Continuum Modeling: Numerical schemes for linear & nonlinear reaction/diffusion systems, Phase fields Models. Application to Batteries and image processing
Part 1: Mathematical Setting

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Plan

1. Introduction

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2. Basics of Elliptic and parabolic PDEs
   - Poisson problems and Boundary conditions
   - Reaction-Equation
   - Diffusion- Equation
   - Reaction-diffusion equations

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   - Allen-Cahn model
   - Allen-Cahn model : Gradient Flow and Minimization of an Energy
   - Cahn-Hilliard Equation

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4. Allen-Cahn vs Cahn Hilliard
5. Annex
   - Stability of a differential system

Phase Fields Models : model natural phenomena

- material science : in complex fluids and soft matter
- Intercalation and conversion in batteries
- Biology, Medecine, Ecology : tumor growth, polluant transport in fluids
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In Mathematics (a very actual and active topic)

- Mathematical analysis (dynamical systems, calculus of variations) [Temam88, Elliott89, Bertozzi, Miranville, Grasselli Pierre *et al*]
- Mathematical modeling
- Numerical analysis[Elliot89,Eyre94] and [Boyer, Shen,Wise,Pierre]

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Goal of the lecture

- Give simple basics of mathematical and numerical approach to phase fields
- Introduce to a friendly numerical environment that allows to produce nice numerical simulations (help in modeling)
- Special focus on Allen-Cahn and Cahn-Hilliard equations

Our approach to Allen-Cahn's model

Two approaches can be considered for studying Allen-Cahn's equation :

Reaction-Diffusion

$$\underbrace{\frac{\partial u}{\partial t} - \Delta u + \frac{1}{\epsilon^2} f(u) = 0 \quad \text{Reaction-diffusion approach}}$$

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$$\underbrace{\frac{\partial u}{\partial t} + \nabla E(u) = 0}_{\text{Gradient Flow approach}}$$

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- Use directly methods devoted to such structured equations
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These two approaches coincide when $-\Delta u + \frac{1}{\epsilon^2} f(u) = \nabla E(u)$

Notations

In $\mathbb{R}^n$

The gradient vector and laplacian of $F : \mathbb{R}^n \to \mathbb{R}$ at $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$

$\nabla F(x) = (\dfrac{\partial F(x)}{\partial x_1}, \cdots, \dfrac{\partial F(x)}{\partial x_n})^T$ and $\Delta F = \displaystyle\sum_{i=1}^{n} \dfrac{\partial^2 F(x)}{\partial x_i^2}$ Taylor $\in \mathbb{R}^n$:

$$F(x + ty) = F(x) + t < \nabla F(x), y > + \text{ lot} \,; < x, y >= \sum_{i=1}^{n} x_i y_i.$$

In a functional space

Let Ω be an open and bounded set of R^n. We define $< u, v >= \displaystyle\int_\Omega uvdx$ If in a addition $\partial\Omega$, the boundary of Ω, is smooth enough, we can define the outise normal $\vec{n}$ of Ω at $x \in \partial\Omega$ as the unit vector pointing outside Ω and orthogonal to the tangent plane at Ω in x.

$\dfrac{\partial u}{\partial n} = \vec{\nabla} u.\vec{n}$: normal derivative of u at x The Green formula :

$$\int_\Omega -\Delta uvdx = \int_\Omega \nabla u \nabla vdx - \int_{\partial\Omega} \frac{\partial u}{\partial n} ud\sigma = \int_\Omega u - \Delta vdx + \int_{\partial\Omega} \frac{\partial v}{\partial n} vd\sigma$$

Elliptic Equation

Let $\Omega \subset \mathbb{R}^n$, $n = 1, 2, 3$ for the applications, the spatial variables are $x_i, i = 1, \cdots, n$. We look to a function satisfying the PDE

$$\alpha u - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{i,j}\frac{\partial}{\partial x_j}\right) = f \quad x \in \Omega, \tag{1}$$

$$+ \text{ Boundary Conditions} \quad \text{on } \partial\Omega \tag{2}$$

$\partial\Omega$ is the boundary of Ω. The Boundary conditions are necessary to give a mathematical sense to this problem. They traduce a physical hypothesis.

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Canonical example

When $a_{i,j}(x) = \delta_{i,j} = \begin{cases} 1 & \text{if} i = j, \\ 0 & \text{else} \end{cases}$ the above equation reads :

$$\alpha u - \Delta u = f \quad x \in \Omega, \tag{3}$$

$$+ \text{ Boundary Conditions} \quad \text{on } \partial\Omega \tag{4}$$

Dirichlet Problem

$$\alpha u - \Delta u = f \quad x \in \Omega, \tag{5}$$

$$u = g \quad \text{on } \partial\Omega \quad \text{Dirichlet Boundary conditions} \tag{6}$$

Here both f and g are given. The **values** of u are **fixed** on $\partial\Omega$. If $g = 0$: Homogeneous Dirichlet BC. This problem possesses a unique solution for every $\alpha \geq 0$.

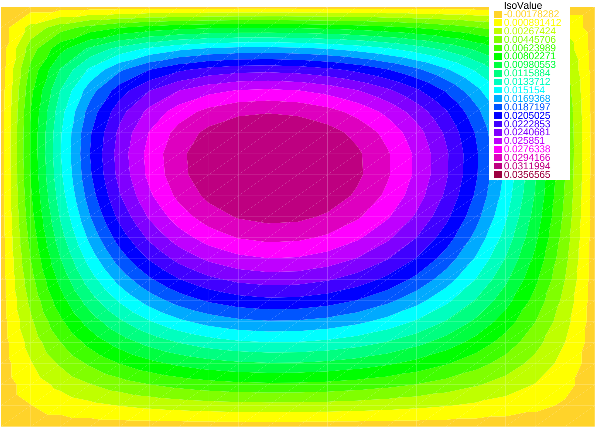

Figure – Isovalues of the Solution of a Poisson problem with Dirichlet BC)

Neumann Problem

$$\alpha u - \Delta u = f \quad x \in \Omega, \tag{7}$$

$$\frac{\partial u}{\partial n} = g \quad \text{on } \partial\Omega \quad \text{Neumann Boundary conditions} \tag{8}$$

Here both f and g are given. The **fluxes** of u are **fixed** on $\partial\Omega$: $\frac{\partial u}{\partial n} = \overrightarrow{\nabla u}\,\overrightarrow{n}$, where $\overrightarrow{n}$ is the external normal vector. If $g = 0$: Homogeneous Neumann BC. This problem possesses a unique solution for every $\alpha > 0$.

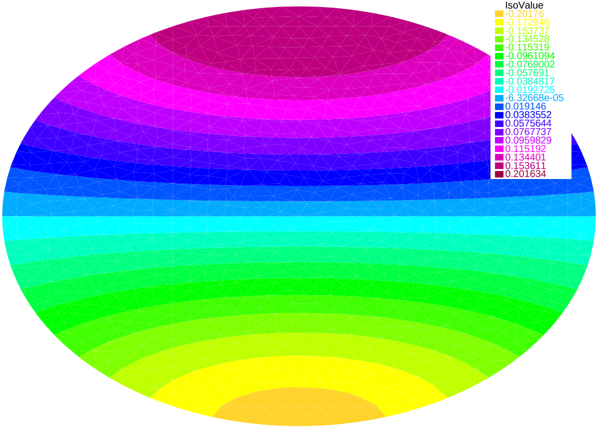

Figure – Isovalues of the Solution of a Poisson problem with Neumann BC

Mixed Dirichlet-Neumann Problem

$$-\Delta u = f \quad x \in \Omega, \tag{9}$$

$$\frac{\partial u}{\partial n} = g \quad \text{on } \partial\Omega_1 \quad \text{Neumann Boundary Conditions} \tag{10}$$

$$u = h \quad \text{on } \partial\Omega_2 \quad \text{Dirichlet Boundary Conditions} \tag{11}$$

