A well-balanced scheme for the Euler equation with a gravitational potential

Vivien Desveaux, Markus Zenk, Christophe Berthon and Christian Klingenberg

Abstract The aim of this work is to derive a well-balanced numerical scheme to approximate the solutions of the Euler equations with a gravitational potential. This system admits an infinity of steady state solutions which are not all known in an explicit way. Among all these solutions, the hydrostatic atmosphere has a special physical interest. We develop an approximate Riemann solver using the formalism of Harten, Lax and van Leer, which takes into account the source term. The resulting numerical scheme is proven to be robust, to preserve exactly the hydrostatic atmosphere and to preserve an approximation of all the other steady state solutions.

1 Introduction

We consider the Euler equations with a gravity source term

$$\begin{cases} \partial_t \rho + \partial_x \rho u = 0, \\ \partial_t \rho u + \partial_x \left(\rho u^2 + p\right) = -\rho \partial_x \phi, \\ \partial_t E + \partial_x (u(E+p)) = -\rho u \partial_x \phi, \end{cases} \tag{1}$$

where $\rho > 0$ denotes the density, $u \in \mathbb{R}$ the velocity, $E > 0$ the total energy and $p > 0$ the pressure. We assume the system is closed by the ideal gas law

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$$p = (\gamma - 1)(E - \rho u^2/2), \quad \text{with } \gamma \in (1,3].$$

Concerning the gravity source term, we assume it derives from a gravitational potential $\phi(x)$, which is a given smooth function. The unknown vector $w = (\rho, \rho u, E)^T$ is assumed to belong to the set of physical admissible states

$$\Omega = \left\{ w \in \mathbb{R}^3 : \quad \rho > 0, \quad E - \rho u^2/2 > 0 \right\}.$$

Following the arguments stated in [9], when dealing with simulations of near equlibrium states of (1), well-balanced numerical schemes are expected to perform better than fractional splitting methods. It means that well-balanced schemes accurately capture the steady state solutions of the system, which is not neccesaraly true for general splitting methods. For the Euler equations with gravity, the steady state solutions at rest are characterized as follows:

$$u = 0, \qquad \partial_x p = -\rho \partial_x \phi. \tag{2}$$

We can exhibit a specific steady state solution of (1) which is of particular physical interest, namely the hydrostatic atmosphere defined for $\alpha > 0$ and $\beta > 0$ by

$$u(x) = 0, \qquad \rho(x) = \alpha e^{-\beta \phi(x)}, \qquad p(x) = \frac{\alpha}{\beta} e^{-\beta \phi(x)}. \tag{3}$$

In the well-known shallow-water model, the lake at rest is the unique steady state at rest (up to a constant) and it finds an explicit definition. In the last decade, numerous numerical schemes were developed to preserve the lake at rest in the shallow-water equations. The reader is referred for instance to [6, 5, 1].

For the Euler equations with gravity (1), the main discrepancy lies in the fact there are an infinity of solutions of (2) and we cannot explicit all of them. Therefore it is very difficult to derive numerical schemes which accurately capture all the solutions of (2). In a recent work [2], Chalons *et al.* succeeded to do so, but only in the case of a constant gravity field. We also mention the work of Käppeli and Mishra [8] where they manage to preserve all the isentropic solutions of (2).

Our aim is thus to derive a numerical scheme which captures exactly the hydrostatic atmosphere (3) and which preserves approximately all the solutions of (2). To address such an issue, we propose to build an approximate Riemann solver, following the formalism of Harten, Lax and van Leer [7] and the extensions introduced by Gallice [4].

The paper is organized as follows. Section 2 is devoted to the derivation of a simple approximate Riemann solver which takes into account the definition of the steady states (2). In Section 3, we present the associated numerical scheme and we establish that it is positive preserving and well-balanced, since it preserves exactly the steady state (3). The relevance of this approach is illustrated in Section 4 with some numerical experiments.

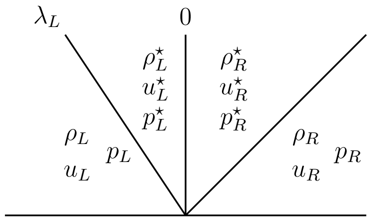

Fig. 1: Structure of the approximate Riemann solver $\widetilde{W}(x/t, w_L, w_R)$

2 The approximate Riemann solver

We now derive an approximate Riemann solver $\widetilde{W}(x/t, w_L, w_R)$ made of three waves with speeds λ_L, 0 and λ_R separating two intermediate states $w_L^\star$ and $w_R^\star$ (see Fig. 1). In order to enforce enough numerical viscosity, these speeds are assumed to satisfy $\lambda_L < 0 < \lambda_R$. As a consequence, this approximate Riemann solver will be fully characterized as soon as the intermediate values $\rho_{L,R}^\star$, $u_{L,R}^\star$ and $p_{L,R}^\star$ are given suitable definitions.

