(joint works with P. Garnier, Y. Mammeri,G. Sadaka (Amiens))

(very) weakly damped KdV equations

J-P. CHEHAB[1]

[1]LAMFA, UMR 7352, Université de Picardie Jules Verne, Amiens, France
(jean-paul.chehab@u-picardie.fr)

Séminaire, Poitiers, March, 17, 2016

Outline

Introduction

Korteweg-de Vries equation (KdV)

Model of one-way propagation of small amplitude and long-wavelength wave.

$$u_t + u_x + u_{xxx} + u^p u_x = 0.$$

($p = 1$: KdV; $p \geq 2$: GKdV))

Benjamin-Bona-Mahony equation (BBM)

Regularized version of KdV equation

$$u_t + u_x - u_{txx} + uu_x = 0.$$

Problem

Modelling natural phenomena needs to take into account damping effects

Damped KdV equations

$$u_t + \alpha_1 u_x + \alpha_2 u_{xxx} + \alpha_3 u u_x + \alpha_4 L[u] = 0 \text{ (or } f(x))$$

$$u(x,0) = u_0(x), x \in \Omega, \alpha \in [0,2], \text{ and } t > 0$$

Operator $L[u]$ is a damping operator when $\int_\Omega L[u]udx \geq 0$ (it makes decrease the L^2 norm in time since $\frac{1}{2}\frac{d\|u\|^2}{dt} + \int_\Omega L[u]udx = 0$).
Damping KdV models: $L[u] = |D|^\alpha u$, where $|D| = \sqrt{-\Delta}$, $\alpha \in [0,2]$.

Modeling

- Ott-Sudan (1970): $L = \gamma \cdot Id$, $\alpha = 0$ and $\Omega = \mathbb{R}$ (Landau damping for ion acoustic wave)
- Dutykh (2007): $L = -\gamma \cdot \partial_x^2$, $\alpha = 2$ (dissipative Tsunami)

We look to the behavior of the solutions for t large. When $L = |D|^\alpha$, $\alpha > 0$ parabolic-like behavior.

Weak damping

When $L = \gamma Id$ and $\Omega = \mathbb{T}$: WEAK DAMPING

Asymptotic behavior

- Goubet (2000, 2002) and Ghidaglia (1988, 1994): $L = \gamma \cdot Id$, $\alpha = 0$.
  Regularity of the global attractor, Asymptotic smoothing.
  Finite dimensional attractor which is in a more regular space than the initial data: this is the asymptotic regularization property
- Rosa-Cabral (2004, numerical study) : $L = \gamma \cdot Id$, $\alpha = 0$ and $\Omega = \mathbb{T}$
  Periodic solutions in times
- Vento (2008): mathematical analysis with $\alpha \in [0, 2]$ and $\Omega = \mathbb{R}$
  (Cauchy Pb and asymptotic behavior)

Building an "economical" damping

- The derivation of KdV equation is valid to capture low frequencial phenomena

Building an "economical" damping

- The derivation of KdV equation is valid to capture low frequencial phenomena (**think in frequencies !!**)

Building an "economical" damping

- The derivation of KdV equation is valid to capture low frequencial phenomena (**think in frequencies !!**)
- Idea : build the damping as a high-pass filter

Building an "economical" damping

- The derivation of KdV equation is valid to capture low frequencial phenomena (**think in frequencies !!**)
- Idea : build the damping as a high-pass filter
- The previous dampings are no high-pass filters when $\alpha \geq 0$.

Building an "economical" damping

- The derivation of KdV equation is valid to capture low frequencial phenomena (**think in frequencies !!**)
- Idea : build the damping as a high-pass filter
- The previous dampings are no high-pass filters when $\alpha \geq 0$.
- We need even weaker damping models and define a "very weak" damping (say $\alpha < 0$?)

