguing the definition of $r_{i+1/2}$ given by (39), the following relation easily holds:

$$f_{i+1/2}^\rho r_{i+1/2} = f_{i+1/2}^\rho \frac{r_i^n + r_{i+1}^n}{2} - \left| f_{i+1/2}^\rho \right| \frac{r_{i+1}^n - r_i^n}{2}.$$

We plug this relation into (38) to get

$$r_i^{n+1} \leq \frac{a}{\rho_i^{n+1}} r_{i-1}^n + \frac{b}{\rho_i^{n+1}} r_i^n + \frac{c}{\rho_i^{n+1}} r_{i+1}^n, \tag{42}$$

where we have set

$$\begin{aligned} a &= \frac{\Delta t}{2\Delta x}\left(f_{i-1/2}^\rho + \left|f_{i-1/2}^\rho\right|\right), \\ b &= \rho_i^n - \frac{\Delta t}{2\Delta x}\left(f_{i+1/2}^\rho + \left|f_{i+1/2}^\rho\right| - f_{i-1/2}^\rho + \left|f_{i-1/2}^\rho\right|\right), \\ c &= \frac{\Delta t}{2\Delta x}\left(\left|f_{i+1/2}^\rho\right| - f_{i+1/2}^\rho\right). \end{aligned} \tag{43}$$

Now, let us notice that

$$\begin{aligned} a + b + c &= \rho_i^n - \frac{\Delta t}{\Delta x}\left(f_{i+1/2}^\rho - f_{i-1/2}^\rho\right), \\ &= \rho_i^{n+1} > 0. \end{aligned}$$

We easily see that a and c are nonnegative. Moreover, the additional CFL like condition (40) enforces the coefficient b to be nonnegative. As a consequence, we have established that the right-hand side of (42) is nothing but a convex combination of r_{i-1}^n, r_i^n and r_{i+1}^n.

According to Lemma 3.1, let us now consider an entropy pair given by

$$(S, G) = (\rho\psi(r), \rho u\psi(r)),$$

with ψ a smooth increasing convex function. The function ψ being increasing, from the inequality (42) we get

$$\psi\left(r_i^{n+1}\right) \leq \psi\left(\frac{a}{\rho_i^{n+1}}r_{i-1}^n + \frac{b}{\rho_i^{n+1}}r_i^n + \frac{c}{\rho_i^{n+1}}r_{i+1}^n\right).$$

By arguing the well-known discrete Jensen inequality, we deduce

$$\psi\left(r_i^{n+1}\right) \leq \frac{a}{\rho_i^{n+1}}\psi\left(r_{i-1}^n\right) + \frac{b}{\rho_i^{n+1}}\psi\left(r_i^n\right) + \frac{c}{\rho_i^{n+1}}\psi\left(r_{i+1}^n\right).$$

Next, by substituting the coefficients a, b and c by their exact value given by (43), we obtain

$$\rho_i^{n+1}\psi(r_i^{n+1}) \leq \rho_i^n\psi(r_i^n) -$$
$$\frac{\Delta t}{2\Delta x}\left(f_{i+1/2}^\rho(\psi(r_i^n) + \psi(r_{i+1}^n)) - |f_{i+1/2}^\rho|(\psi(r_{i+1}^n) - \psi(r_i^n))\right.$$
$$\left. -f_{i-1/2}^\rho(\psi(r_{i-1}^n) + \psi(r_i^n)) + |f_{i-1/2}^\rho|(\psi(r_i^n) - \psi(r_{i-1}^n))\right),$$

which can rewrite as follows:

$$\rho_i^{n+1}\psi(r_i^{n+1}) \leq \rho_i^n\psi(r_i^n) - \frac{\Delta t}{\Delta x}\left(f_{i+1/2}^\rho\psi_{i+1/2}^n - f_{i-1/2}^\rho\psi_{i-1/2}^n\right),$$

where $\psi_{i+1/2}$. is defined by (41). The proof is thus achieved. $\qquad\square$

4 The e-MOOD scheme for the Euler equations

In this section, we derive space high-order numerical schemes, given by (10), which satisfy the required discrete entropy inequalities (15). For the sake of simplicity in the forthcoming theoretical developments, we restrict ourselves to time first-order numerical methods. However, after Lemma 2.2, the space high-order scheme, now detailed, will easily extend by Runge-Kutta procedure to obtain time high-order schemes, which remain entropy preserving. Time high-order extensions will be used to perform numerical experiments.

To impose the expected inequalities (15), we now suggest to introduce an additional *a posteriori* limitation when reconstructing both states $w_{i-1/2}^+$ and $w_{i+1/2}^-$ on the cell $(x_{i-1/2}, x_{i+1/2})$. This *a posteriori* limitation technique was recently introduced by Clain et al. [17, 21], the so-called MOOD schemes.

The MOOD technique allows to extend any first-order scheme, which satisfies some required properties, to get space high-order scheme preserving the same properties. It is based on an iterative process to determine, locally on each cell, the better reconstruction according to the imposed properties (here, robustness and stability).

Let us consider a first-order conservative scheme given by (7). Under a standard CFL-like condition (8), we assume that this first-order scheme satisfies the needed robustness and stability properties:

Robustness If all the initial states w_i^n are in Ω then the evolved states w_i^{n+1} remain in Ω.

