Development of DDFV Methods for the Euler Equations

Christophe Berthon, Yves Coudière and Vivien Desveaux

Abstract We propose to extend some recent gradient reconstruction, the so–called DDFV approaches, to derive accurate finite volume schemes to approximate the weak solutions of the 2D Euler equations. A particular attention is paid on the limitation procedure to enforce the required robustness property. Some numerical experiments are performed to highlight the relevance of the suggested MUSCL–DDFV technique.

Key words: Finite volume methods for hyperbolic problems, Euler equations, DDFV reconstruction, MUSCL reconstruction, Robustness
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1 Introduction

This work is devoted to the numerical approximation of the 2–D Euler equations, given as follows:

$$\partial_t \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ E \end{bmatrix} + \partial_x \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho u v \\ u(E+p) \end{bmatrix} + \partial_y \begin{bmatrix} \rho v \\ \rho u v \\ \rho v^2 + p \\ v(E+p) \end{bmatrix} = 0, \tag{1}$$

where $\rho > 0$ denotes the density, $(u,v) \in \mathbb{R}^2$ the velocity vector and $E > 0$ the total energy. For the sake of the simplicity in the presentation, the pressure is given by the perfect gas law $p = (\gamma - 1)\left[E - \frac{\rho}{2}(u^2 + v^2)\right]$. The forthcoming developments will

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easily extend to general pressure laws. To shorten the notations, the system can be rewritten as follows:

$$\partial_t W + \partial_x f(W) + \partial_y g(W) = 0, \tag{2}$$

where $W = {}^t(\rho, \rho u, \rho v, E) : \mathbb{R}^2 \times \mathbb{R}^+ \to \Omega$ is the unknown state vector and $f(W) : \Omega \to \mathbb{R}^4$ and $g(W) : \Omega \to \mathbb{R}^4$ are the flux functions which find clear definitions. The convex set of admissible states is defined by:

$$\Omega = \left\{ W \in \mathbb{R}^4; \rho > 0, (u,v) \in \mathbb{R}^2, E - \frac{\rho}{2}\left(u^2 + v^2\right) > 0 \right\}. \tag{3}$$

When approximating (1), several strategies have been proposed to increase the accuracy of the numerical solutions among which the most popular is certainly the MUSCL scheme (for instance see [16, 15, 12, 13]). This scheme extends any first–order scheme into a second–order approximation using a piecewise linear reconstruction. In the 2–D case, the main difficulty is to find a technique to reconstruct gradients that can be extended to unstructured meshes (see [4]).

The DDFV (Discrete Duality Finite Volume) method was introduced in the field of elliptic equations in order to reconstruct gradients on distorted meshes (see [9, 6, 1, 10]). The idea of this method is to combine two distinct finite volume schemes on two overlapping meshes: the primal mesh and the dual mesh whose cells are built around the vertices of the primal mesh. This process adds new numerical unknowns at the vertices of the primal mesh, but it will allow to reconstruct very accurate gradients.

It was first proposed to take advantage of the DDFV gradient in order to built second order schemes for the linear convection–diffusion equation in [5]. In this paper, new values of the unknown are built at the midpoint of the interfaces by mean of some averages of the DDFV gradient. The resulting scheme is proved to be of second order in the diffusive regime.

The aim of this work is to extend DDFV–like methods to the case of the Euler equations. As a first step, we have only developed such a method on structured meshes in order to simplify the computation and to check its efficiency. On unstructured meshes, the extension of the DDFV gradient is straightforward. Our reconstruction and limitation procedures generalize although being more technical. Note that the vertices of the primal cells do not coincide with the center of gravity of the dual cells. It might influence the accuracy of the method and some alternatives will be considered in future work.

The paper is organized as follows. In Section 2, we introduce the dual mesh and we describe the reconstruction process and the limitation process of our scheme. Section 3 concerns the robustness of our scheme. Indeed, with most of first–order schemes, if a numerical solution is initially valued in Ω, then it remains in Ω. Such a property must be preserved by the second–order accurate scheme. Section 4 is devoted to numerical experiments to illustrate the relevance of DDFV approach when evaluating second–order reconstructions. We give some conclusions and future developments in Section 5.