According to the work by Harten, Lax and van Leer [7], the approximate solver must satisfy the consistency relation

$$\frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2} \widetilde{W}\left(\frac{x}{\Delta t}, w_L, w_R\right) dx = \frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2} W_{\mathscr{R}}\left(\frac{x}{\Delta t}, w_L, w_R\right) dx, \tag{4}$$

where $W_{\mathscr{R}}(x/t, w_L, w_R)$ denotes the exact solution of the Riemann problem for (1). If the CFL restriction $\frac{\Delta t}{\Delta x}\max(|\lambda_L|, |\lambda_R|) \leq \frac{1}{2}$ is satisfied, we can compute the average of the approximate Riemann solver $\widetilde{W}$ to get an equivalent formulation to (4):

$$\left(\frac{1}{2} + \lambda_L \frac{\Delta t}{\Delta x}\right) w_L - \lambda_L \frac{\Delta t}{\Delta x} w_L^\star + \lambda_R \frac{\Delta t}{\Delta x} w_R^\star + \left(\frac{1}{2} - \lambda_R \frac{\Delta t}{\Delta x}\right) w_R \\ = \frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2} W_{\mathscr{R}}\left(\frac{x}{\Delta t}, w_L, w_R\right) dx. \tag{5}$$

First, we deal with the momentum equation by integrating the momentum component of the Riemann solution $W_{\mathscr{R}}^{\rho u}$ associated to (1). Provided that the wave velocities involved in the exact Riemann solution $W_{\mathscr{R}}$ stay within (λ_L, λ_R), we get

$$\frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2} (\rho u)_{\mathscr{R}}\left(\frac{x}{\Delta t}, w_L, w_R\right) dx = \frac{\rho_L u_L + \rho_R u_R}{2} - \frac{\Delta t}{\Delta x}(\rho_R u_R^2 + p_R - \rho_L u_L^2 - p_L) \\ - \frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2}\int_0^{\Delta t} \rho_{\mathscr{R}}\left(\frac{x}{t}, w_L, w_R\right) \partial_x \phi \, dt dx. \tag{6}$$

For the sake of simplicity in the notations, we set

$$\widehat{q} = \frac{\lambda_R \rho_R u_R - \lambda_L \rho_L u_L}{\lambda_R - \lambda_L} - \frac{1}{\lambda_R - \lambda_L}(\rho_R u_R^2 + p_R - \rho_L u_L^2 - p_L).$$

Plugging (6) into relation (5) gives

$$\frac{\lambda_R \rho_R^\star u_R^\star - \lambda_L \rho_L^\star u_L^\star}{\lambda_R - \lambda_L} = \widehat{q} - \frac{1}{(\lambda_R - \lambda_L)\Delta t} \int_{-\Delta x/2}^{\Delta x/2} \int_0^{\Delta t} \rho_{\mathscr{R}}\left(\frac{x}{t}, w_L, w_R\right) \partial_x \phi \, dt dx$$

The integral of the source term is usually difficult to compute exactly, so we propose the following approximation:

$$\frac{1}{\Delta t} \int_{-\Delta x/2}^{\Delta x/2} \int_0^{\Delta t} \rho_{\mathscr{R}}\left(\frac{x}{t}, w_L, w_R\right) \partial_x \phi \, dt dx \approx \overline{\rho}(\phi_R - \phi_L). \tag{7}$$

Here, $\overline{\rho}$ represents an average between ρ_L and ρ_R that will be defined later in order to preserve the steady states. Finally, we get the equation

$$\frac{\lambda_R \rho_R^\star u_R^\star - \lambda_L \rho_L^\star u_L^\star}{\lambda_R - \lambda_L} = \widehat{q} - \frac{1}{(\lambda_R - \lambda_L)} \overline{\rho}(\phi_R - \phi_L). \tag{8}$$

We adopt the same strategy for the total energy. We introduce the intermediate total energy as follows:

$$\widehat{E} = \frac{\lambda_R E_R - \lambda_L E_L}{\lambda_R - \lambda_L} - \frac{1}{\lambda_R - \lambda_L}(u_R(E_R + p_R) - u_L(E_L + p_L)).$$