Stability For all $i \in \mathbb{Z}$, the following discrete entropy inequality is satisfied:

$$\rho_i^{n+1} r_i^{n+1} \leq \rho_i^n r_i^n - \frac{\Delta t}{\Delta x} \left(f_{\Delta x}^\rho \left(w_i^n, w_{i+1}^n \right) r_{i+1/2}^n - f_{\Delta x}^\rho \left(w_{i-1}^n, w_i^n \right) r_{i-1/2}^n \right), \quad (44)$$

with $r_{i+1/2}^n$ defined as follows

$$r_{i+1/2}^n = \begin{cases} r_{i+1}^n & \text{if} \quad f_{\Delta x}^\rho \left(w_i^n, w_{i+1}^n \right) < 0, \\ r_i^n & \text{if} \quad f_{\Delta x}^\rho \left(w_i^n, w_{i+1}^n \right) > 0. \end{cases} \quad (45)$$

Once again, let us emphasize that such a first-order scheme exists. For instance, the reader is referred to the HLLC scheme or the Suliciu relaxation scheme [15, 48].

Next, we adopt a reconstruction procedure given by (16). If the increment reconstruction μ_i^n, is defined by (17), we stay within the standard MUSCL procedure, but μ_i^n can be associated to higher accurate reconstruction approaches. In the sequel, the reconstruction is imposed to be Ω-preserving:

$$w_{i-1/2}^+ = w_i^n - \frac{1}{2}\mu_i^n \in \Omega \quad \text{and} \quad w_{i+1/2}^- = w_i^n + \frac{1}{2}\mu_i^n \in \Omega \quad \text{for all } i \in \mathbb{Z}.$$

We notice that the reconstruction satisfies the following property:

$$w_i^n = \frac{1}{2}w_{i-1/2}^+ + \frac{1}{2}w_{i+1/2}^-. \quad (46)$$

It is possible to avoid this restriction on the reconstruction. Indeed, invoking arguments stated in [5], we can consider a reconstruction such that w_i^n is not a convex combination of $w_{i-1/2}^+$ and $w_{i+1/2}^-$:

$$w_i^n \neq \alpha w_{i-1/2}^+ + (1 - \alpha)w_{i+1/2}^-, \qquad \alpha \in (0, 1).$$

However, the relation (46) makes easier to obtain the robustness requirements. As a consequence, for the sake of simplicity, we adopt the relation (46).

We are now able to present the suggested e-MOOD scheme.

1. **Reconstruction step:** For all i in $\mathbb{Z}$, on each side of the interface $x_{i+1/2}$, we evaluate high-order states, given by

$$w_{i+1/2}^- = w_i^n + \frac{1}{2}\mu_i^n \in \Omega \quad \text{and} \quad w_{i+1/2}^+ = w_{i+1}^n - \frac{1}{2}\mu_{i+1}^n \in \Omega. \quad (47)$$

2. **Evolution step:** The reconstructed approximate solution is evolved as follows:

$$w_i^{n+1,\star} = w_i^n - \frac{\Delta t}{\Delta x} \left(f_{\Delta x} \left(w_{i+1/2}^-, w_{i+1/2}^+ \right) - f_{\Delta x} \left(w_{i-1/2}^-, w_{i-1/2}^+ \right) \right). \quad (48)$$

3. *A posteriori* **limitation step:** We have the following alternative.

- If for all $i \in \mathbb{Z}$, we have

$$\rho^{n+1,\star} r_i^{n+1,\star} \leq \rho_i^n r(w_i^n) - \frac{\Delta t}{\Delta x} \left(f_{\Delta x}^\rho \left(w_{i+1/2}^-, w_{i+1/2}^+ \right) r_{i+1/2}^n \right.$$
$$\left. - f_{\Delta x}^\rho \left(w_{i-1/2}^-, w_{i-1/2}^+ \right) r_{i-1/2}^n \right), \quad (49)$$

  where $r_{i+1/2}^n$ is defined by (45), then the solution is valid and the updated approximation at time $t^n + \Delta t$ is defined by

$$w_i^{n+1} = w_i^{n+1,\star}, \quad \forall i \in \mathbb{Z}.$$

- Otherwise, for all $i \in \mathbb{Z}$ such that (49) is not satisfied, we set

$$w_{i-1/2}^+ = w_i^n \quad \text{and} \quad w_{i+1/2}^- = w_i^n,$$

  and we go back to step 2.

Before we establish the robustness and stability properties satisfied by the above e-MOOD scheme, we underline several important points coming with this numerical procedure beside the initial MOOD introduced in [17].

First of all, we recall that the initial MOOD schemes consider an iterative procedure over the order of accuracy involved in the reconstruction step. In [17, 21], the authors adopt a sequence of reconstructions indexed by the degree $0 \leq d_i \leq d_{\max}$ of the polynomial reconstruction, where $\mu_i^n = 0$ as soon as $d_i = 0$. It is worth noticing that the degree d_i is locally defined over the cell $(x_{i-1/2}, x_{i+1/2})$. Next, during the *a posteriori* limitation step, if the property (here, entropy preserving property) is not satisfied, the order of accuracy is decreased and the MOOD technique is once again performed but for a smaller value of d_i. This iterative procedure on d_i stops with $d_i = 0$ since a first-order scheme is recovered, and by assumption, this first-order scheme must preserve the expected property.

For the sake of simplicity in the e-MOOD presentation, we have stopped the iterative procedure at the end of the first iteration. Of course, it is possible to adopt a procedure made of several iteration from $d_i = d_{\max}$ to $d_i = 0$. The robustness and stability results, stated below, will be preserved.

The second point to be emphasized concerns the effective order of accuracy. Indeed, the e-MOOD scheme, but also the initial MOOD scheme, substitutes the high-order scheme by a first-order method as soon as the required properties are not satisfied. Clearly, if the imposed property is *too strong*, the limitation will be active over the whole domain of computation and the resulting approximation will turn out to be first-order accurate. In practice, we have considered a reconstruction step given by a usual MUSCL approach and the resulting numerical improvements are obvious. However, it seems impossible to rigorously prove the order of accuracy (excepted first-order). From our point of view, the derived e-MOOD scheme must be understood as a stabilization technique and not only as a space high-order procedure.