Problem

Let L be a weak damping operator i.e.

$$0 \leq \int L[u]udx \leq c\|u\|_{L^2}^2$$

and consider the equation

$$u_t + u_{xxx} + L[u] + uu_x = f, \;\; x \in \mathbb{T}, t > 0 \tag{1}$$

$$u(x, 0) = u_0(x) \tag{2}$$

Do still have we the phenomena proven/ pointed out by Ghidaglia, Goubet Rosa, Cabral for even more weak dampings ?

$$\exists c > 0 \; s.t. \; (L[u], u)_{L^2} \leq c|u|_{L^2}^2 \quad \text{and} \quad \nexists d > 0 (L[u], u)_{L^2} \geq d|u|_{L^2}^2$$

The model (damping in frequencies)

Let $u \in L^2(\mathbb{T})$, and consider its Fourier expansion, we have:

$$u(x,t) = \sum_{k \in \mathbb{Z}} \hat{u}_k(t) e^{i\frac{2\pi kx}{L}}.$$

The damping

We define the nonlocal damping in space (or dissipative) term as :

$$L_\gamma u = \sum_{k \in \mathbb{Z}} \gamma_k \hat{u}_k(t) e^{i\frac{2\pi kx}{L}} \quad \gamma_k = \gamma_{-k}, \forall k \in \mathbb{Z}, \gamma_k \geq 0$$

associated to the energy space $H^\gamma = \{u \in L^2 / \sum_{k \in \mathbb{Z}} \gamma_k |\hat{u}_k|^2 < +\infty\}$ with $|u|_\gamma^2 = \sum_{k \in \mathbb{Z}} \gamma_k |\hat{u}_k|^2 = \int L_\gamma(u) u dx$

The model (damping in frequencies)

Let $u \in L^2(\mathbb{T})$, and consider its Fourier expansion, we have:

$$u(x,t) = \sum_{k \in \mathbb{Z}} \hat{u}_k(t) e^{i\frac{2\pi kx}{L}}.$$

The damping

We define the nonlocal damping in space (or dissipative) term as :

$$L_\gamma u = \sum_{k \in \mathbb{Z}} \gamma_k \hat{u}_k(t) e^{i\frac{2\pi kx}{L}} \quad \gamma_k = \gamma_{-k}, \forall k \in \mathbb{Z}, \gamma_k \geq 0$$

associated to the energy space $H^\gamma = \{u \in L^2 / \sum_{k \in \mathbb{Z}} \gamma_k |\hat{u}_k|^2 < +\infty\}$ with $|u|_\gamma^2 = \sum_{k \in \mathbb{Z}} \gamma_k |\hat{u}_k|^2 = \int L_\gamma(u) u dx$

Remark: if $\gamma_k = \left|\frac{2\pi k}{L}\right|^\alpha$ then $L_\gamma[u] = |D|^\alpha u$

Special attention to the case

$$\lim_{k \to +\infty} \gamma_k = 0$$

The damping is then weaker than when $\gamma_k = \gamma \Longrightarrow$ high-pass-like filter.

Approach

Analysis

- Cauchy problem (Garnier (KdV, ArXiV 2015), C-Garnier Mammeri (BBM, DCDS-B, 2015))

Numerics and focus on :

- Test different damping in band of frequency
- Rate of damping
- Long time behavior, time regularization (Sobolev).
- computation of Steady state and periodic solution
- Hierarchy of dampings, inverse problem
- Prevent Blow up for GKdV

Damping

When the KdV equation is nor forced nor damped, it possesses an infinite number of invariants, the first ones being

- The mass : $I_0(u) = \bar{u}(t) = \int_0^L u(x,t)dx = \int_0^L u_0(x)dx$
- The L^2 norm : $I_1(u) = \int_0^L (u(x,t))^2 dx = \int_0^L (u_0(x)^2 dx$
- The Energy : $I_2(u) = \int_0^L \left(\frac{\partial u(x,t)}{\partial x}\right)^2 dx - \frac{1}{6}\int_0^L (u(x,t))^3 dx$

We'll look to the time behavior of $|u|_2$ and $|u|_\gamma$ as $t \to +\infty$.

Which damping ?