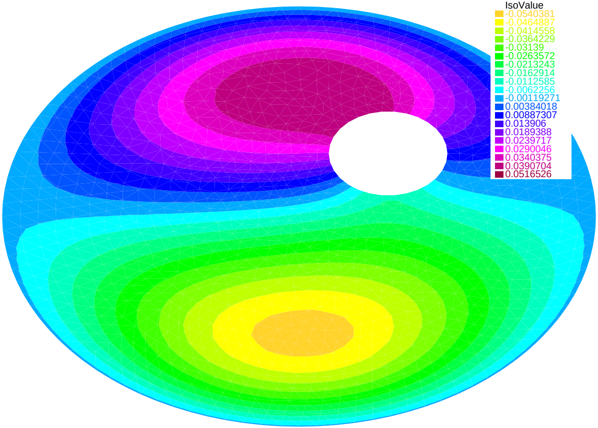

Figure – Isovalues of the Solution of a Poisson problem with mixed BC

Consider the scalar ODE

$$\frac{du}{dt} = F(u) \tag{12}$$

$$u(0) = u_0 \text{ (Initial Condition)} \tag{13}$$

It is an ODE.

Properties

- Existence and uniqueness (under conditions on F) (without exsitence nothing can be done ... without unicity you don't know what your are computing...)
- Regularity (important for justifying approximation techniques)

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Long time behaviour

- Steady state $u^*/F(u^*) = 0$
- Periodic solutions in time
- Chaos

Stablility of Steady states (see Annex for the higher dimensional case)

- u^* is **stable** if for every $u(0)$ close enough to u^*, $u(t)$ remains close to u^*
- u^* is **asymptotically stable** if for every $u(0)$ close enough to u^*, $u(t) \rightarrow u^*$ as $t \rightarrow +\infty$
- u^* is **unstable** in the other cases
- Linear stability
  - Write the ODE near a steady state u^* : set $u(t) = u^* + \omega(t)$

$$f(u(t)) \quad = f(u^* + \omega(t)) = \underbrace{\begin{array}{l} f(u^*) + f'(u^*)\omega(t) + \text{lower terms} \\ \text{by Taylor's formula} \end{array}}_{\simeq f'(u^*)\omega(t)}$$

  Since $\dfrac{du}{dt} = \dfrac{du^* + \omega(t)}{dt} = \dfrac{d\omega(t)}{dt}$, we can write the linearized ODE at u^*

$$\frac{d\omega(t)}{dt} = f'(u^*)\omega(t)$$

  whose solution is $\omega(t) = e^{f'(u^*)t}\omega(0)$.
  - Hence u^* is stable if $f'(u^*) \leq 0$ (asymptotically stable if $f'(u^*) < 0$) , it is unstable otherwise.

Logistic equation

$$\frac{du}{dt} = \frac{1}{\epsilon^2} u(1 - u^2), \tag{14}$$

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- Steady states are $u = 0$, $u = 1$ and $u = -1$
- Let $f(u) = \frac{1}{\epsilon^2} u(1-u^2)$. We have $f'(u) = \frac{1}{\epsilon^2}(1-3u^2)$

u^*	$f'(u^*)$	Stability
0	$\frac{1}{\epsilon^2} > 0$	unstable
1	$-\frac{2}{\epsilon^2} < 0$	asymp. stable
-1	$-\frac{2}{\epsilon^2} < 0$	asymp. stable

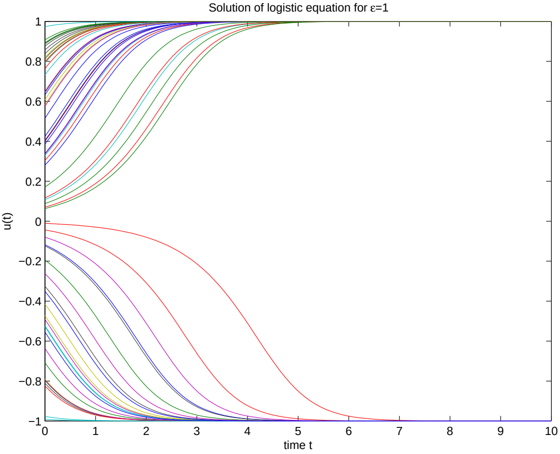

Figure – Evolution of the solution for different initial data for $\epsilon = 1$

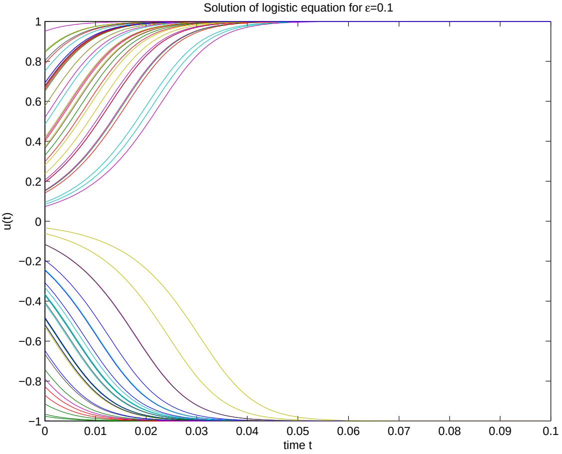

Figure – Evolution of the solution for different initial data for $\epsilon = 0.1$

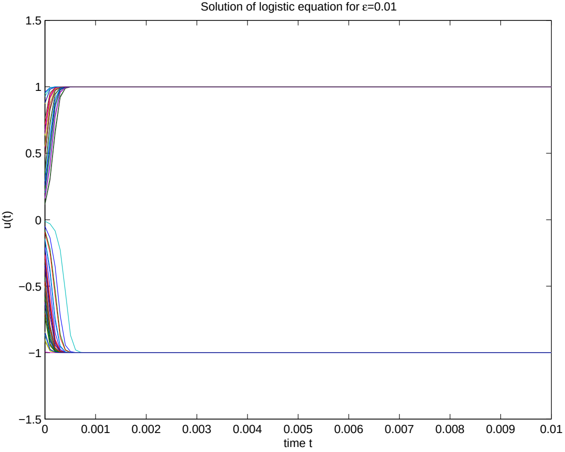

Figure – Evolution of the solution for different initial data for $\epsilon = 0.01$

Consider now a functional ODE

$$\frac{du}{dt} = \frac{1}{\epsilon^2} u(1 - u^2), \tag{16}$$

$$u(x, 0) = u_0(x) \tag{17}$$

where $u(x, 0)$ is now a given function of the variable $x \in [0, 1]$

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The solution of this equation can be expressed as

$$u(x,t) = \frac{u_0(x)}{\sqrt{u_0(x)^2 + (1-u_0(x)^2) * e^{-2\frac{t}{\epsilon^2}}}}$$

We see that $u(x,t) \to sign(u_0(x))$ as $t \to +\infty$, so exactly what was observed in the scalar case.