The last concern is devoted to the choice of the *a posteriori* limitation and its practical consequence. Indeed, the initial MOOD scheme [17, 21] considers an *a posteriori* limitation based on the robustness and on a maximum principle. In

fact, considering a maximum principle, several difficulties arise (see [5, 7, 55, 54]) associated to detection of local extrema. Here, the entropy *a posteriori* limitation turns out to be very easily implemented.

In addition, it is important to notice that in the original MOOD method, enforcing a constant reconstruction on a cell $(x_{i-1/2}, x_{i+1/2})$ (i.e. $d_i = 0$) is not sufficient to ensure that the maximum principle is satisfied on this cell. Indeed, all the states involved in the evolution step (48) have to be constant reconstructions. This includes the states $w_{i-1/2}^-$ and $w_{i+1/2}^+$ which have respectively to be equal to w_{i-1}^n and w_{i+1}^n. In practice, this implies that the original MOOD method needs two different reconstructions on each cell: one for each interface. This is not the case for the e-MOOD scheme, since as soon as the *a posteriori* limitation procedure has been activated on a cell, the evolved state on this cell satisfies the required robustness and stability properties, regardless of the reconstruction used on the neighbouring cells. Indeed, inside a cell $(x_{i-1/2}, x_{i+1/2})$ where the *a posteriori* limitation procedure has been activated, the e-MOOD scheme rewrites

$$w_i^{n+1} = w_i^n - \frac{\Delta t}{\Delta x} \left(f_{\Delta x}\left(w_i^n, w_R\right) - f_{\Delta x}\left(w_L, w_i^n\right) \right).$$

We underline that this first-order scheme satisfies (44)-(45), independently of the definition of w_L and w_R. This remark is essential since it makes the method very attractive and computationally costless when compared to the initial MOOD scheme.

Now, we are able to state the robustness and the stability properties satisfied by the e-MOOD scheme.

THEOREM 4.1 Assume the time step Δt satisfies the two following CFL like conditions:

$$\frac{\Delta t}{\Delta x} \max_{i \in \mathbb{Z}} \left(\left| \lambda^\pm \left(w_{i+1/2}^+, w_{i+1/2}^- \right) \right|, \left| \lambda^\pm \left(w_{i-1/2}^+, w_{i+1/2}^- \right) \right| \right) \leq \frac{1}{4}, \qquad (50)$$

$$\frac{\Delta t}{\Delta x} \left(\max(0, f_{i+1/2}^\rho) - \min(0, f_{i-1/2}^\rho) \right) \leq \rho_i^n. \qquad (51)$$

Assume that w_i^n and all the reconstructed states $w_{i+1/2}^\pm$, defined by (47), belong to Ω for all i in $\mathbb{Z}$. Then the updated state w_i^{n+1}, given by the e-MOOD belongs to Ω for all i in $\mathbb{Z}$. Moreover, for all smooth increasing convex function ψ, the e-MOOD scheme satisfies

$$\frac{1}{\Delta t} \left(\rho_i^{n+1} \psi(r_i^{n+1}) - \rho_i^n \psi(r_i^n) \right) +$$
$$\frac{1}{\Delta x} \left(f_{\Delta x}^\rho(w_{i+1/2}^-, w_{i+1/2}^+) \psi(r_{i+1/2}^n) - f_{\Delta x}^\rho(w_{i-1/2}^-, w_{i-1/2}^+) \psi(r_{i-1/2}^n) \right) \leq 0,$$
$$(52)$$

where $r_{i+1/2}^n$ is defined by (45). As a consequence the e-MOOD scheme is entropy preserving.

Proof First, we establish the robustness of the e-MOOD scheme. Since no *a posteriori* limitation is devoted to enforce w_i^{n+1} to stay in Ω, we have to prove that $w_i^{n+1,\star}$, defined by (48), belongs to Ω for all i in $\mathbb{Z}$. If the limitation was activated

on the cell $(x_{i-1/2}, x_{i+1/2})$, then we have $w^{+}_{i-1/2} = w^n_i$ and $w^{-}_{i+1/2} = w^n_i$, so the reconstructed states are in Ω. If the limitation was not activated, then the states $w^{+}_{i-1/2}$ and $w^{-}_{i+1/2}$ are obtained by the reconstruction procedure which is assumed to preserve Ω. In both cases, the states $w^{+}_{i-1/2}$ and $w^{-}_{i+1/2}$ are in Ω.

Let us define the two following intermediate states:

$$w^{n+1,+}_{i-1/2} = w^{+}_{i-1/2} - \frac{\Delta t}{\Delta x/2} \left(f_{\Delta x}\left(w^{+}_{i-1/2}, w^{-}_{i+1/2} \right) - f_{\Delta x}\left(w^{-}_{i-1/2}, w^{+}_{i-1/2} \right) \right),$$

$$w^{n+1,-}_{i+1/2} = w^{-}_{i+1/2} - \frac{\Delta t}{\Delta x/2} \left(f_{\Delta x}\left(w^{-}_{i+1/2}, w^{+}_{i+1/2} \right) - f_{\Delta x}\left(w^{+}_{i-1/2}, w^{-}_{i+1/2} \right) \right).$$

In fact, we notice that both intermediate updated states, $w^{n+1,+}_{i-1/2}$ and $w^{n+1,-}_{i+1/2}$, are evaluated by involving a first-order scheme with a mesh size given by $\Delta x/2$. Since the first-order scheme is Ω-preserving, we immediately get $w^{n+1,+}_{i-1/2}$ and $w^{n+1,-}_{i+1/2}$ in Ω as long as the CFL-like condition (50) is satisfied. Now, by involving (46), we have

$$w^{n+1,\star}_i = \frac{1}{2} w^{n+1,+}_{i-1/2} + \frac{1}{2} w^{n+1,-}_{i+1/2},$$

to immediately deduce that $w^{n+1,\star}_i$ belongs to the convex set Ω.

