Dynamics and Keane's Theorem

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Outline

Motivations and context

- K simplicial complex on a set S,
- $|K|_1$ geometric realization with $|\alpha - \beta|_1 = \sum_{s \in S} |\alpha(s) - \beta(s)|$,
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<u>Goal</u>: study the orbits $\{f^n(x)\}_{n\in\mathbb{N}}$ for some $x \in X$.

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Take $X = \mathbb{F}_p$. By Fermat's little theorem, we have $Per(z^p, \mathbb{F}_p) = \mathbb{F}_p$ and $Per(z^{p-1}, \mathbb{F}_p) = \{0, 1\}$.

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  $\to$ Ergodicity and Birchoff's theorem.

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  This is an automorphism of the 2-dimensional torus.
- $X = \{0, 1\}^{\mathbb{N}}$ and $T : (x_n)_{n \geq 0} \mapsto (x_{n+1})_{n \geq 0}$. This is known as the shift operator.

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We say that μ is a T-invariant measure (or, equivalently, T is a measure-preserving transformation) if $\mu(T^{-1}B) = \mu(B)$ for all $B \in \mathcal{B}$.

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- $X = \mathbb{T} \simeq {}^{\mathbb{R}}\!/_{\mathbb{Z}}$, $T : x \mapsto x + \alpha \mod 1$ with $\alpha \in \mathbb{R}$ and μ the Lebesgue measure. This is a rotation on the circle.

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We can know look at a remarkable result due to Poincaré !

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Let $T : X \to X$ be a measure preserving transformation of the probability space $(X, \mathcal{B}, \mu)$. Let $B \in \mathcal{B}$ be such that $\mu(B) > 0$. Then for μ-a.e. $x \in B$, the orbit $\{T^n x\}_{n \in \mathbb{N}}$ returns to B infinitely often.

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Remark

That is to say that it exists $E \subset B$ with $\mu(E) = \mu(B)$ such that for every $x \in E$, it exists integers $n_1 < n_2 < \cdots < n_j < \cdots$ such that $T^{n_i} x \in E$ for all $i \geq 1$.

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Whence $\mu(F) = 0$ and $\mu(B \setminus E) = 0$.

References

To know more about problems and ideas in dynamics :

- J. Milnor, *Dynamics in One Complex Variable*.
- J. H. Silverman, *The Arithmetic of Dynamical Systems*.
- C. Walkden, *Ergodic Theory*.

Keane's theorem

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It uses $m(\Omega)$ finite and Ω is nonatomic. The idea is that we can find an increasing sequence of sets and a right-inverse to the measure m.

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<u>Goal</u> : Find $\varphi : B \times G \to G$ continuous. Then, set $\theta(t, T) = \varphi(\Psi(t), T)$ for $(t, T) \in [0, 1] \times G$.

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- $E \cap (F_1 \Delta F_2) = (E \cap F_1) \Delta (E \cap F_2)$,

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Goal : Find $\varphi : B \times G \to G$ continuous. Then, set $\theta(t, T) = \varphi(\Psi(t), T)$ for $(t, T) \in [0, 1] \times G$.
$\to$ This is the big part of Keane's article. It uses Poincaré's recurrence theorem !

Symmetric difference operator :
$E \Delta F := (E \setminus F) \cup (F \setminus E) = (E \cup F) \setminus (E \cap F)$.

- Commutative, associative, $E \Delta \varnothing = E$ and $E \Delta E = \varnothing$,
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Remark

Ω_k depends continuously on (E, T).

Proof

Lemma

The map $\varphi : B \times G \to G$ *defined by* $\varphi(E, T) = T_E$ *is continuous.*

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Sketch of the proof for the weak topology on G :

Fix $F \in B$ and $(E^*, T^*) \in B \times G$.

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The statement is true for both topologies on G.

Sketch of the proof for the weak topology on G :
Fix $F \in B$ and $(E^*, T^*) \in B \times G$. Choose $\varepsilon > 0$.
Take an integer k_0 large enough such that

$$m\left(\bigcup_{k>k_0} \Omega_k^*\right) < \varepsilon.$$

Proof

Let N be a *weak* neighborhood of (E^*, T^*) such that for each $(E, T) \in N$ and each $1 \leq k \leq k_0$,

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Set $\Lambda_k := \Omega_k^* \cap \Omega_k$.

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Set $\Lambda_k := \Omega_k^* \cap \Omega_k$. Then,

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In the following, let $A_{k_0} = \bigcup_{k=1}^{k_0} \Lambda_k$ and $A^{\complement}_{k_0} = \bigcup_{k>k_0} \Lambda_k$.

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In the following, let $A_{k_0} = \bigcup_{k=1}^{k_0} \Lambda_k$ and $A^{\complement}_{k_0} = \bigcup_{k>k_0} \Lambda_k$.
Remember that $m(A^{\complement}_{k_0}) < \varepsilon$.