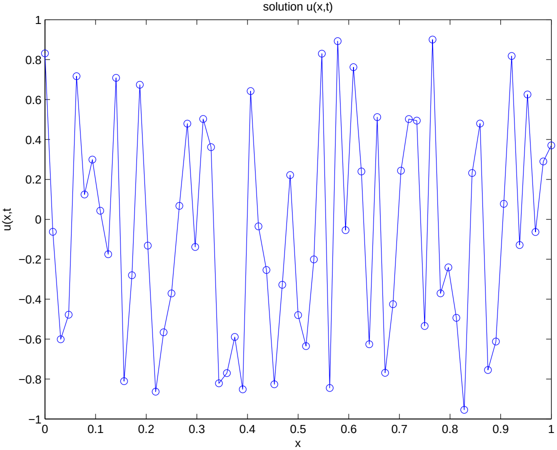

Figure – Solution at $t = 0$ $\epsilon = 1$ (random initial datum)

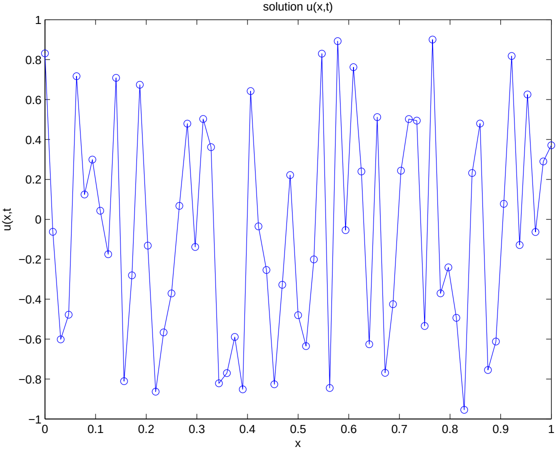

Figure – Solution at $t = 2$ $\epsilon = 1$

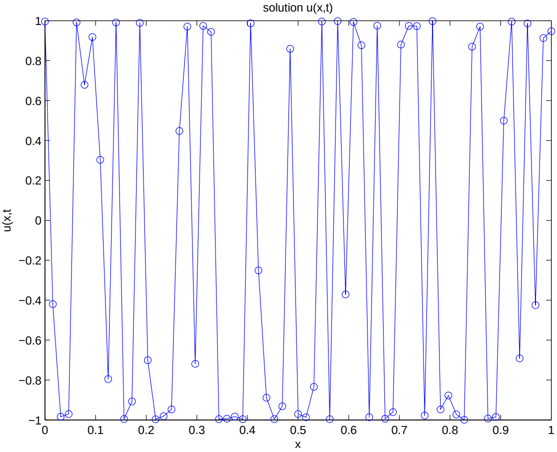

Figure – Solution at $t = 4$ $\epsilon = 1$

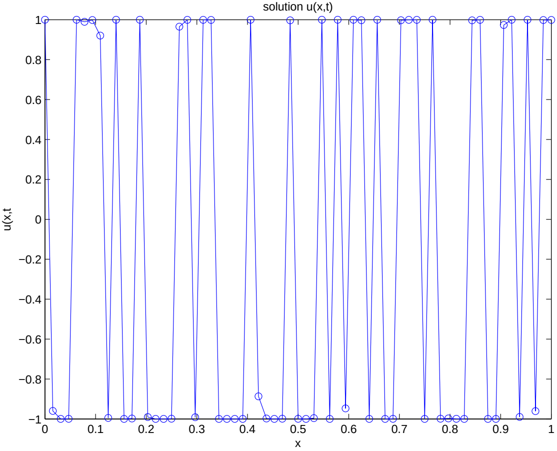

Figure – Solution at $t = 6$ $\epsilon = 1$

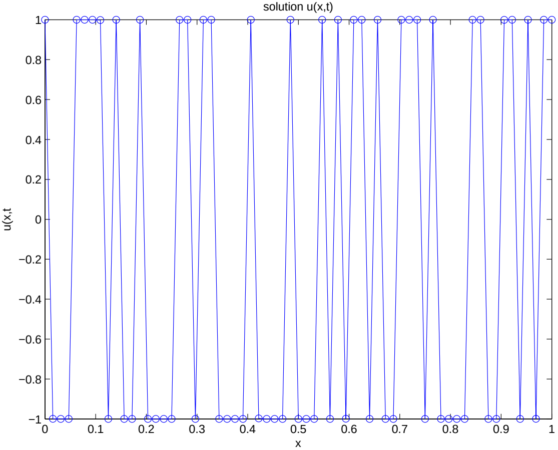

Figure – Solution at $t = 8$ $\epsilon = 1$

The prototype is the Heat equation : it models the evolution of the distribution of heat in a domain, with given boundary condition and external source :

$$\frac{\partial u}{\partial t} - \Delta u = f \qquad \Omega, \tag{18}$$

$$\tag{19}$$

$$+ \text{ Boundary Conditions} \quad \text{ on } \partial\Omega, \tag{20}$$

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Mathematical Properties

- Existence and Uniqueness
- Regularization
- When f is not time-dependent : when $t \to +\infty$, convergence to a steady state u^* solution of an elliptic equation

  $$-\Delta u = f$$

  with proper boundary conditions
- Stability (energy inequality)

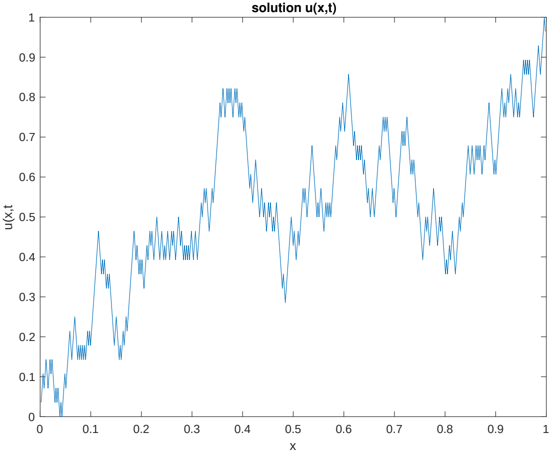

Figure – Solution at $t = 0$ (brownian-type initial datum)