Next, by definition of the e-MOOD scheme, the following discrete entropy inequality is satisfied for all $i \in \mathbb{Z}$:

$$\frac{1}{\Delta t} \left(\rho^{n+1}_i r^{n+1}_i - \rho^n_i r^n_i \right) + \\ \frac{1}{\Delta x} \left(f^{\rho}_{\Delta x}(w^{-}_{i+1/2}, w^{+}_{i+1/2}) r^n_{i+1/2} - f^{\rho}_{\Delta x}(w^{-}_{i-1/2}, w^{+}_{i-1/2}) r^n_{i-1/2} \right) \leq 0, \tag{53}$$

where $r^n_{i+1/2}$ is defined by (45).

Under the CFL condition (51), we can apply Theorem 3.2 and the e-MOOD scheme satisfy all the required entropy inequalities (52). The proof is thus achieved. $\square$

To conclude this section, let us underline that the robustness of the e-MOOD schemes comes from the CFL condition (50) and the relation (46). In fact, if (46) is not satisfied, after [5], additional CFL restrictions and reconstruction limitations can be imposed to enforce the required robustness. This Ω-preserving property, naturally satisfied by the e-MOOD scheme, is another understanding of the initial MOOD method [17] where an additional *a posteriori* limitation is imposed to satisfy the expected robustness property.

5 Numerical experiments

For the sake of consistency, the numerical experiments now detailed follow the same strategy as imposed in Section 2.4. To validate the e-MOOD scheme, we adopt a numerical flux function involved in (48) given by the HLLC scheme [48, 49]. Concerning the e-MOOD reconstruction step (47), MUSCL limiters are considered. Here, we only deal with minmod and superbee limiter functions. Indeed, after Tables 1-8,

the minmod function gives relevant approximations while the superbee limiter develops numerical blowup. Time first- and second-order are systematically compared. Regarding the CFL condition, we adopt the restrictions (50)-(51).

The first numerical test is devoted to illustrate the relevance of the e-MOOD procedure by approximating a smooth periodic solution of (1). Over a periodic domain of computation $(0, 1)$, the initial data is given by

$$\rho_0(x) = \begin{cases} 1 & \text{if } x < 0.2 \text{ or } x > 0.8, \\ 1 + \exp\left(\frac{(x-0.5)^2}{(x-0.2)(x-0.8)}\right) & \text{if } 0.2 < x < 0.8. \end{cases}$$
$$u_0(x) = 1,$$
$$p_0(x) = 1.$$

The exact solution is given by

$$w(x, t) = w_0(x - at), \qquad \text{with } a = 1$$

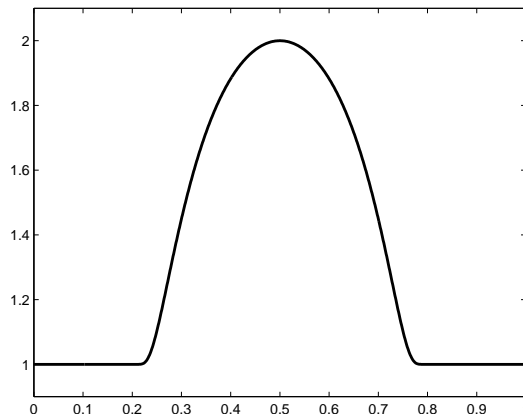and is displayed Figure 5 at time $t = 1$.

Figure 5: Initial and final density solution for the smooth problem

Table 9 and Figure 6 give the L^1-errors obtained by the minmod and the superbee limiters with a time second-order Runge-Kutta technique. Concerning the minmod results, we notice that both MUSCL and e-MOOD approximations give the same error. In fact, in this numerical test, the minmod MUSCL scheme turns out to be positive entropy preserving. As a consequence, e-MOOD procedure stays inactivated and we get the same results. Now, concerning the superbee function, the e-MOOD procedure enforces some numerical regularizations and no blowup appears at the discrepancy with the superbee MUSCL scheme. To complete this validation benchmark, we also give the results obtained by involving a fourth-order minmod technique [8, 53]. It is worth noticing that both MUSCL and e-MOOD schemes give similar results. As a consequence, we claim that the presented e-MOOD technique does not turn out to be too diffusive and it preserves the order of accuracy satisfied

Nb of	MUSCL scheme			e-MOOD scheme		
cells	minmod	superbee	minmod4	minmod	superbee	minmod4
125	3.75E-3	7.62E-4	1.22E-4	3.75E-3	7.62e-4	1.24e-4
250	1.43E-3	4.98E-4	4.80E-5	1.43E-3	4.98e-4	4.81e-5
500	4.94E-4	2.41E-4	7.16E-6	4.94E-4	2.41e-4	7.18e-6
1000	1.46E-4	7.73E-5	4.92E-7	1.49E-4	7.70e-5	4.93e-7
2000	3.91E-5	2.13E-5	1.48E-8	3.91E-5	2.11e-5	1.49e-8
4000	9.60E-6	5.44E-6	3.11E-10	9.60E-6	5.49e-5	3.17e-10
8000	2.40E-6	1.47E-6	1.89E-11	2.40E-6	1.67e-6	1.92e-11
16000	6.02E-7	2.55E-2	3.39E-12	6.02E-7	1.12e-6	3.40e-12
32000	1.51E-7			1.51E-7	1.55e-6	

