Université de Picardie Jules Verne
Cours de l'Ecole Doctorale/Doctoral School's Lectures
Calculs Numériques/Numerical Computations
2019-2020

Solving Boundary value problem

Consider the Boundary values problem

$$-u"(x) = f(x),\ x \in]0,1[, \tag{1}$$
$$u(0) = u(1) = 0. \tag{2}$$

A nice way to build a finite differences scheme for solving this problem is the following:

- Consider N regularly spaced point in $[0,1]$, $x_i = ih$ (also called the grid-points), with $h = \dfrac{1}{N+1}$. Assume that the equation holds for every points of $[0,1]$, particularly at $x_i, i = 1, \cdots, N$. We can write

$$-u"(x_i) = f(x_i),\ i = 1, \cdots, N.$$

- Of course, we must approach the quantities $-u"(x_i)$. A way to do that is to use Taylor's expansion:

$$\begin{aligned} u(x_{i+1}) &= u(x_i + h) &= u(x_i) + hu'(x_i) + \frac{h^2}{2!}u"(x_i) + \frac{h^3}{3!}u^{(3)}(x_i) + \frac{h^4}{4!}u^{(4)}(\xi_1), \\ u(x_{i-1}) &= u(x_i - h) &= u(x_i) - hu'(x_i) + \frac{h^2}{2!}u"(x_i) - \frac{h^3}{3!}u^{(3)}(x_i) + \frac{h^4}{4!}u^{(4)}(\xi_2), \end{aligned}$$

  Summing these two expression, one gets after the usual simplifications

$$\frac{2u(x_i) - u(x_{i+1}) - u(x_{i-1})}{h^2} = -u"(x_i) + \frac{1}{4!}(u^{(4)}(\xi_1) + u^{(4)}(\xi_2)) = -u"(x_i) + O(h^2)$$

  Replacing $u"(x_i)$ by this expression in the equation, we obtain

$$\frac{2u(x_i) - u(x_{i+1}) - u(x_{i-1})}{h^2} = f(x_i) + O(h^2)$$

  The lower order term $O(h^2)$ is of course never known.

- Now, we can describe the approximation method: we define $U_i \approx u(x_i)$ as

$$\frac{2u_i - u_{i+1} - u_{i-1}}{h^2} = f(x_i), i = 1, \cdots, N$$

  and $u_0 = u(0) = 0$, $u_{N+1} = u(1) = 0$.

- This last system of equation is equivalent to

$$AU = F$$

with

$$A = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & & \\ 0 & \ddots & \ddots & \ddots & \\ & & & & \\ 0 & & & -1 & 2 \end{pmatrix}, U = \begin{pmatrix} u(x_1) \\ \\ \vdots \\ \\ u(x_N) \end{pmatrix} \text{ and } F = \begin{pmatrix} f(x_1) \\ \\ \vdots \\ \\ f(x_N) \end{pmatrix}$$

- We can build the matrix A in Scilab as follows `n=5;`
  `A = 2?eye(n,n)?diag(ones(n?1,1),1)?diag(ones(n?1,1),?1);`

Exercice 1 (Simple Poisson Problem)

1. We take $f(x) = \pi^2 \sin(\pi x)$, so the corresponding solution to the poisson problem is $u(x) = \sin(\pi x)$. Compute the numerical approximations with the finite difference scheme for ($n = 10$, $n = 50$, $n = 100$). Plot on the same graphic the error at the grid points.

2. Same questions with $f(x) = 25\pi^2 \sin(5\pi x)$. What can you say ?

3. We now consider a nonregular data.

   (a) We take now $f(x) = \begin{cases} 1 & x \in]0, 1/2[\ , \\ 0 & x \in [1/2, 1[, \end{cases}$.

   (b) Plot the solution. What do you observe ?

Exercice 2 (Wave and damped wave equation)
Consider the wave equation

$$\frac{\partial^2 u}{\partial t^2} + \nu \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0 \quad x \in]0, 1[, \ t \in)0, T), \tag{3}$$

$$u0(0, t) = 0, u(1, t) = 0 \qquad t \in)0, T), \tag{4}$$

$$u(x, 0) = u_0(x) \qquad x \in]0, 1[, \tag{5}$$

$$u_t(x, 0) = u_1(x) \qquad x \in]0, 1[. \tag{6}$$

We admit that the problem possesses a unique solution. To approach the solution numerically, we display a finite difference scheme in both space and time: we look to numerical approximations of $u_i^k \approx u(x_i, t_k)$, with $t_k = k\Delta t$. At first we discretize the problem in space to obtain a (second order) differential system

$$\frac{d^2 U}{dt^2} + \nu \frac{du}{dt} AU = 0$$

Then we approach each term at the discrete times t_k by finite differences, we obtain the numerical scheme

$$\frac{U^{k+1} - 2U^k + U^{k-1}}{\Delta t^2} + \nu \frac{U^{k+1} - U^{k-1}}{2\Delta t} \frac{1}{2} \left(AU^{k+1} + AU^{k-1} \right) = 0$$

(a) We take in this question $\nu = 0$.

   i. We start from $u_0(x) = \sin(\pi x)$. Simulate the equation and make an animation. What do you observe ?

ii. We look to the quantity

$$E(U^{k+1}) = \frac{\|U^{k+1} - U^k\|^2}{2\Delta t} + \frac{1}{4}\Big(< AU^{k+1}, U^{k+1} > + < AU^k, U^k >\Big)$$

Plot $E(U^k)$ for each k. What can you say ?

iii. The physical problem corresponds to a model of vibration of a string (of a violin for instance). Do the numerical results fit with the physical observations ?

(b) We now take into account a damping term with $\nu = 0$.

i. Make the simulation for different values of ν. What do you observe ?

ii. Does it coincide now with that you observe in the real world ?