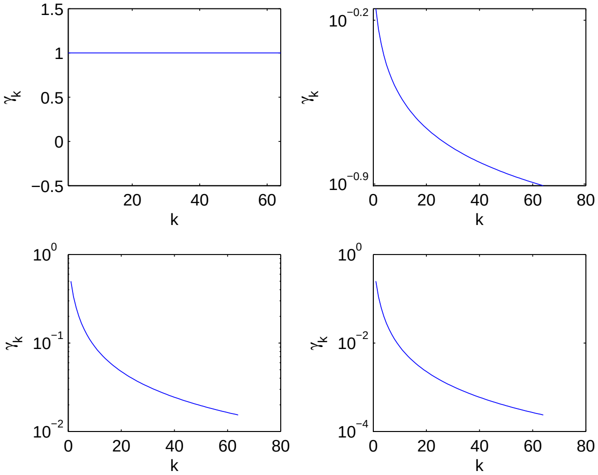


Figure : Various frequency damping profiles

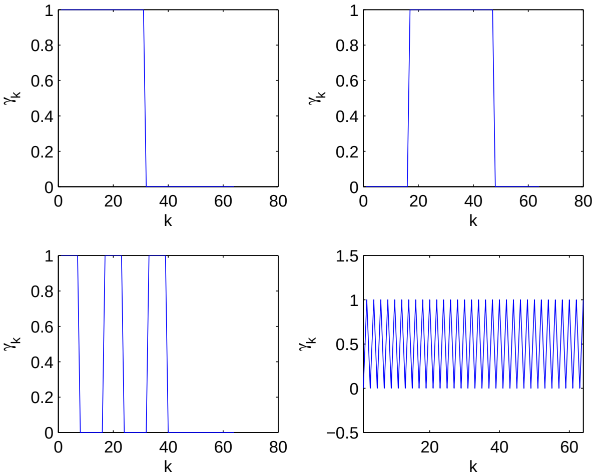


Figure : Various frequency damping profiles

It is necessary to have $\gamma_k > 0$: indeed

lemma

Let $u, v \in L^\infty(\mathbb{T})$. Assume that $\widehat{u}_{2k+1} = \widehat{v}_{2k+1} = 0$. Then

$$\widehat{uv}_{2k+1} = 0, \ k \in \mathbb{Z}.$$

It follows by induction that if $\widehat{u}_{2k+1} = 0$, then $\widehat{u}^p_{2k+1} = 0$.
So if $\widehat{u}_{0\,2k+1} = 0$ and $\gamma_{2k} = 0$ (comb-like filter) then the solution is not damped

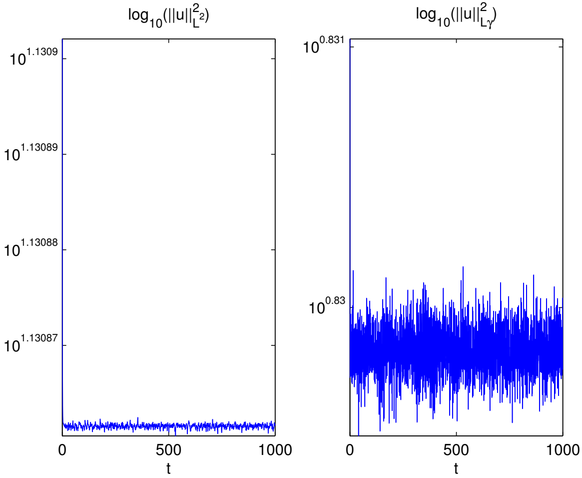


Figure : $f = 0$, $\hat{(u_0)}_{2k+1} = 0$ and $\gamma_{2k} = 0, \gamma_{2k+1} = 1$

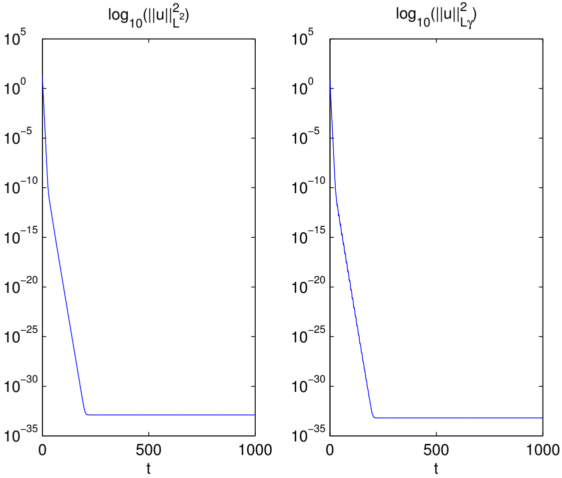


Figure : $f = 0,\ \hat{(u_0)}_{2k+1} = 0$ and $\gamma_{2k+1} = 0, \gamma_{2k} = 1$

Damping Properties

Proposition : (The linear Homogeneous equation)

$$u_t + L_\gamma[u] = 0$$

Assume that $\gamma_k > 0, \forall k \in \mathbb{Z}$ and that $u_0 \in H^{\beta/\gamma}$. Then : $|u|^2_\beta \leqslant \frac{e^{-1}}{2t}|u_0|^2_{\frac{\beta}{\gamma}}$

More generally, assume that $\gamma_k \in [0,1], \forall k \in \mathbb{Z}$ and that $u_0 \in H_{\frac{1}{\gamma^s}}$.