Table 9: L^1 error for the smooth problem using the MUSCL scheme and the e-MOOD scheme
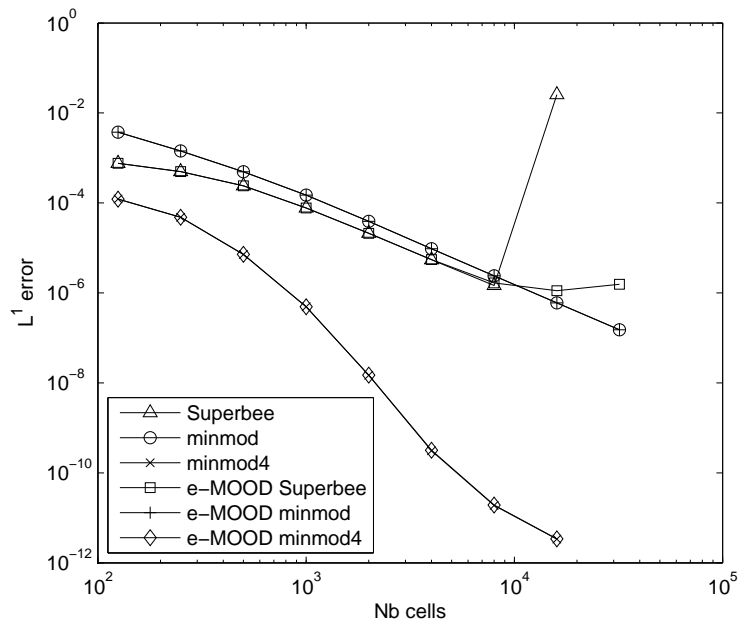

Figure 6: Comparison between MUSCL scheme and e-MOOD scheme – L^1 error for the smooth problem

by the MUSCL scheme, while being now entropy preserving according to Theorem 4.1.

Next, we consider the approximation of the solution of Riemann problems with an initial data given by (33). In Table 10 and Figure 7, we present the numerical results obtained by simulating a 1-rarefaction with an initial data given by (34). We immediately remark that there is no longer any numerical blow-up. But the e-MOOD scheme preserves the required order of accuracy. In Table 11 and Figure 8, similar results are obtained by simulating a 1-shock with an initial data defined by (35).

Nb of cells	first-order time		second-order time	
	minmod	superbee	minmod	superbee
125	2.33E-2	3.17E-2	1.53E-2	2.05E-2
250	1.51E-2	1.99E-2	1.10E-2	1.20E-3
500	9.26E-3	1.23E-2	6.30E-3	7.18E-3
1000	5.63E-3	7.45E-3	3.70E-3	4.23E-3
2000	3.33E-3	4.44E-3	2.13E-3	2.44E-3
4000	1.93E-3	2.66E-3	1.22E-3	1.40E-3
8000	1.11E-3	1.71E-3	6.88E-4	7.89E-4
16000	6.34E-4	1.14E-3	3.84E-4	4.39E-4
32000	3.62E-4	8.14E-4	2.16E-4	2.43E-4

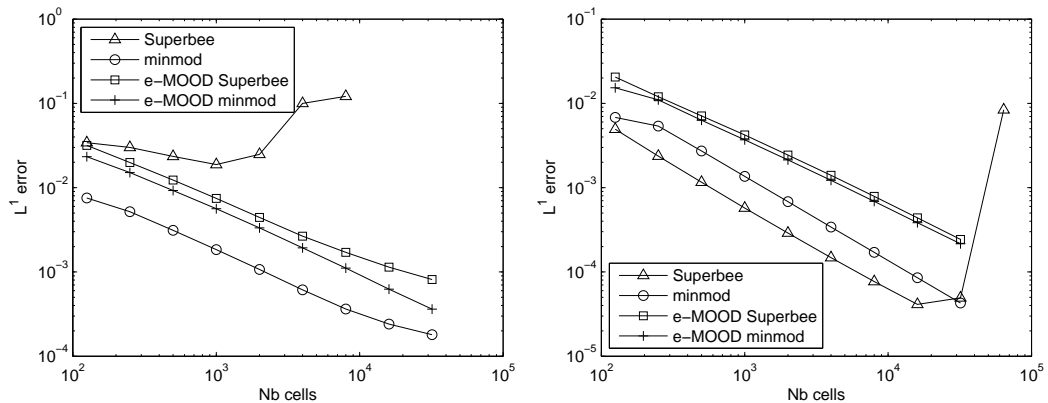Table 10: L^1 error for the 1-rarefaction using the e-MOOD scheme

Figure 7: Comparison between MUSCL scheme and e-MOOD scheme – L^1 error for the 1-rarefaction – Left: first-order time scheme. Right: second-order time scheme

Nb of	first-order time		second-order time	
cells	minmod	superbee	minmod	superbee
125	4.05E-2	3.99E-2	4.12E-2	4.12E-2
250	1.63E-2	1.60E-2	1.66E-2	1.67E-3
500	6.98E-3	7.17E-3	7.16E-3	7.31E-3
1000	2.74E-3	2.77E-3	2.82E-3	2.80E-3
2000	2.60E-3	2.57E-3	2.65E-3	2.63E-3
4000	1.51E-3	1.49E-3	1.53E-3	1.52E-3
8000	9.43E-4	9.45E-4	9.59E-4	9.58E-4
16000	2.60E-4	2.53E-4	2.68E-4	2.56E-4
32000	1.14E-5	1.17E-4	1.19E-4	1.16E-4