2 Presentation of the scheme

First let us introduce the main notations. We consider a primal mesh composed of rectangular cells

$$K_{i,j} = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}] \times [y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}], \quad i,j \in \mathbb{Z}. \tag{4}$$

For the sake of simplicity, we will assume that the mesh is uniform, and we enforce $x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}} = y_{j+\frac{1}{2}} - y_{j-\frac{1}{2}} = h$, for all $i,j \in \mathbb{Z}$, where $h > 0$ is fixed.

Let $W_{i,j}^n$ stand for an approximation of the mean value of W on the cell $K_{i,j}$ at time t^n. We denote by $\Delta t > 0$ the time increment. At time $t^{n+1} = t^n + \Delta t$, the updated first–order approximation is given by (see [7, 13, 12]):

$$\begin{aligned} W_{i,j}^{n+1} = W_{i,j}^n - \frac{\Delta t}{h} \big(& F(W_{i,j}^n, W_{i+1,j}^n) - F(W_{i-1,j}^n, W_{i,j}^n) \\ & + G(W_{i,j}^n, W_{i,j+1}^n) - G(W_{i,j-1}^n, W_{i,j}^n) \big), \end{aligned} \tag{5}$$

where $F : \Omega \times \Omega \to \mathbb{R}^4$ and $G : \Omega \times \Omega \to \mathbb{R}^4$ are consistent numerical flux functions. In addition, to avoid some instabilities [12, 13], the time step is restricted according to a CFL–like condition given as follows:

$$\frac{\Delta t}{h} \max_{(i,j)\in\mathbb{Z}^2} \left(\left|\lambda_F^{\pm}(W_{i,j}^n, W_{i+1,j}^n)\right|, \left|\lambda_G^{\pm}(W_{i,j}^n, W_{i,j+1}^n)\right| \right) \leq \frac{1}{4}, \tag{6}$$

where $\lambda_\Phi^{\pm}(W_L, W_R)$ denotes suitable numerical wave velocities associated to the numerical flux function $\Phi(W_L, W_R)$.

2.1 The dual mesh

We denote by $B_{i+\frac{1}{2},j+\frac{1}{2}} = \left(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}} \right)$ the vertices of the primal mesh and by $B_{i,j} = (x_i, y_j)$ the center of the primal cell $K_{i,j}$. Around each vertex of the primal mesh $B_{i+\frac{1}{2},j+\frac{1}{2}}$, we construct a dual cell $K_{i+\frac{1}{2},j+\frac{1}{2}} = [x_i, x_{i+1}] \times [y_j, y_{j+1}]$. The set of the dual cells $\left(K_{i+\frac{1}{2},j+\frac{1}{2}} \right)_{i,j\in\mathbb{Z}}$ constitutes a second mesh which we call dual mesh. The centers of the dual cells are the vertices of the primal mesh and conversely.

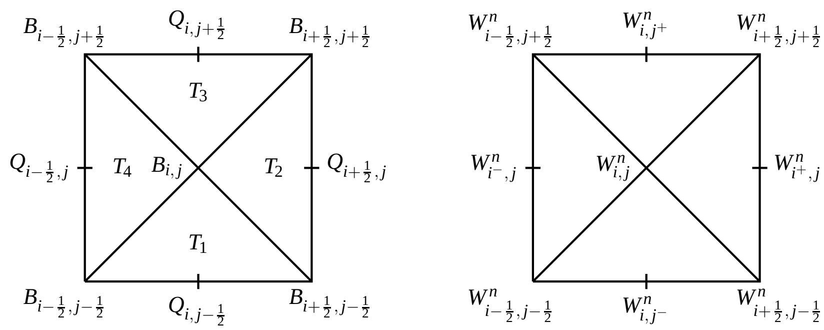
At time t^n, we assume known approximations $W_{i+\frac{1}{2},j+\frac{1}{2}}^n$ of the mean values of W on cells $K_{i+\frac{1}{2},j+\frac{1}{2}}$. As a consequence, at time t^n, on each primal or dual cell, we know four approximate values at the vertices and one approximate value at the center (see Fig.1b).