Then, an integration of the E-component of the Riemann solution associated to (1) leads to the following relation:

$$\frac{\lambda_R E_R^\star - \lambda_L E_L^\star}{\lambda_R - \lambda_L} = \widehat{E} - \frac{1}{(\lambda_R - \lambda_L)\Delta t} \int_{-\Delta x/2}^{\Delta x/2} \int_0^{\Delta t} (\rho u)_{\mathscr{R}}\left(\frac{x}{t}, w_L, w_R\right) \partial_x \phi \, dt dx.$$

According to (7), we approximate the integral of the source term by

$$\frac{1}{\Delta t} \int_{-\Delta x/2}^{\Delta x/2} \int_0^{\Delta t} (\rho u)_{\mathscr{R}}\left(\frac{x}{t}, w_L, w_R\right) \partial_x \phi \, dt dx \approx \overline{\rho} \frac{u_L + u_R}{2}(\phi_R - \phi_L). \tag{9}$$

It is worth noticing that we could replace $(u_L + u_R)/2$ by any consistent average between u_L and u_R, like we did for ρ. However this choice will not intervene into the preservation of the steady states, so for the sake of simplicity, we use the arithmetic mean value. We finally obtain the equation

$$\frac{\lambda_R E_R^\star - \lambda_L E_L^\star}{\lambda_R - \lambda_L} = \widehat{E} - \frac{1}{(\lambda_R - \lambda_L)} \overline{\rho} \frac{u_L + u_R}{2}(\phi_R - \phi_L). \tag{10}$$

Concerning the density, we suggest the three following Rankine-Hugoniot jump relations through the waves of speed λ_L, 0 and λ_R:

$$\rho_L^\star u_L^\star - \rho_L u_L = \lambda_L(\rho_L^\star - \rho_L), \tag{11}$$

$$\rho_R^\star u_R^\star = \rho_L^\star u_L^\star, \tag{12}$$

$$\rho_R u_R - \rho_R^\star u_R^\star = \lambda_R(\rho_R - \rho_R^\star). \tag{13}$$

Let us notice that the consistency relation (5) for the density component is automatically satisfied as soon as the three relations (11), (12) and (13) hold.

To complete the solver, there is one missing equation. We decide to choose a linearization of the equation (2) describing the steady states:

$$p_R^\star - p_L^\star = -\overline{\rho}(\phi_R - \phi_L). \tag{14}$$

The system formed by equations (8), (10), (11), (12), (13) and (14) is easily solved to find

$$\rho_{L,R}^\star = \rho_{L,R} + \frac{1}{\lambda_{L,R}}(q^\star - \rho_{L,R} u_{L,R}), \qquad u_{L,R}^\star = \frac{q^\star}{\rho_{L,R}^\star},$$

$$E_L^\star = \widehat{E} + \frac{\lambda_R}{\lambda_R - \lambda_L}\left(\frac{\rho_L^\star (u_L^\star)^2}{2} - \frac{\rho_R^\star (u_R^\star)^2}{2}\right) + \frac{\overline{\rho}(\phi_R - \phi_L)}{\lambda_R - \lambda_L}\left(\frac{\lambda_R}{\gamma - 1} - \frac{u_L + u_R}{2}\right),$$

$$E_R^\star = \widehat{E} + \frac{\lambda_L}{\lambda_R - \lambda_L}\left(\frac{\rho_L^\star (u_L^\star)^2}{2} - \frac{\rho_R^\star (u_R^\star)^2}{2}\right) + \frac{\overline{\rho}(\phi_R - \phi_L)}{\lambda_R - \lambda_L}\left(\frac{\lambda_L}{\gamma - 1} - \frac{u_L + u_R}{2}\right),$$

where we have set

$$q^\star = \widehat{q} - \frac{1}{\lambda_R - \lambda_L}\overline{\rho}(\phi_R - \phi_L).$$

The characterisation of the approximate Riemann solver will be achieved as soon as the density average $\overline{\rho}$ will be stated. The precise definition of $\overline{\rho}$ will be given in the next section, accordingly to the well-balanced property.

3 The numerical scheme

Now, we describe the numerical scheme associated with the approximate Riemann solver $\widetilde{W}$. We consider a mesh of $\mathbb{R}$ made of cells $[x_{i-1/2}, x_{i+1/2})$ for $i \in \mathbb{Z}$, with constant size $\Delta x = x_{i+1/2} - x_{i-1/2}$. We search an update w_i^{n+1} of the solution at time t^{n+1}, knowing an approximation w_i^n at time t^n and on the cell $[x_{i-1/2}, x_{i+1/2})$. We also introduce a discretization of the gravitational potential ϕ as follows:

$$\phi_i = \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} \phi(x)dx.$$

To evolve this approximate solution from t^n to $t^n+\Delta t$, we consider the juxtaposition of Riemann problems located at the interfaces $x_{i+1/2}$. We denote by $\lambda^{L,R}_{i+1/2}$ the left and right speed and by $\overline{\rho}^n_{i+1/2}$ the average value of the density which appear in the approximate Riemann solver $\widetilde{W}\left(\frac{x-x_{i+1/2}}{t-t^n}, w_i^n, w_{i+1}^n\right)$. To ensure that the approximate Riemann solvers do not interact, we enforce the CFL condition

$$\frac{\Delta t}{\Delta x}\max_{i\in\mathbb{Z}}\left|\lambda^{L,R}_{i+1/2}\right| \le \frac{1}{2}.$$