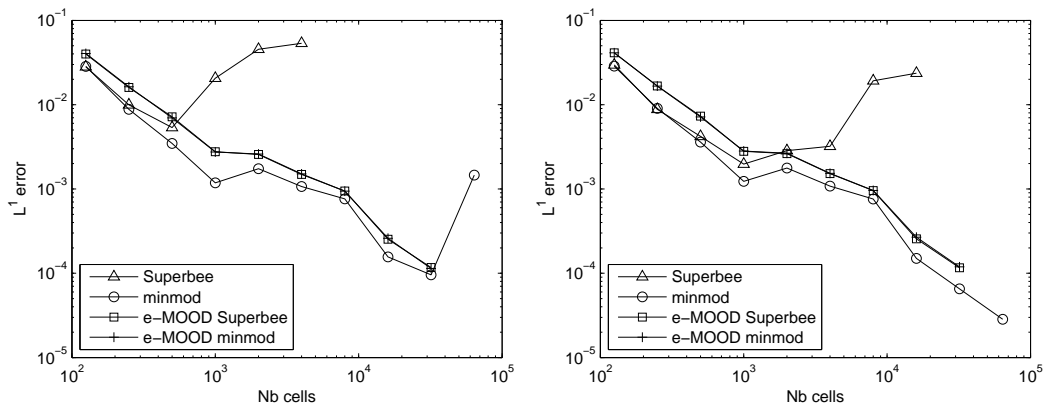Table 11: L^1 error for the shock-shock using the e-MOOD scheme

Figure 8: Comparison between MUSCL scheme and e-MOOD scheme – L^1 error for the shock-shock – Left: first-order time scheme. Right: second-order time scheme

A Proof of the Lax-Wendroff Theorem

For the sake of completeness of the present paper, we now give the proof of the Lax-Wendroff Theorem 2.1. In fact, from [36] (see also [39, 24]), we have the required proof with space high-order numerical flux functions. We here give a direct extension by considering time high-order discretization with approximation in a convex set.

We define two kinds of rectangular cells in the (x, t) plane:

$$R_i^n = [x_{i-1/2}, x_{i+1/2}) \times [t^n, t^{n+1}),$$

$$\widetilde{R}_{i+1/2}^n = [x_i, x_{i+1}) \times [t^n, t^{n+1}).$$

Arguing this notation, we introduce the piecewise constant functions

$$f^{\Delta,(l)}(x, t) = f_{i+1/2}^{n,(l)}, \quad \text{for } (x, t) \in \widetilde{R}_{i+1/2}^n.$$

In the sequel, convergences are implicitly considered up to a subsequence. With some abuse, we state that convergence in L_{loc}^1 implies convergence a.e. We shall see at the end of the proof of Theorem 2.1 that it is not a restriction.

The proof is organized as follows: Lemma A.1 is a technical result about the convergence of a shifted sequence. In Lemma A.2, we prove that the convergence of $w^{\Delta,(j)}$ to w in L_{loc}^1 implies the convergence a.e. of $f^{\Delta,(j)}$ to $f(w)$. From this result, we deduce in Lemma A.3 that all the $w^{\Delta,(j)}$ converge to w in L_{loc}^1. Finally, thanks to Lemma A.2 and Lemma A.3, we can achieve the proof of Theorem 2.1.

LEMMA A.1 We consider a sequence of functions $u^\Delta : \mathbb{R} \to \Omega$ satisfying the following hypotheses:

  (i) there exists a compact $K \subset \Omega$ such that u^Δ is valued in K;

 (ii) u^Δ converges in $L_{\text{loc}}^1(\mathbb{R}; \Omega)$ to a function u.

Then for all $\xi \in \mathbb{R}$, the quantity $u^\Delta(x + \xi \Delta x)$ converges to $u(x)$ for a.e. $x \in \mathbb{R}$.

Proof Let $a < b$ be two reals. We define $I^\Delta = \int_a^b \left\| u^\Delta(x + \xi \Delta x) - u(x) \right\| dx$. A triangle inequality gives

$$I^\Delta \leq \int_a^b \left\| u^\Delta(x + \xi \Delta x) - u(x + \xi \Delta x) \right\| dx + \int_a^b \left\| u(x + \xi \Delta x) - u(x) \right\| dx.$$

We denote respectively by I_1^Δ and I_2^Δ the two integrals in the last inequality. We are going to show that both these integrals converge to 0.

For I_1^Δ, a substitution gives $I_1^\Delta = \int_{a+\xi\Delta x}^{b+\xi\Delta x} \left\| u^\Delta(x) - u(x) \right\| dx$, so

$$I_1^\Delta = \int_a^b \left\| u^\Delta(x) - u(x) \right\| dx -$$
$$\int_a^{a+\xi\Delta x} \left\| u^\Delta(x) - u(x) \right\| dx + \int_b^{b+\xi\Delta x} \left\| u^\Delta(x) - u(x) \right\| dx.$$

Thanks to hypothesis (i), we have

$$\int_a^{a+\xi\Delta x} \left\| u^\Delta(x) - u(x) \right\| dx \leq 2\xi\Delta x \sup_{w \in K} \|w\| \to 0,$$

and the same goes on for $\int_b^{b+\xi\Delta x} \left\| u^\Delta(x) - u(x) \right\| dx$. Next, by arguing hypothesis (ii), the integral $\int_a^b \left\| u^\Delta(x) - u(x) \right\| dx$ converge to 0 and thus $I_1^\Delta \to 0$.