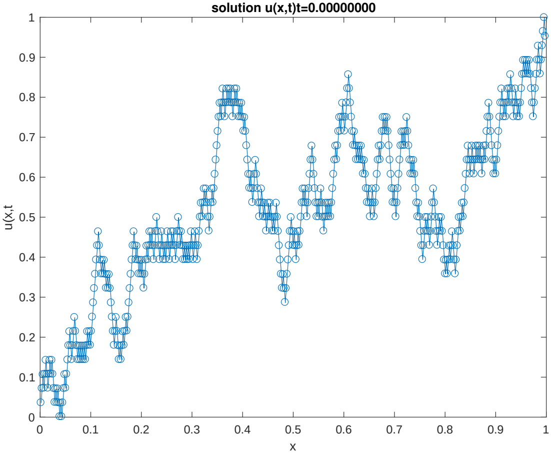

Figure – Solution at $t = 0.1$

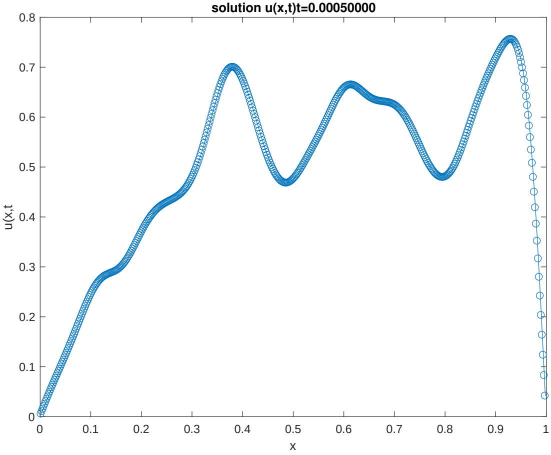

Figure – Solution at $t = 0.2$

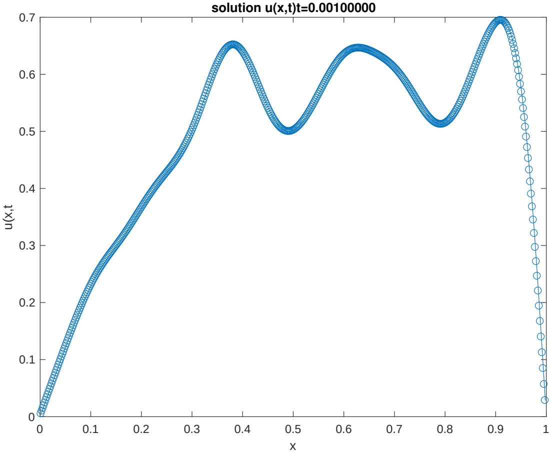

Figure – Solution at $t = 0.3$

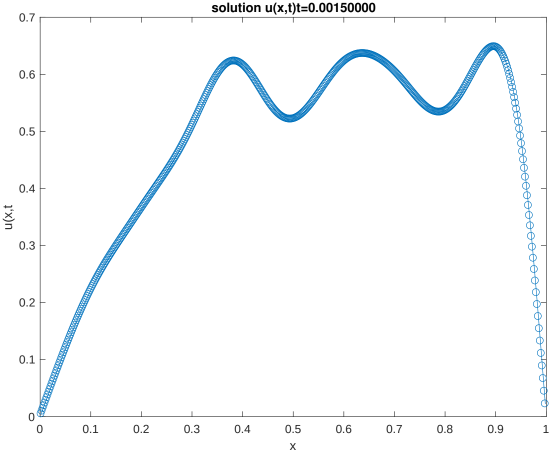

Figure – Solution at $t = 0.4$

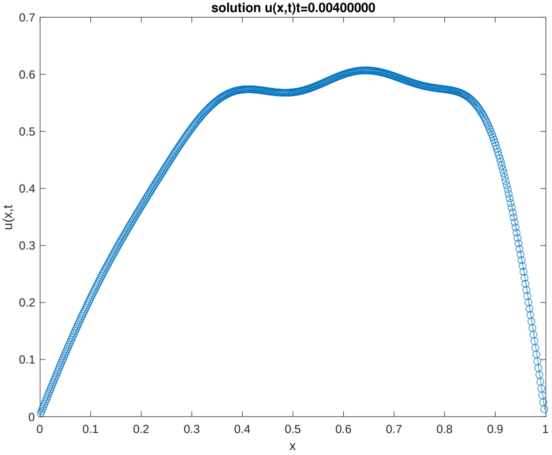

Figure – Solution at $t = 0.5$

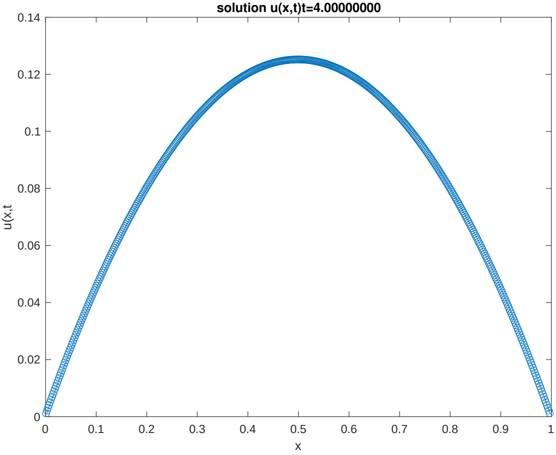

Figure – Solution at $t = 0.6$

We can write them into the form

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$$\tag{22}$$

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The source term $F(u)$ (the reaction term) depends on u There is a competition between the two terms $-\Delta u$ and $F(u)$. As in the nonlinear ODE case

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By definition, a phase field model is a mathematical model for solving interfacial problems

Main characteristics

they model the dynamical minimization of a physical free energy. According to the conservation of mass property, we distinguish two type of equations

- Allen-Chan, gradient flow with no mass conservation ; It models phase transition
- Cahn-Hilliard, which is mass conservative and models phase separation

Application domains

- material science : in complex fluids and soft matter (interfacial fluid flow, polymer science and in industrial applications).
- Intercalation and conversion in batteries
- Biological Science, Medecine and ecology : tumor growth, polluant transport in fluids
- Image processing : inpainting, image segmentation

Allen-Cahn equation is a and writes as

$$\frac{\partial u}{\partial t} + M(-\Delta u + \frac{1}{\epsilon^2} f(u)) = 0 \quad (24)$$

$$\frac{\partial u}{\partial n} = 0 \quad (25)$$

$$u(0,x) = u_0(x) \quad (26)$$

- It describes the process of phase separation in iron alloys [Allen-Cahn, 1972, 1973], including order-disorder transitions : M is the **mobilty** (taken to be 1 for simplicity), $F = \int_{-\infty}^{u} f(v)dv$ is the free energy, u is the (non-conserved) **order parameter**, ϵ is the **interface length**.
- The homogenous Neumann boundary condition implies that there is not a loss of mass outside the domain Ω
- there is a competition between the potential term and the diffusion term : regularization in phase transition
- Maximum principle : if $|u_0(x)| \leq \beta$ then $|u(x,t)| \leq \beta$, where β is the magnitude of largest zero of f.

It is a reaction-diffusion equation ! !

Derivation

Minimize the Energy $E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{1}{\epsilon}\int_\Omega F(u)dx$, where F is the potential of the free energy. The first term (Dirichlet energy) makes the phase transition smooth.

some potentials

- Double Well potential $F(u) = \frac{1}{4}(1-u^2)^2$ or Ginzburg-Landau double Well potential
- Truncated double-well potential

$$\tilde{F}(u) = \begin{cases} \dfrac{3M^2-1}{2}u^2 2M^3 u + \frac{1}{4}(3M^4+1) & \text{if } u > M \\ \frac{1}{4}(1-u^2)^2 & \text{if } u \in [-M, M] \\ \dfrac{3M^2-1}{2}u^2 + 2M^3 u + \frac{1}{4}(3M^4+1) & \text{if } u < -M \end{cases}$$

- Logarithmic free energy

$$F(u) = \frac{\theta}{2}\left((1+u)\ln 1 + u + (1-u)\ln 1 - u\right) - \frac{\theta_c}{2}u^2$$

Gradient Flow basics : $\frac{du}{dt} = -\nabla E(u)$

Definition and properties

- Energy decreasing : using the rule $\frac{dE(u)}{dt} =< \nabla E(u), \frac{du}{dt} >$, we have

$$\frac{dE(u)}{dt} = - < \nabla E(u), \nabla E(u) >= -\|\nabla E(u)\|^2 \leq 0$$

- The steady state of the system are the critical points of $E(u)$
- When E is coercive ($E(u) \to +\infty$ as $\|u\| \to +\infty$), the stable steady state are the minima of $E(u)$

Allen-Cahn as a Gradient flow

$$\text{We let } E(u) = \frac{1}{2}\int_\Omega \|\nabla u\|^2 dx + \frac{1}{\epsilon^2}\int_\omega F(u)dx$$

We have $\nabla E(u) = -\Delta u + \frac{1}{\epsilon^2} f(u)$ with $f = F'$. We recover Allen-Cahn's equation.

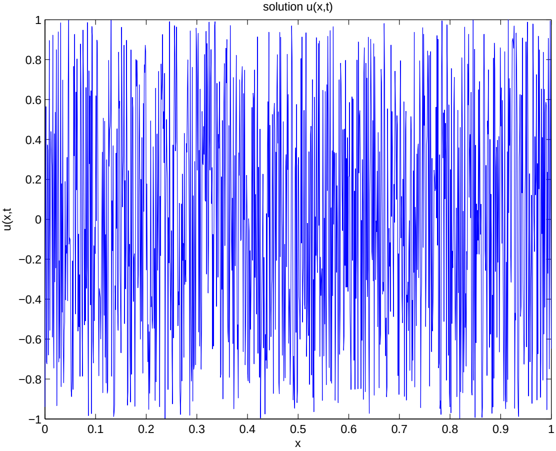


Figure – Solution of AC : initial data (uniform randomly in $[0, 1]$)