Next, we follow the classical procedure for Godunov-type schemes to obtain a numerical scheme. It consists of a step of evolution using the approximate Riemann solver, followed by a step of projection on the space of piecewise constant functions. The update approximation at time $t^{n+1}=t^n+\Delta t$ is thus given by

$$w_i^{n+1} = \frac{1}{\Delta x}\int_{-\Delta x/2}^{0} \widetilde{W}\left(\frac{x}{\Delta t}, w_{i-1}^n, w_i^n\right)dx + \frac{1}{\Delta x}\int_{0}^{\Delta x/2} \widetilde{W}\left(\frac{x}{\Delta t}, w_i^n, w_{i+1}^n\right)dx.$$

After straightforward computations, the numerical scheme associated with the approximate Riemann solver developed in the Section 2 can be written as follows:

$$\begin{cases} \rho_i^{n+1} = \rho_i^n - \frac{\Delta t}{\Delta x}\left(F^{\rho}_{i+1/2} - F^{\rho}_{i-1/2}\right), \\ \rho_i^{n+1}u_i^{n+1} = \rho_i^n u_i^n - \frac{\Delta t}{\Delta x}\left(F^{\rho u}_{i+1/2} - F^{\rho u}_{i-1/2}\right) \\ \qquad\qquad - \frac{\Delta t}{2}\left(\overline{\rho}_{i-1/2}\frac{\phi_i-\phi_{i-1}}{\Delta x} + \overline{\rho}_{i+1/2}\frac{\phi_{i+1}-\phi_i}{\Delta x}\right), \\ E_i^{n+1} = E_i^n - \frac{\Delta t}{\Delta x}\left(F^{E}_{i+1/2} - F^{E}_{i-1/2}\right) \\ \qquad\qquad - \frac{\Delta t}{2}\left(\overline{\rho}_{i-1/2}\frac{u_{i-1}^n+u_i^n}{2}\frac{\phi_i-\phi_{i-1}}{\Delta x} + \overline{\rho}_{i+1/2}\frac{u_i^n+u_{i+1}^n}{2}\frac{\phi_{i+1}-\phi_i}{\Delta x}\right), \end{cases} \tag{15}$$

where the numerical flux is defined by

$$\left(F^{\rho}_{i+1/2}, F^{\rho u}_{i+1/2}, F^{E}_{i+1/2}\right) = \left(F^{\rho}, F^{\rho u}, F^{E}\right)(w_i^n, w_{i+1}^n), \tag{16}$$

$$F^{\rho}(w_L,w_R) = \frac{\rho_L u_L + \rho_R u_R}{2} + \frac{\lambda_L}{2}(\rho_L^\star - \rho_L) + \frac{\lambda_R}{2}(\rho_R^\star - \rho_R), \tag{17}$$

$$F^{\rho u}(w_L,w_R) = \frac{\rho_L u_L^2 + p_L + \rho_R u_R^2 + p_R}{2} + \frac{\lambda_L}{2}(q^\star - \rho_L u_L) + \frac{\lambda_R}{2}(q^\star - \rho_R u_R), \tag{18}$$

$$F^{E}(w_L,w_R) = \frac{u_L(E_L+p_L) + u_R(E_R+p_R)}{2} + \frac{\lambda_L}{2}(E_L^\star - E_L) + \frac{\lambda_R}{2}(E_R^\star - E_R). \tag{19}$$

Now, we present the properties satisfied by the scheme (15). The first two results deal with the well-balanced properties and are straightforward according to the derivation of the scheme. The last result concerns the robustness of the scheme. The proof is more technical and the reader is referred to [3] for the details.