Now we study the convergence of I_2^Δ. The set of continuous functions being dense in $L^1(\mathbb{R}; \Omega)$, for all $\epsilon > 0$, we can find a continuous function ψ such that $\|\psi - u\|_{L^1([a,b+\xi\Delta x];\Omega)} \leq \epsilon$. The triangle inequality gives

$$I_2^\Delta \leq \int_a^b \|u(x + \xi\Delta x) - \psi(x + \xi\Delta x)\| \, dx$$

$$+ \int_a^b \|\psi(x + \xi\Delta x) - \psi(x)\| \, dx + \int_a^b \|\psi(x) - u(x)\| \, dx. \quad (54)$$

The first integral of (54) writes

$$\int_a^b \|u(x + \xi\Delta x) - \psi(x + \xi\Delta x)\| \, dx = \int_{a+\xi\Delta x}^{b+\xi\Delta x} \|u(x) - \psi(x))\| \, dx,$$

and is lower than ϵ, so is the last integral of (54), thanks to the definition of ψ. Finally, the second integral of (54) converges to 0 by continuity of ψ. We have thus $I_2^\Delta \to 0$ and so $I^\Delta \to 0$.

This means that $x \mapsto u^\Delta(x + \xi\Delta x)$ converges in L^1_{loc} to u. As a consequence, $u^\Delta(x + \xi\Delta x)$ converges to $u(x)$ for a.e. $x \in \mathbb{R}$. $\qquad \square$

LEMMA A.2 Under the hypotheses of Theorem 2.1, if $w^{\Delta,(j)}$ converges in $L^1_{\text{loc}}(\mathbb{R} \times \mathbb{R}^+; \Omega)$ to w, then $f^{\Delta,(j)}$ converges a.e. to $f(w)$.

Proof We first notice that if $x \in \widetilde{R}^n_{i+1/2}$, then $x + (k - 1/2)\Delta x \in R^n_{i+k}$. As a consequence, we can rewrite equation (14) into

$$f^{\Delta,(j)}(x,t) = F\left(w^{\Delta,(j)}\left(x - (s - 1/2)\Delta x, t\right), \cdots, w^{\Delta,(j)}\left(x + (s - 1/2)\Delta x, t\right) \right).$$

The sequence of functions $x \mapsto w^{\Delta,(j)}(x,t)$ satisfies the hypotheses of Lemma A.1 for a.e. $t \in \mathbb{R}^+$. So for all $\xi \in \mathbb{R}$, $w^{\Delta,(j)}(x + \xi\Delta x, t)$ converges to $w(x,t)$ for a.e. $(x,t) \in \mathbb{R} \times \mathbb{R}^+$. Thanks to the continuity and the consistency of F, we deduce that $f^{\Delta,(j)}$ converges to $f(w)$ a.e. $\qquad \square$

LEMMA A.3 Under the hypotheses of Theorem 2.1, $w^{\Delta,(\ell)}$ converges to w in $L^1_{\text{loc}}(\mathbb{R} \times \mathbb{R}^+; \Omega)$, for $\ell = 0, \cdots, m - 1$.

Proof We are going to prove this result by induction on ℓ. The result is true by hypothesis for $\ell = 0$, since $w^{\Delta,(0)} = w^\Delta$. Let us assume that $w^{\Delta,(j)}$ converges in $L^1_{\text{loc}}(\mathbb{R} \times \mathbb{R}^+; \Omega)$ to w, for $j = 0, \cdots, \ell - 1$. Lemma A.2 ensures that $f^{\Delta,(j)}$ converges a.e. to $f(w)$, for $j = 0, \cdots, \ell - 1$. Besides $f^{\Delta,(j)}$ is valued in the compact

$F(K, \cdots, K)$, so Lemma A.1 ensures that $(x,t) \mapsto f^{\Delta,(j)}(x + \Delta x/2, t)$ and $(x,t) \mapsto f^{\Delta,(j)}(x - \Delta x/2)$ both converges to $f(w)$ a.e.

Equation (12) rewrites

$$w^{\Delta,(\ell)}(x,t) = w^\Delta(x,t) - \frac{\Delta t}{\Delta x} \sum_{j=0}^{\ell-1} c_{\ell,j} \left(f^{\Delta,(j)}(x + \Delta x/2, t) - f^{\Delta,(j)}(x - \Delta x/2) \right).$$

Each term of the sum converges to 0 a.e. and w^Δ converges to w a.e., so $w^{\Delta,(\ell)}$ converges to w a.e. The sequence $w^{\Delta,(\ell)}$ being uniformly bounded, the dominated convergence theorem ensures that $w^{\Delta,(\ell)}$ converges to w in $L^1_{\text{loc}}(\mathbb{R} \times \mathbb{R}^+; \Omega)$. $\qquad\square$

We can now prove the Theorem 2.1.

Proof [Proof of Theorem 2.1] Let $\phi \in C^1_c(\mathbb{R} \times \mathbb{R}^+; \mathbb{R}^d)$ be a compactly supported smooth function. For $i \in \mathbb{Z}$ and $n \in \mathbb{N}$, we define $\phi_i^n = \phi(x_i, t^n)$. Multiplying the last iteration ($\ell = m$) of the scheme (12) by $\Delta x \phi_i^n$ and summing over i and n, we get

$$\Delta x \sum_{i,n} \left(w_i^{n+1} - w_i^n \right) \cdot \phi_i^n + \Delta t \sum_{i,n} \phi_i^n \cdot \sum_{j=0}^{m-1} c_{m,j} \left(f_{i+1/2}^{n,(j)} - f_{i-1/2}^{n,(j)} \right) = 0.$$