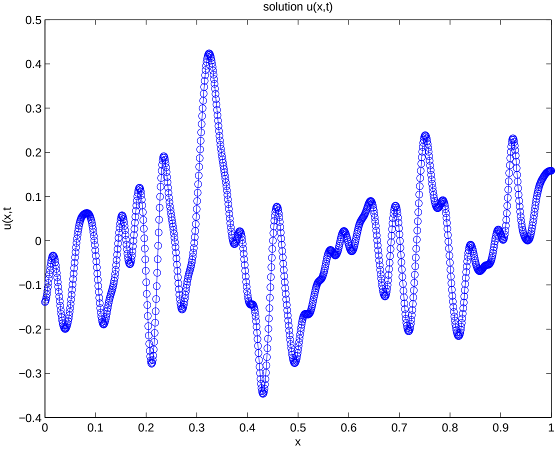


Figure – Solution of AC at time $t =$

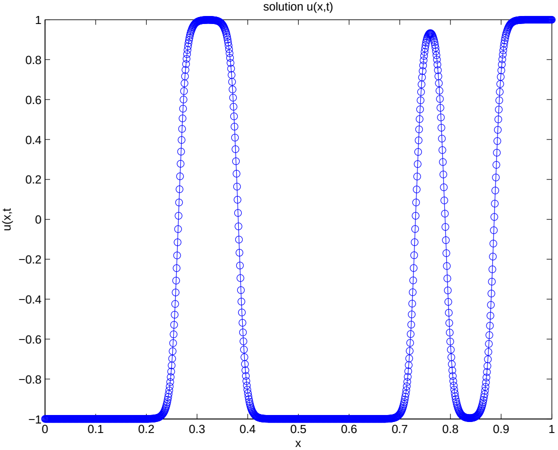


Figure – Solution of AC at time $t =$

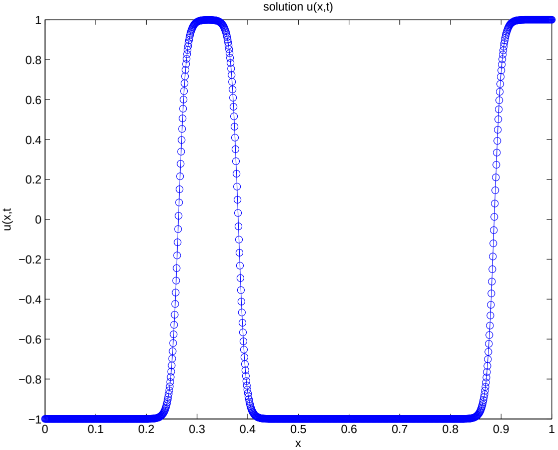


Figure – Solution of AC at time $t =$

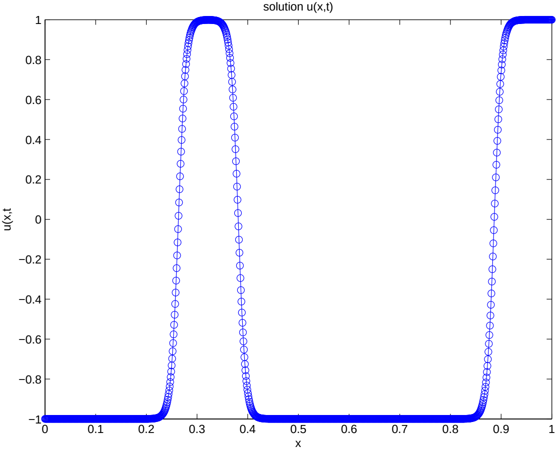


Figure – Solution of AC at time $t =$

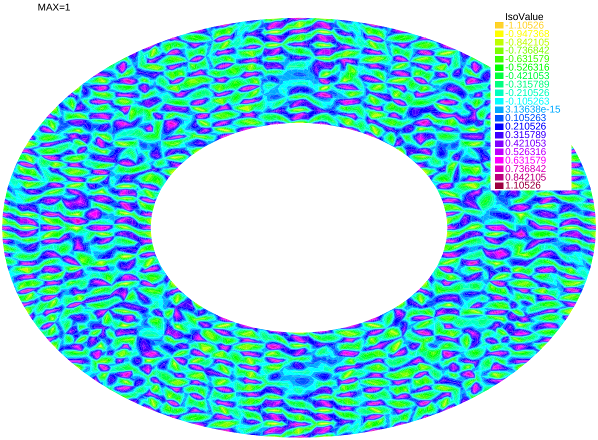


Figure – Allen Cahn : Initial datum

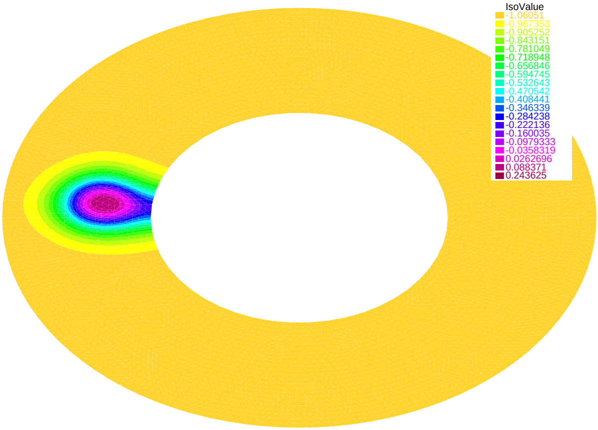


Figure – Solution at $t =$ after phase separation

A first application in image processing : the image segmentation

$$\frac{\partial \phi}{\partial t} - \Delta\phi + \frac{F'(\phi)}{\epsilon^2} + \lambda\left((1+\phi)(f_0-c_1)^2 - (1-\Phi)(f_0-c_2)^2\right), \quad x \in \Omega \tag{27}$$

$$\frac{\partial \Phi}{\partial n} = 0, \qquad \partial\Omega \tag{28}$$

If C is the segmenting curve, then the phase ϕ corresponds to the situations

$$\phi(x) = \begin{cases} > 0 & \text{if } x \text{ is inside } C, \\ = 0 & \text{if } x \in C, \\ < 0 & \text{if } x \text{ is outside } C, \end{cases}$$

Here $\epsilon > 0$, $F'(\phi) = \phi(\phi^2 - 1)$, λ is a nonnegative parameter, f_0 is the given image.The terms c_1 and c_2 are the averages of f_0 in the regions $(\phi \geq 0)$ and $(\phi < 0)$, say

$$c_1 = \frac{\int_\Omega f_0(x)(1+\phi(x))dx}{\int_\Omega (1+\phi(x))dx} \text{ and } c_2 = \frac{\int_\Omega f_0(x)(1-\phi(x))dx}{\int_\Omega (1-\phi(x))dx}$$

original image

segmented image

Figure – Photograph. Segmentation $\Delta t = 5.e - 7$, $\epsilon = 0.04$, $\tau = 1$, $\lambda = 10^{10}$

Figure – House. Segmentation $\Delta t = 5.e-7$, $\epsilon = 0.04$, $\tau = 1$, $\lambda = 10^{10}$

The CH equation describes the process of phase separation, by which the two components of a binary fluid spontaneously separate and form domains pure in each component. It writes as

$$\frac{\partial u}{\partial t} - \Delta(-\Delta u + \frac{1}{\epsilon^2} f(u)) = 0, \quad (29)$$

$$\frac{\partial u}{\partial n} = 0, \quad (30)$$

$$\frac{\partial}{\partial n}\left(\Delta u - \frac{1}{\epsilon^2} f(u)\right) = 0, \quad (31)$$

$$u(0,x) = u_0(x) \quad (32)$$

Derivation

Minimize the Energy $E(u)$ with a conservation of the mass. Write the equation as a gradient flow :

$$\frac{\partial u}{\partial t} = \mathcal{L}(\frac{\partial}{\partial u}E(u))$$

Where $\mathcal{L}$ is an operator such that $\int_\Omega \mathcal{L}(\frac{\partial}{\partial u}E(u))dx = 0$. A simple choice is $\mathcal{L} = D\nabla.[\phi(u)\nabla(.)]$, where D is the diffusion and ϕ is the mobility, eg, when $\phi = 1$, $\mathcal{L} = D\Delta$.