In the sequel, we will deal simultaneously with primal and dual cells. We thus define the set of the indexes of primal and dual cells $\mathbb{S} = \mathbb{Z}^2 \cup (\mathbb{Z} + \frac{1}{2})^2$. The set of primal and dual cells is then $\{K_{i,j}\}_{(i,j)\in\mathbb{S}}$. For $(i,j) \in \mathbb{S}$, we denote by

$Q_{i+\frac{1}{2},j} = (x_{i+\frac{1}{2}}, y_j)$, the middle of the interface between the cells $K_{i,j}$ and $K_{i+1,j}$ and by $Q_{i,j+\frac{1}{2}} = (x_i, y_j + \frac{1}{2})$, the middle of the interface between the cells $K_{i,j}$ and $K_{i,j+1}$ (see Fig. 1a). On each cell $K_{i,j}$ for $(i,j) \in \mathbb{S}$, we reconstruct values $W^n_{i^\pm,j}$ and $W^n_{i,j^\pm}$ at points $Q_{i\pm\frac{1}{2},j}$ and $Q_{i,j\pm\frac{1}{2}}$ (see Fig. 1b). Arguing these notations, the second order scheme reads as follows:

$$W^{n+1}_{i,j} = W^n_{i,j} - \frac{\Delta t}{h}\Big[F\left(W^n_{i^+,j}, W^n_{i+1^-,j}\right) - F\left(W^n_{i-1^+,j}, W^n_{i^-,j}\right) + G\left(W^n_{i,j^+}, W^n_{i,j+1^-}\right) - G\left(W^n_{i,j-1^+}, W^n_{i,j^-}\right)\Big]. \quad (7)$$

We now detail the evaluation of $W^n_{i^\pm,j}$ and $W^n_{i,j^\pm}$. We recall that both the primal



(a) (b)

Fig. 1: (Left) Geometry of the cell $K_{i,j}$. (Right) Location of the known states and of the reconstructed states.

and dual unknowns are solutions of a finite volume scheme. The two schemes are coupled through the gradient reconstruction.

2.2 Gradient reconstruction

As a first step, we perform a gradient reconstruction. To address such an issue, we derive a relevant cell splitting. We consider a primal or dual cell $K_{i,j}$, $(i,j) \in \mathbb{S}$. The cell can be decomposed into four triangles using the four vertices and the center. We denote by T_1 the bottom triangle and the other ones are denoted by T_2, T_3 and T_4, clockwise (see Fig. 1a).

We define a function $\widehat{W} : K_{i,j} \to \mathbb{R}^4$ piecewise linear on the T_l and which coincides with the approximate values at the four vertices and at the center.

Next, we project each coordinate $\widehat{W}_k$ of $\widehat{W}$ on the space of linear function which takes the value $(W_{i,j})^n{}_k$ at the point $B_{i,j}$. This means that for all integers $k \in [1,4]$, we seek $\mu_k \in \mathbb{R}^2$ which minimizes the functional $E_k(\nu) : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$E_k(\nu) = \int_{K_{i,j}} \left| \widehat{W}_k(X) - \left[\left(W_{i,j}^n\right)_k + \nu \cdot (X - B_{i,j}) \right] \right|^2 dX. \qquad (8)$$

Existence and uniqueness of the minimum are immediate since the functional is strictly convex. The numerical computation of the minimum is quite easy since we only need to compute the Jacobian of E_k and to find its zero. For the sake of simplicity in the notations, we denote by $\mu = {}^t(\mu_1, \mu_2, \mu_3, \mu_4)$, the vector of the solutions of these minimization problems. Hence, we define $\widetilde{W}_\mu(X) : K \to \mathbb{R}^4$ the function whose k–th coordinate is $\left(W_{i,j}^n\right)_k + \mu_k \cdot (X - B_{i,j})$.