A summation by parts gives

$$\Delta x \sum_{i,n} w_i^{n+1} \cdot \left(\phi_i^{n+1} - \phi_i^n \right) + \Delta x \sum_i w_i^0 \cdot \phi_i^0 + \Delta t \sum_{i,n} \left(\phi_{i+1}^n - \phi_i^n \right) \cdot \sum_{j=0}^{m-1} c_{m,j} f_{i+1/2}^{n,(j)} = 0.$$
(55)

We define the piecewise constant function $\phi^\Delta(x,t) = \phi_i^n = \phi(x_i, t^n)$ for $(x,t) \in R_i^n$. We can then put equation (55) into an integral form

$$\int_{\mathbb{R} \times [\Delta t, +\infty)} w^\Delta(x,t) \cdot \frac{\phi^\Delta(x,t) - \phi^\Delta(x, t - \Delta t)}{\Delta t} dx dt + \int_{\mathbb{R}} w_0(x) \cdot \phi^\Delta(x,0) dx = 0$$

$$+ \int_{\mathbb{R} \times \mathbb{R}^+} \frac{\phi^\Delta(x + \Delta x/2, t) - \phi^\Delta(x - \Delta x/2, t)}{\Delta x} \cdot \sum_{j=0}^{m-1} c_{m,j} f^{\Delta,(j)}(x,t) dx dt. \quad (56)$$

The function ϕ being smooth, ϕ^Δ uniformly converges to ϕ and since w_0 is essentially bounded, we have

$$\int_{\mathbb{R}} w_0(x) \cdot \phi^\Delta(x,0) dx \rightarrow \int_{\mathbb{R}} w_0(x) \cdot \phi(x,0) dx. \quad (57)$$

We denote respectively by I_1^Δ and I_2^Δ the first integral of (56) which corresponds to the time derivative and the third integral of (56) which corresponds to the space derivative.

Convergence of the time discretization I_1^Δ

The integral I_1^Δ writes

$$I_1^\Delta = \int_{\mathbb{R} \times \mathbb{R}^+} w^\Delta(x,t) \cdot 1_{\mathbb{R} \times [\Delta t, +\infty)}(x,t) \frac{\phi^\Delta(x,t) - \phi^\Delta(x, t - \Delta t)}{\Delta t} dx dt.$$

The function $(x,t) \mapsto 1_{\mathbb{R} \times [\Delta t, +\infty)}(x,t) \frac{\phi^\Delta(x,t) - \phi^\Delta(x,t-\Delta t)}{\Delta t}$ uniformly converges to $\partial_t \phi$ and the functions w^Δ are uniformly essentially bounded, so we have

$$\int_{\mathbb{R} \times \mathbb{R}^+} w^\Delta(x,t) \cdot \left(1_{[\Delta t, +\infty)} \frac{\left(\phi^\Delta(x,t) - \phi^\Delta(x,t-\Delta t) \right)}{\Delta t} - \partial_t \phi(x,t) \right) dx dt \to 0.$$

Since w^Δ converges in $L^1_{\text{loc}}(\mathbb{R} \times \mathbb{R}^+)$ to w, we have

$$\int_{\mathbb{R} \times \mathbb{R}^+} w^\Delta(x,t) \cdot \partial_t \phi(x,t) dx dt \to \int_{\mathbb{R} \times \mathbb{R}^+} w(x,t) \cdot \partial_t \phi(x,t) dx dt.$$

The last two limits imply

$$I_1^\Delta \to \int_{\mathbb{R} \times \mathbb{R}^+} w(x,t) \cdot \partial_t \phi(x,t) dx dt. \tag{58}$$

Convergence of the space discretization I_2^Δ

Arguing again the smoothness of the function ϕ, the sequence

$$(x,t) \mapsto \frac{\phi^\Delta(x + \Delta x/2, t) - \phi^\Delta(x - \Delta x/2, t)}{\Delta x}$$

uniformly converges to $\partial_x \phi$. Moreover the functions $f^{\Delta,(j)}$ are all uniformly bounded since they are valued in the compact $F(K, \cdots, K)$. As a consequence, we have

$$\int_{\mathbb{R} \times \mathbb{R}^+} \left(\frac{\phi^\Delta(x + \Delta x/2, t) - \phi^\Delta(x - \Delta x/2, t)}{\Delta x} - \partial_x \phi(x,t) \right) \cdot \\ \sum_{j=0}^{m-1} c_{m,j} f^{\Delta,(j)}(x,t) dx dt \to 0. \tag{59}$$

Combining Lemma A.3 and Lemma A.2, we get that $f^{\Delta,(j)}$ converges a.e. to $f(w)$, for $j = 0, \cdots, m-1$. Using (13), we deduce that $\sum_{j=0}^{m-1} c_{m,j} f^{\Delta,(j)}$ converges a.e. to $f(w)$.

The dominated convergence theorem ensures that

$$\int_{\mathbb{R} \times \mathbb{R}^+} \partial_x \phi(x,t) \cdot \sum_{j=0}^{m-1} c_{m,j} f^{\Delta,(j)}(x,t) dx dt \to \int_{\mathbb{R} \times \mathbb{R}^+} f(w(x,t)) \cdot \partial_x \phi(x,t) dx dt. \tag{60}$$

From (59) and (60), we deduce

$$I_2^\Delta \to \int_{\mathbb{R} \times \mathbb{R}^+} f(w(x,t)) \cdot \partial_x \phi(x,t) dx dt. \tag{61}$$

The three limits (57), (58), (61) are true up to a subsequence. Obviously we can find a joint subsequence that satisfy both three limits. Taking the limit for this subsequence in equation (56) prove that w is a weak solution of (11).

The proof for the entropy part is almost the same and present no additional difficulty. $\qquad \square$

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