Properties

- Conservation of the mass : $\bar{u} = \int_\Omega u(x,t)dx = \int_\Omega u_0(x)dx$
- Decay of the energy in time

$$\frac{\partial E(u)}{\partial t} = -\int_\Omega |\nabla(-\Delta u + \frac{1}{\epsilon^2}f(u))|^2 dx \leq 0$$

CH as coupled system

A nice way to study and to simulate CH is to decouple the equation as follows :

$$\frac{\partial u}{\partial t} - \Delta\mu = 0, \tag{33}$$

$$\mu = -\Delta u + \frac{1}{\epsilon^2} f(u), \tag{34}$$

$$\frac{\partial u}{\partial n} = 0, \tag{35}$$

$$\frac{\partial \mu}{\partial n} = 0, \tag{36}$$

$$u(0,x) = u_0(x) \tag{37}$$

Application domains

material science : in complex fluids and soft matter (interfacial fluid flow, polymer science and in industrial applications).

Biological Science and Medicine : tumor growth

Image processing : inpainting (see also hereafter)

Time Simulations

- 2D phase separation `https://www.youtube.com/watch?v=52ZDH9mzDtc` Link
- 3D phase separation (in a cube) `https://www.youtube.com/watch?v=ROd6EMoLdjQ` Link (in a sphere) `https://www.youtube.com/watch?v=CrGatXppcrc` Link
- mussels pattern `https://www.youtube.com/watch?v=u-mEjfBaYks` Link

Pattern dynamics, corsening

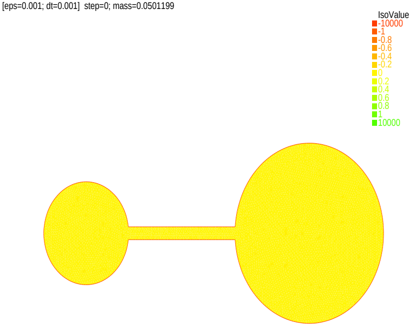


Figure – Cahn Hilliard coarsening : t=0 $\epsilon = 0.001$, $(\Delta t = 0.001)$

Pattern dynamics, corsening

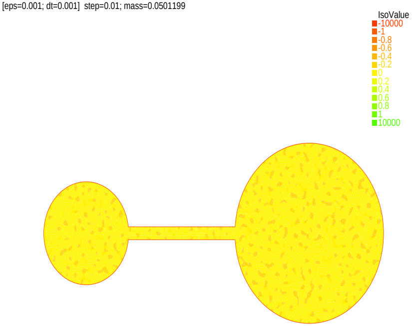


Figure – Cahn Hilliard coarsening : t=0.001 $\epsilon = 0.001$, $(\Delta t = 0.001)$

Pattern dynamics, corsening

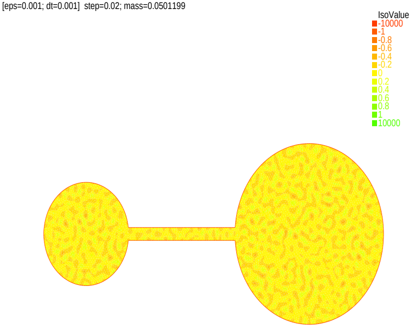


Figure – Cahn Hilliard coarsening : t=0.002 $\epsilon = 0.001$, $(\Delta t = 0.001)$

Pattern dynamics, corsening

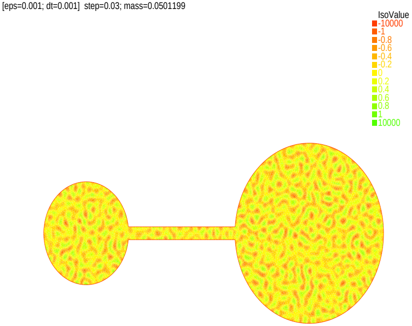


Figure – Cahn Hilliard coarsening : t=0.003 $\epsilon = 0.001$, $(\Delta t = 0.001)$

Pattern dynamics, corsening

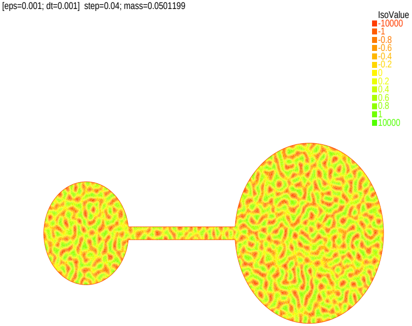


Figure – Cahn-Hilliard coarsening : t=0.003 $\epsilon = 0.001$, $(\Delta t = 0.001)$

Pattern dynamics, corsening

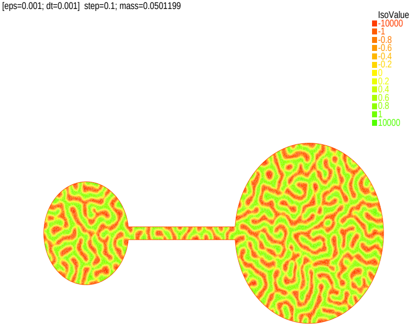


Figure – Cahn Hilliard coarsening : t=0.01 $\epsilon = 0.001$ ($\Delta t = 0.001$)

Pattern dynamics, corsening

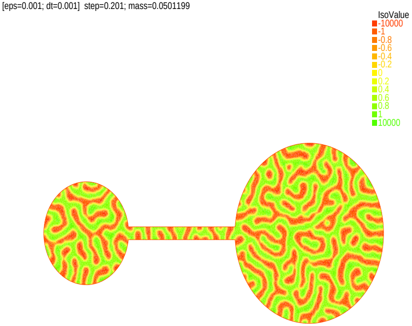


Figure – Cahn Hilliard coarsening : t=0.02 $\epsilon = 0.001$, $(\Delta t = 0.001)$

Pattern dynamics, corsening

Figure – Cahn Hilliard coarsening : t=0.03 $\epsilon = 0.001$, $(\Delta t = 0.001)$

Pattern dynamics, corsening

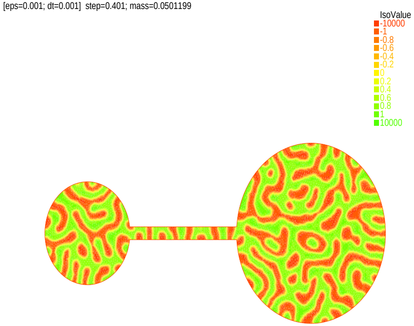


Figure – Cahn Hilliard coarsening : t=0.04 $\epsilon = 0.001$ $(\Delta t = 0.001)$

A first application in image processing : the inpainting

The inpainting is the process of reconstructing lost or deteriorated parts of images (photos as well as videos) : the idea is to take off the detoriated part (mask) and to recover in this region the rest of the original image. This can be done by using several techniques, one of them being based on Cahn-Hilliard equations. Link

A Cahn-Hilliard mathematical model for the inpainting

Let g be the original image and $D \subset \Omega$ the region of Ω in which the image is deterred. The idea is to add a penalty term that forces the image to remain unchanged in $\Omega \setminus D$ and to reconnect the fields of g inside D. Let $\lambda >> 1$

$$\underbrace{\frac{\partial u}{\partial t} - \Delta(-\epsilon\Delta u + \frac{1}{\epsilon} f(u))}_{\text{Cahn-Hilliard equation}} \quad \underbrace{+\lambda\chi_{\Omega\setminus D}(x)(u-g)}_{\text{Fidelity term}} = 0, \tag{38, 39}$$

$$\frac{\partial u}{\partial n} = 0 \quad \frac{\partial}{\partial n}\left(\Delta u - \frac{1}{\epsilon^2} f(u)\right) = 0, \tag{40}$$

$$u(0,x) = u_0(x) \tag{41}$$

Here $\chi_{\Omega\setminus D}(x) = \begin{cases} 1 & \text{if } x \in \Omega \setminus D, \\ 0 & \text{else} \end{cases}$

- The presence of the penalization term $\lambda\chi_{\Omega\setminus D}(x)(u-g)$ forces the solution to be close to g in $\Omega \setminus D$ when $\lambda >> 1$
- The Cahn-Hilliard flow has as effect to connect the fields inside D
- here ϵ will play the role of the "contrast". A post-processing is possible using a thresholding procedure.