2.3 Limitation

We assume that the states $W_{i,j}^n$, $(i,j) \in \mathbb{S}$, are in Ω. Let us remark that the reconstructed function $\widetilde{W}_\mu$ does not necessarily remain in Ω. As a consequence, we have to limit the slopes μ_k. To address such an issue, we propose to substitute the slope μ by $\theta\mu$ where $\theta \in [0,1]$ is a limitation parameter to be fixed according to the required robustness property. To ensure existence and uniqueness of an optimal limited slope, we have to restrict Ω to a close set. We fix a small parameter $\varepsilon > 0$ and we define

$$\Omega_\varepsilon = \left\{ W \in \mathbb{R}^4; \rho \geq \varepsilon, (u,v) \in \mathbb{R}^2, E - \frac{\rho}{2}\left(u^2 + v^2\right) \geq \varepsilon \right\}. \qquad (9)$$

Since we need the values of the reconstructed function only at points $B_{i\pm\frac{1}{2},j}$ and $B_{i,j\pm\frac{1}{2}}$, we require $\widetilde{W}_{\theta\mu}(B_{i\pm\frac{1}{2},j}) \in \Omega_\varepsilon$ and $\widetilde{W}_{\theta\mu}(B_{i,j\pm\frac{1}{2}}) \in \Omega_\varepsilon$. We thus define the optimal slope limiter by

$$\theta = \max\left\{ t \in [0,1]; \widetilde{W}_{t\mu}(B_{i\pm\frac{1}{2},j}) \in \Omega_\varepsilon, \widetilde{W}_{t\mu}(B_{i,j\pm\frac{1}{2}}) \in \Omega_\varepsilon \right\}. \qquad (10)$$

We emphasize that this set is nonempty since it contains 0. Besides, the maximum is reached because Ω_ε is a close set and $t \mapsto \widetilde{W}_{t\mu}(B_{l,m})$ is continuous. Solving for θ requires to find the roots of some quadratic functions (the energy). Finally, the reconstructed states are given by $W_{i^\pm,j}^n = \widetilde{W}_{\theta\lambda}(B_{i\pm\frac{1}{2},j})$ and $W_{i,j^\pm}^n = \widetilde{W}_{\theta\lambda}(B_{i,j\pm\frac{1}{2}})$.

3 Robustness

We now establish the robustness of the proposed reconstruction. First, let us assume that the directional flux functions F and G are first–order robust on both primal and dual meshes. Indeed, under the CFL condition

$$\frac{\Delta t}{h} \max_{(i,j)\in\mathbb{S}} \left(\left|\lambda_F^{\pm}(W_{i,j}^n, W_{i+1,j}^n)\right|, \left|\lambda_G^{\pm}(W_{i,j}^n, W_{i,j+1}^n)\right| \right) \leq \frac{1}{4}, \tag{11}$$

we assume that the updated states, given by (5) for all pairs (i,j) in $\mathbb{S}$, stay in Ω. Now, let us recall the following statements (for instance see [2, 12]) about robustness of the directional numerical flux functions:

Theorem 1. *Let us consider a robust numerical flux Φ. Assume that W_1, W_2 and W_3 are in Ω. Let W_2^- and W_2^+ be two reconstructed states in Ω such that $W_2 = \frac{W_2^- + W_2^+}{2}$. Assume the CFL condition*

$$\frac{\Delta t}{h} \max \left(|\lambda_\Phi^+(W_1, W_2^-)|, |\lambda_\Phi^{\pm}(W_2^-, W_2^+)|, |\lambda_\Phi^-(W_2^+, W_3)| \right) \leq \frac{1}{4}. \tag{12}$$

Then we have $W_2 - \frac{\Delta t}{h} \left(\Phi(W_2^+, W_3) - \Phi(W_1, W_2^+) \right) \in \Omega$.

We assume that the 1D numerical fluxes F and G are robust. In addition, we assume that the states $W_{i,j}^n$, $(i,j) \in \mathbb{S}$ are in Ω, so that the limitation procedure described in section 2.3 ensures that the reconstructed states $W_{i^\pm,j}^n$ and $W_{i,j^\pm}^n$, $(i,j) \in \mathbb{S}$, remain in Ω. To shorten the notations, we set

$$\begin{aligned}
\Lambda_F &= \max_{(i,j)\in\mathbb{S}} \left(|\lambda_F^{\pm}(W_{i^-,j}^n, W_{i^+,j}^n)|, |\lambda_F^{\pm}(W_{i^+,j}^n, W_{i+1^-,j}^n)| \right), \\
\Lambda_G &= \max_{(i,j)\in\mathbb{S}} \left(|\lambda_G^{\pm}(W_{i,j^-}^n, W_{i,j^+}^n)|, |\lambda_G^{\pm}(W_{i,j^+}^n, W_{i,j+1^-}^n)| \right).
\end{aligned}$$