Keane's theorem

$$m(T_E(F) \triangle T^*_{E^*}(F)) \leqq m\left(T_E\left(F \cap \sum_{k=1}^{k_0} \Lambda_k\right) \triangle T^*_{E^*}\left(F \cap \sum_{k=1}^{k_0} \Lambda_k\right)\right) + 2\epsilon$$

$$\leqq 2\epsilon + \sum_{k=1}^{k_0} m(T^k(F \cap \Lambda_k) \triangle T^{*k}(F \cap \Lambda_k)).$$

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$$m\big(T_E(F)\Delta T^*_{E^*}(F)\big) = m\left([T_E(F)\Delta T^*_{E^*}(F)]\cap T_E(A^{\complement}_{k_0})\right) + m\left([T_E(F)\Delta T^*_{E^*}(F)]\cap T_E(A_{k_0})\right)$$

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$$\begin{aligned} m(T_E(F) \triangle T^*_{E^*}(F)) &\leqq m\left(T_E\left(F \cap \sum_{k=1}^{k_0} \Lambda_k\right) \triangle T^*_{E^*}\left(F \cap \sum_{k=1}^{k_0} \Lambda_k\right)\right) + 2\epsilon \\ &\leqq 2\epsilon + \sum_{k=1}^{k_0} m(T^k(F \cap \Lambda_k) \triangle T^{*k}(F \cap \Lambda_k)). \end{aligned}$$

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$$=\underbrace{T_E(F\cap A_{k_0})\Delta[T^*_{E^*}(F)\cap T_E(A_{k_0})]}$$
$$\subset T_E(F\cap A_{k_0})\Delta T^*_{E^*}(F)$$

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$$m\big(T_E(F)\Delta T^*_{E^*}(F)\big) \leq 2\varepsilon + m\Big(T_E(F\cap A_{k_0})\Delta T^*_{E^*}(F\cap A_{k_0})\Big)$$

On Λ_k, $T_E = T^k$ and $T^*_{E^*} = T^{*k}$ and the measure is subadditive.

Proof

$$\begin{aligned}
m(T^k(F \cap \Lambda_k) \triangle T^{*k}(F \cap \Lambda_k)) \leqq\ & m(T^k(F \cap \Lambda_k) \triangle T^k(F \cap \overset{*}{\Omega}_k)) \\
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Now, using the preservation of the measure, we may observe that the first and third terms have the same weight !

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Now, using the preservation of the measure, we may observe that the first and third terms have the same weight ! The second term is controlled by the estimate given by our choice of the neighborhood N.

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$$m(T_E(F) \triangle T^*_{E^*}(F)) \leqq 2\epsilon + k_0 \cdot \frac{\epsilon}{k_0} + 2\sum_{k=1}^{k_0} m((F \cap \Lambda_k) \triangle (F \cap \Omega^*_k)),$$
$$\leqq 3\epsilon + 2m\left(\sum_{k=1}^{k_0} \Lambda_k \triangle \sum_{k=1}^{k_0} \Omega^*_k\right) \leqq 5\epsilon.$$

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It suffices to understand the last "sum".

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Note that $(F \cap \Lambda_k)\Delta(F \cap \Omega_k^*) = F \cap (\Lambda_k \Delta \Omega_k^*)$

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Note that $(F \cap \Lambda_k)\Delta(F \cap \Omega_k^*) = F \cap (\Lambda_k \Delta \Omega_k^*)$ and $(\Omega_k^*)_{1 \leq k \leq k_0}$ are pairwise disjoint (same goes for $(\Lambda_k)_{1 \leq k \leq k_0}$).

Keane's theorem

$$m(T_E(F) \triangle T^*_{E^*}(F)) \leqq 2\epsilon + k_0 \cdot \frac{\epsilon}{k_0} + 2 \sum_{k=1}^{k_0} m((F \cap \Lambda_k) \triangle (F \cap \Omega^*_k)),$$

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It suffices to understand the last "sum".
Note that $(F \cap \Lambda_k)\Delta(F \cap \Omega_k^*) = F \cap (\Lambda_k \Delta \Omega_k^*)$ and $(\Omega_k^*)_{1 \leq k \leq k_0}$ are pairwise disjoint (same goes for $(\Lambda_k)_{1 \leq k \leq k_0}$). Thus, we have

$$\bigcup_{k=1}^{k_0} \Lambda_k \Delta \bigcup_{k=1}^{k_0} \Omega_k^* = \bigcup_{k=1}^{k_0} \Omega_k^* \setminus \bigcup_{k=1}^{k_0} \Lambda_k,$$

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$\to$ We will see that Ψ is a Serre fibration in the next talk.

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Thank you !