Illustration with a simple example : the inpainting of a triangle.

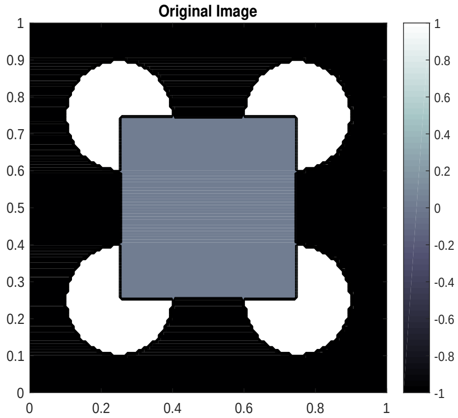
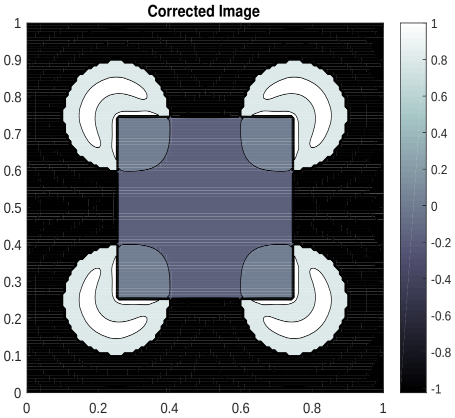


Figure – Inpainting with-C-H. $\Delta t = 0.001$, $\epsilon = 0.05$, $N = 128$ - Initial inpainted image, left ($t = 0$), right ($t = 0.05$), $\sigma = 2$, $\lambda = 100000$

Illustration with a simple example : the inpainting of a triangle.



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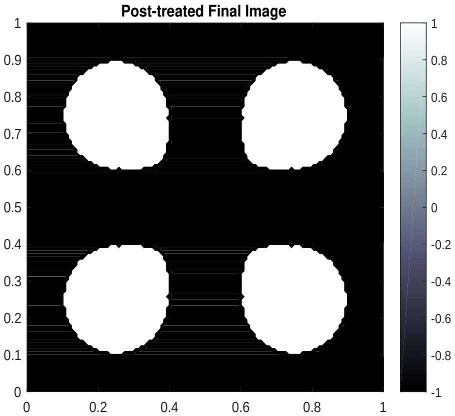

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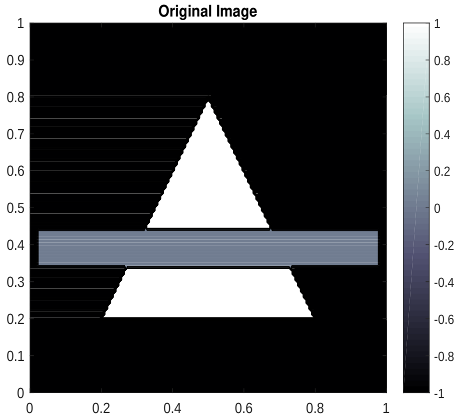

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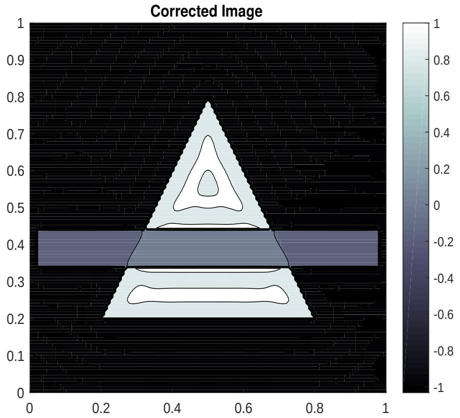


Figure – Inpainting with-C-H. $\Delta t = 0.001$, $\epsilon = 0.05$, $N = 128$ - Initial inpainted

Illustration with a simple example : the inpainting of a triangle.

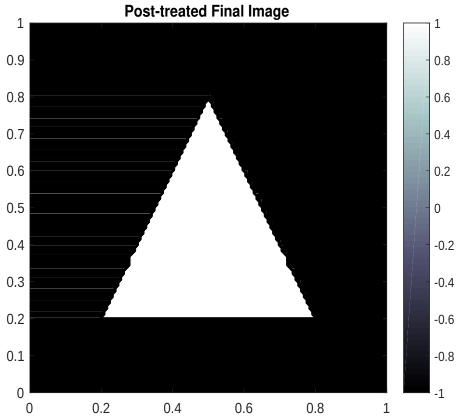


Figure – Inpainting with-C-H. $\Delta t = 0.001$, $\epsilon = 0.05$, $N = 128$ - Initial inpainted image, left ($t = 0$), right ($t = 0.05$), $\sigma = 2$, $\lambda = 100000$

Illustration with a 3D example : the inpainting of a box

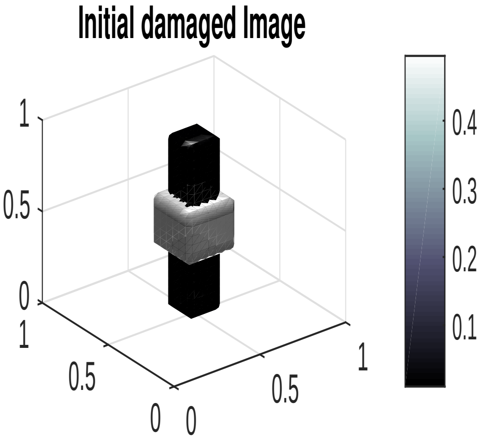
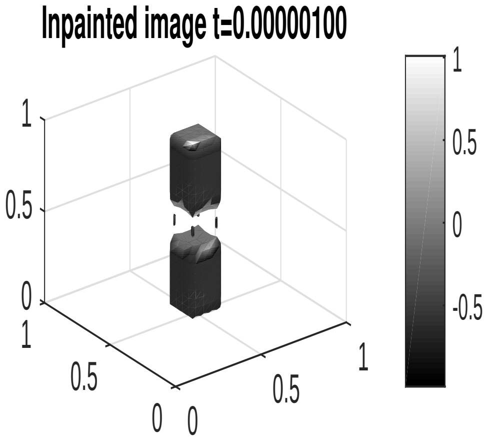


Figure – 3D Inpainting with C-H and classical scheme. $\Delta t = 1.e-6$, $\epsilon = 0.01$,
$N = 20$, $\lambda = 10000000$. Initial damaged image (top left) and at inpainted image at

Illustration with a 3D example : the inpainting of a box



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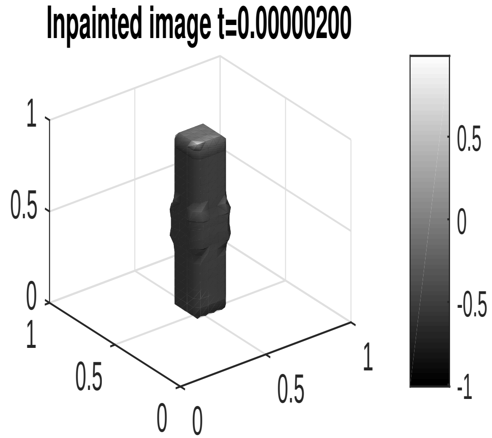

Figure – Inpainting with-C-H. $\Delta t = 0.001$, $\epsilon = 0.05$, $N = 128$ - Initial inpainted image, left $(t = 0)$, right $(t = 0.05)$, $\sigma = 2$, $\lambda = 100000$