By applying Theorem 1 we have

$$W_{i,j}^n - \frac{\Delta t}{h} \left[F\left(W_{i^+,j}^n, W_{i+1^-,j}^n\right) - F\left(W_{i-1^+,j}^n, W_{i^-,j}^n\right) \right] \in \Omega, \tag{13}$$

as soon as the CFL restriction $\frac{\Delta t}{h}\Lambda_F \leq \frac{1}{4}$ holds, and we get

$$W_{i,j}^n - \frac{\Delta t}{h} \left[G\left(W_{i,j^+}^n, W_{i,j+1^-}^n\right) - G\left(W_{i,j-1^+}^n, W_{i,j^-}^n\right) \right] \in \Omega, \tag{14}$$

under the CFL condition $\frac{\Delta t}{h}\Lambda_G \leq \frac{1}{4}$.

Considering half sum of (13) and (14), we finally obtain $W_{i,j}^{n+1} \in \Omega$, for all $(i,j) \in \mathbb{S}$ under the CFL condition [12] $\frac{\Delta t}{h} \max(\Lambda_F, \Lambda_G) \leq \frac{1}{8}$. The robustness of the proposed numerical method is thus established.

4 Numerical tests

We have chosen two cases from the collection of 2D Riemann problems proposed by [11], namely configuration 3 (p. 594) and 6 (p. 596). They are called case 1 and case 2. These problems are solved on the square $[0,1] \times [0,1]$ divided in four quadrants by lines $x = 1/2$ and $y = 1/2$. The Riemann problems are defined by initial constant states on each quadrant. All four 1D Riemann Problems between quadrants have exactly one wave: four shocks for the case 1 and four contact discontinuities for the case 2. Both cases were computed with primal grids of 200×200 cells which represent about 80,000 cells counting the dual mesh. In order to complete the scheme (7), the adopted numerical flux functions F and G are given by the well–known HLLC approximate Riemann solver (see [8, 14, 3]). The results are displayed for density in Fig. 2. We also provide a comparison with the classical MUSCL scheme on the line $y = x$ and a comparison of the CPU time between the two methods.