Theorem 1. *Assume there are positive constants α and β such that the initial data satisfies for all $i \in \mathbb{Z}$:*

$$u_i^0 = 0, \qquad \rho_i^0 = \alpha e^{-\beta\phi_i}, \qquad p_i^0 = \frac{\alpha}{\beta} e^{-\beta\phi_i}.$$

Assume the ρ*-average is defined by* $\overline{\rho} = \begin{cases} \frac{\rho_R - \rho_L}{\ln(\rho_R) - \ln(\rho_L)} & \text{if } \rho_L \neq \rho_R, \\ \rho_L & \text{if } \rho_L = \rho_R. \end{cases}$
Then the approximation given by (15) stays at rest: $w_i^n = w_i^0$*, for all* $n \in \mathbb{N}$ *and* $i \in \mathbb{Z}$*.*

Theorem 2. *Assume the initial data satisfies the following approximation of (2) for all* $i \in \mathbb{Z}$*:*

$$u_i^0 = 0, \qquad \frac{p_{i+1} - p_i}{\Delta x} + \overline{\rho}_{i+1/2} \frac{\phi_{i+1} - \phi_i}{\Delta x} = 0. \tag{20}$$

Then the approximation given by (15) stays at rest: $w_i^n = w_i^0$*, for all* $n \in \mathbb{N}$ *and* $i \in \mathbb{Z}$*.*

We underline that this result holds true independently of the definition of $\overline{\rho}$. In fact, Theorem 2 states a preservation of approximations of the solutions of (2), according to the discretization (20).

Finally, we establish the robustness of the scheme (15).

Theorem 3. *For all* $i \in \mathbb{Z}$*, assume* $|\lambda_{i+1/2}^L|$ *and* $\lambda_{i+1/2}^R$ *are large enough such that*

- $\left|\lambda_{i+1/2}^R / \lambda_{i+1/2}^L\right|$ *is large enough if* $\phi_{i+1} > \phi_i$*;*
- $\left|\lambda_{i+1/2}^L / \lambda_{i+1/2}^R\right|$ *is large enough if* $\phi_{i+1} < \phi_i$*.*

Then the scheme (15) preserves the set Ω*:* $\quad \forall i \in \mathbb{Z}, w_i^n \in \Omega \quad \Rightarrow \quad w_i^{n+1} \in \Omega.$

4 Numerical results

We present now two numerical experiments to underline the relevance of the designed scheme.

The first experiment is taken from [10]. We consider here a constant gravity field given by the potential $\phi(x) = x$. We start with a hydrostatic atmosphere with a perturbation in pressure:

$$\rho_0(x) = e^{-x}, \qquad u_0(x) = 0, \qquad p_0(x) = e^{-x} + 0.01 e^{-100(x-0.5)^2}.$$

This initial data is evolved on the computational domain $[0,1]$ using 100 cells until time $t = 0.25$. The obtained perturbation in pressure is presented in Fig. 2, where it is compared to a reference solution computed using 30.000 cells.

The second test is devoted to illustrate the behaviour of the scheme (15) around a non-hydrostatic steady state. Moreover, this experiment also emphasizes that the scheme can deal with more complex gravitational fields than the constant one. Indeed, we consider a gravitational potential given by $\phi(x) = -\sin(2\pi x)$ on the domain $[0,1]$ with periodic boundary conditions. We can easily check that the solution

$$\rho(x) = 3 + 2\sin(2\pi x), \quad u(x) = 0, \quad p(x) = 3 + 3\sin(2\pi x) - 0.5\cos(4\pi x) \tag{21}$$

is a non-hydrostatic steady state of (1). We evolve the initial data given by (21) until time $t = 1$ for different values of the number of cells N. The L^1 errors in density and velocity are shown in Table 1 and we observe that although this steady state is not exactly preserved, a second-order convergence is achieved. Let us notice that the scheme (15) is first-order, but this particular steady state is captured up to second-order. This is due to the fact that equation (14) is a second-order approximation of (2).

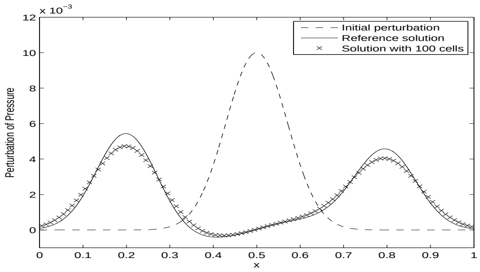

Fig. 2: Pressure perturbation for the hydrostatic atmosphere

N	Density		Velocity	
100	2.68E-05	–	2.11E-05	–
200	6.05E-06	2.15	5.40E-06	1.97
400	1.09E-06	2.47	1.36E-06	1.99
800	2.20E-07	2.31	3.39E-07	2.00
1600	4.86E-08	2.18	8.46E-08	2.00
3200	1.14E-08	2.09	2.11E-08	2.00

Table 1: L^1 error and convergence rates for the density and the velocity

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