Illustration with a 3D example : the inpainting of a box

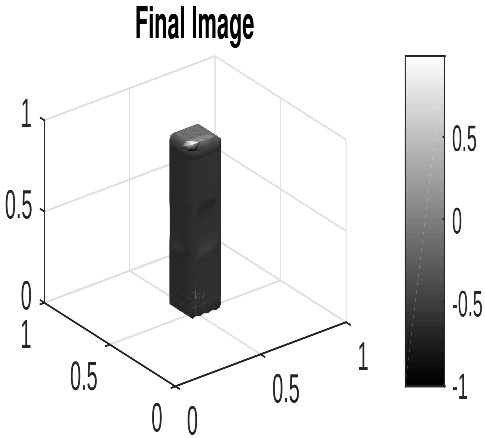

Figure – Inpainting with-C-H. $\Delta t = 0.001$, $\epsilon = 0.05$, $N = 128$ - Initial inpainted image, left ($t = 0$), right ($t = 0.05$), $\sigma = 2$, $\lambda = 100000$

Allen-Cahn

$$\frac{\partial u}{\partial t} - \Delta u + \frac{1}{\epsilon^2} f(u) = 0$$

Allen-Cahn

$$\frac{\partial u}{\partial t} - \Delta u + \frac{1}{\epsilon^2} f(u) = 0$$

Allen-Cahn

$$E(u) = \frac{1}{2}\int_\Omega \|\nabla u\|^2 dx + \frac{1}{\epsilon^2}\int_\Omega F(u)dx$$

Allen-Cahn

$$\frac{\partial u}{\partial t} - \Delta u + \frac{1}{\epsilon^2} f(u) = 0$$

Allen-Cahn

$$E(u) = \frac{1}{2}\int_\Omega \|\nabla u\|^2 dx + \frac{1}{\epsilon^2}\int_\Omega F(u)dx$$

u is **not** conserved

$$\frac{du}{dt} = -\nabla E(u)$$

Allen-Cahn

$$\frac{\partial u}{\partial t} - \Delta u + \frac{1}{\epsilon^2} f(u) = 0$$

Allen-Cahn

$$E(u) = \frac{1}{2} \int_\Omega \|\nabla u\|^2 dx + \frac{1}{\epsilon^2} \int_\Omega F(u) dx$$

u is **not** conserved

$$\frac{du}{dt} = -\nabla E(u)$$

Double Well Potential

$$F(u) = \frac{1}{\epsilon^2} \frac{1}{4} (1 - u^2)^2$$

$$f(u) = F'(u)$$

Allen-Cahn

$$\frac{\partial u}{\partial t} - \Delta u + \frac{1}{\epsilon^2} f(u) = 0$$

Allen-Cahn

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$$F(u) = \frac{1}{\epsilon^2}\frac{1}{4}(1-u^2)^2$$

$$f(u) = F'(u)$$

Cahn-Hilliard

$$\frac{\partial E(u)}{\partial t} = -\int_\Omega |\nabla(-\Delta u + \frac{1}{\epsilon^2} f(u))|^2 dx \leq 0$$

Allen-Cahn

$$\frac{\partial u}{\partial t} - \Delta u + \frac{1}{\epsilon^2} f(u) = 0$$

Allen-Cahn

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$$\frac{\partial E(u)}{\partial t} = -\int_\Omega |\nabla(-\Delta u + \frac{1}{\epsilon^2} f(u))|^2 dx \leq 0$$

u is locally conserved, according to Fick's second law $\frac{du}{dt} = -\nabla J$

Allen-Cahn

$$\frac{\partial u}{\partial t} - \Delta u + \frac{1}{\epsilon^2} f(u) = 0$$

Allen-Cahn

$$E(u) = \frac{1}{2}\int_\Omega \|\nabla u\|^2 dx + \frac{1}{\epsilon^2}\int_\Omega F(u)dx$$

u is **not** conserved

$$\frac{du}{dt} = -\nabla E(u)$$

Double Well Potential

$$F(u) = \frac{1}{\epsilon^2}\frac{1}{4}(1-u^2)^2$$

$$f(u) = F'(u)$$

Cahn-Hilliard

$$\frac{\partial E(u)}{\partial t} = -\int_\Omega |\nabla(-\Delta u + \frac{1}{\epsilon^2} f(u))|^2 dx \leq 0$$

u is locally conserved, according to Fick's second law $\frac{du}{dt} = -\nabla J$
Define the potential
$\mu = -\Delta u + F'(u)$

Allen-Cahn

$$\frac{\partial u}{\partial t} - \Delta u + \frac{1}{\epsilon^2} f(u) = 0$$

Allen-Cahn

$$E(u) = \frac{1}{2}\int_\Omega \|\nabla u\|^2 dx + \frac{1}{\epsilon^2}\int_\Omega F(u)dx$$

u is **not** conserved

$$\frac{du}{dt} = -\nabla E(u)$$

Double Well Potential

$$F(u) = \frac{1}{\epsilon^2}\frac{1}{4}(1-u^2)^2$$

$$f(u) = F'(u)$$

Cahn-Hilliard

$$\frac{\partial E(u)}{\partial t} = -\int_\Omega |\nabla(-\Delta u + \frac{1}{\epsilon^2} f(u))|^2 dx \leq 0$$

u is locally conserved, according to Fick's second law $\frac{du}{dt} = -\nabla J$
Define the potential
$\mu = -\Delta u + F'(u)$
Constitutive equation

$$J = -M(u)\nabla \mu$$

Allen-Cahn

$$\frac{\partial u}{\partial t} - \Delta u + \frac{1}{\epsilon^2} f(u) = 0$$

Allen-Cahn

$$E(u) = \frac{1}{2}\int_\Omega \|\nabla u\|^2 dx + \frac{1}{\epsilon^2}\int_\Omega F(u)dx$$

u is **not** conserved

$$\frac{du}{dt} = -\nabla E(u)$$

Double Well Potential

$$F(u) = \frac{1}{\epsilon^2}\frac{1}{4}(1-u^2)^2$$

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Cahn-Hilliard

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Define the potential
$\mu = -\Delta u + F'(u)$
Constitutive equation

$$J = -M(u)\nabla\mu$$

$$f(u) = F'(u)$$

In higher dimension we consider the differential system :

$$\frac{du}{dt} = F(u) \tag{42}$$

$$u(0) = u_0 \in \mathbb{R}^n \text{ (Initial Condition)} \tag{43}$$

Her $F : \mathbb{R}^n \to \mathbb{R}^n$ is a regular function. We will assume that properties of existence and uniqueness of the solution as well as those of regularity are satisfied. We just here extend the analysis of stability of the steady states of the differential system.

The linear case

Consider the case $F(u) = Au - b$. When A is invertible, the only steady state is the solution of the linear system $Au^* = b$. We can write any vector $v \in \mathbb{R}^n$ as $v = u^* + w$. Then

$$\frac{dw}{dt} = Aw$$

We have formally $w(t) = exp(tA)w(0)$. Then $w(t)$ remains bounded iff the eigenvalues of A are of negative real part. If their real part is strictly negative, $w(t)$ converges to 0 as $t \to +\infty$: this is the asymptotic stability.

Linear stability in dimension n

Let u^* be a steady state, i.e. $F(u^*) = 0$. As in the scalar case, we write the system near u^* : set $u(t) = u^* + \omega(t)$

$$F(u(t)) = F(u^* + \omega(t))$$
$$= F(u^*) + JF(u^*)\omega(t) + \text{lower terms}$$
$$\underbrace{\text{by Taylor's formula}}$$
$$\simeq JF'(u^*)\omega(t)$$

Here $JF(u^*)$ is the jacobian matrix of F at u^*, say $(JF(u^*))_{i,j} = \dfrac{\partial F_i(u^*)}{\partial x_j}$.

- If none of the eigenvalues of $JF(u^*)$ is pure imaginary, then the local stability is similar to the one in the linear case
- We cannot conclude with the spectral argument if there is at least a couple of pure imaginary eigenvalues.