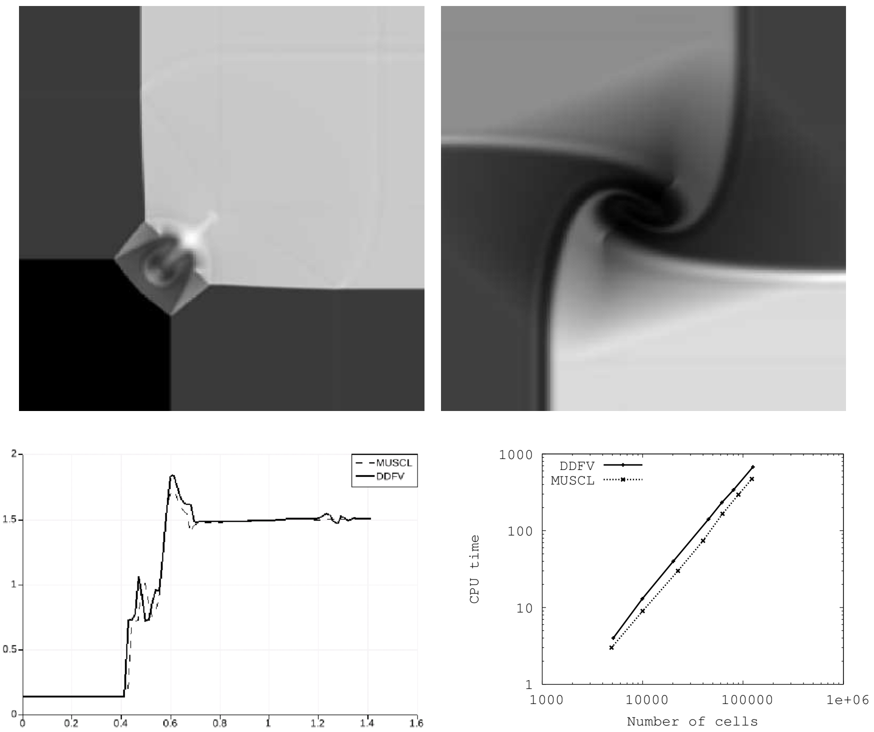


Fig. 2: Results for the 2D Riemann Problem Case 1 (top left) and case 2 (top right) obtained by the derived MUSCL–DDFV scheme. Comparison between the MUSCL–DDFV scheme and the classical MUSCL scheme for case 1: density on the line $y = x$ (bottom left) and CPU time (bottom right).

5 Conclusion

We have presented a second–order robust scheme to approximate the solutions of the 2D Euler equations. The main novelty of this work lies in the gradient reconstruction based on the DDFV methods and the use of two overlapping meshes. We have shown that the method gives good results on structured meshes. Arguing the properties of the DDVF approach, unstructured mesh extensions will be easily obtained.

In order to ensure the robustness, we have enforced that the reconstructed state vectors remain conservative. Another improvement must be performed to propose robust non–conservative reconstructions.

References

1. Andreianov, B., Boyer, F., Hubert, F.: Discrete duality finite volume schemes for Leray-Lions-type elliptic problems on general 2D meshes. Numerical Methods for Partial Differential Equations **23**(1), 145–195 (2007)
2. Berthon, C.: Stability of the MUSCL schemes for the Euler equations. Comm. Math. Sci **3**, 133–158 (2005)
3. Bouchut, F.: Nonlinear stability of finite volume methods for hyperbolic conservation laws and well-balanced schemes for sources. Frontiers in Mathematics. Birkhäuser Verlag, Basel (2004)
4. Buffard, T., Clain, S.: Monoslope and multislope MUSCL methods for unstructured meshes. Journal of Computational Physics **229**(10), 3745–3776 (2010)
5. Coudière, Y., Manzini, G.: The Discrete Duality Finite Volume Method for Convection-diffusion Problems. SIAM Journal on Numerical Analysis **47**(6), 4163–4192 (2010)
6. Domelevo, K., Omnes, P.: A finite volume method for the Laplace equation on almost arbitrary two-dimensional grids. Mathematical Modelling and Numerical Analysis **39**(6), 1203–1249 (2005)
7. Godlewski, E., Raviart, P.A.: Numerical approximation of hyperbolic systems of conservation laws, *Applied Mathematical Sciences*, vol. 118. Springer-Verlag, New York (1996)
8. Harten, A., Lax, P., Van Leer, B.: On upstream differencing and Godunov-type schemes for hyperbolic conservation laws. SIAM review pp. 35–61 (1983)
9. Hermeline, F.: A finite volume method for the approximation of diffusion operators on distorted meshes. Journal of computational Physics **160**(2), 481–499 (2000)
10. Hermeline, F.: Approximation of 2-D and 3-D diffusion operators with variable full tensor coefficients on arbitrary meshes. Computer Methods in Applied Mechanics and Engineering **196**(21-24), 2497–2526 (2007)
11. Kurganov, A., Tadmor, E.: Solution of two-dimensional Riemann problems for gas dynamics without Riemann problem solvers. Numerical Methods for Partial Differential Equations **18**(5), 584–608 (2002)
12. LeVeque, R.: Finite volume methods for hyperbolic problems. Cambridge Univ Pr (2002)
13. Toro, E.: Riemann solvers and numerical methods for fluid dynamics: a practical introduction. Springer Verlag (2009)
14. Toro, E., Spruce, M., Speares, W.: Restoration of the contact surface in the HLL-Riemann solver. Shock waves **4**(1), 25–34 (1994)
15. Van Leer, B.: Towards the ultimate conservative difference scheme. V. A second-order sequel to Godunov's method. Journal of Computational Physics **32**(1), 101–136 (1979)
16. Van Leer, B.: A historical oversight: Vladimir P. Kolgan and his high-resolution scheme. Journal of Computational Physics (2010)

The paper is in final form and no similar paper has been or is being submitted elsewhere.