Then, for every $s > 0$, $|u|^2_{L^2} \leqslant \min\left(e^{-s}\left(\frac{s}{2t}\right)^s |u_0|^2_{\frac{1}{\gamma^s}}, |u_0|^2_{L^2}\right)$

Proposition

Assume that there exist α, β and C, three strictly positive real numbers s.t.

i. $|(\widehat{u_0})_k|^2 \leq C\gamma_k^{2\delta}$, with $\delta = \alpha + \beta$.

ii. $\displaystyle\sum_{k\in\mathbb{Z}} \gamma_k^{2\beta} < +\infty$

Then $|u|^2_{L^2} \leq Ce^{-1}\left(\frac{\alpha}{t}\right)^{2\alpha} \displaystyle\sum_{k\in\mathbb{Z}} \gamma_k^{2\beta} = \mathcal{O}\left(\frac{1}{t^{2\alpha}}\right)$.

Proposition : (The nonlinear Homogeneous equation)

Assume that $\bar{u}(0) = 0$ and $\gamma_k > 0$

i. $\lim_{t \to +\infty} |u|_{L^2} = 0.$

ii. In addition, if $\bar{u}(0) = 0$ and if $\exists c > 0$ such that $\gamma_k \geq c > 0, \forall k \in Z$ then $|u|_{L^2} \leq \kappa e^{-ct} |u|_L^2.$

Remark When $\gamma_k > 0$, $\lim_{k \to +\infty} \gamma_k = 0$, orbit converge to 0 in L^2, but it can be at an arbitrary slow rate, it depends on how γ_k converge to 0 as k goes to infinity.

Damping Properties

The Energy Ratio Function

$$\text{Let } G(u,t) \mapsto G(u,t) = \frac{|u|_\gamma}{|u|_{L^2}} = \sqrt{\frac{\sum_{k\in\mathbb{Z}} \gamma_k |\hat{u}_k|^2}{\sum_{k\in\mathbb{Z}} |\hat{u}_k|^2}}$$

so $\frac{1}{2}\frac{d\|u\|^2}{dt} + G(u,t)^2\|u\|^2 = 0$

Damping Properties

The Energy Ratio Function

$$\text{Let } G(u,t) \mapsto G(u,t) = \frac{|u|_\gamma}{|u|_{L^2}} = \sqrt{\frac{\sum_{k\in\mathbb{Z}} \gamma_k |\hat{u}_k|^2}{\sum_{k\in\mathbb{Z}} |\hat{u}_k|^2}}$$

so $\frac{1}{2}\frac{d\|u\|^2}{dt} + G(u,t)^2\|u\|^2 = 0$

Proposition : (The nonlinear forced equation)

Assume that f belongs to $H_{\frac{1}{\gamma}} \bigcap L^2$ Then

$$|u(t)|_{L^2}^2 \leq e^{-\int_0^t G^2(s)ds} |u_0|_{L^2}^2 + \int_0^t e^{-\int_s^t G^2(\tau)d\tau} |f|_{\frac{1}{\gamma}}^2 ds.$$

If $f = 0$ then $|u(t)|_{L^2}^2 = e^{-2\int_0^t G^2(s)ds} |u_0|_{L^2}^2$. It follows that $\lim_{t\to+\infty} |u|_{L^2} = 0 \iff G \notin L_t^2(0,+\infty)$.

Spatial and time discretization

Pseudospectral (Fourier)

Let $\Omega = [0, L]$, $N \in \mathbb{N}^*$ we consider the expansion of L-periodic function u as

$$u_N(x) = \sum_{k=-N/2+1}^{N/2} \hat{u}_k e^{i\frac{2k\pi x}{L}}, \text{ with } \hat{u}_k = \frac{2}{L}\int_0^L u(x) e^{-i\frac{2k\pi x}{L}} dx$$

Then

$$(u_N)_x = \sum_{k=-N/2+1}^{N/2} \left(i\frac{2k\pi}{L}\right) \hat{u}_k e^{i\frac{2k\pi x}{L}}, \ (u_N)_{xxx} = \sum_{k=-N/2+1}^{N/2} \left(i\frac{2k\pi}{L}\right)^3 \hat{u}_k e^{i\frac{2k\pi x}{L}}$$

$$L_\gamma[u] = \sum_{k=-N/2+1}^{N/2} \gamma_k \hat{u}_k e^{\frac{2i\pi kx}{L}}$$

Time marching schemes

i. Backward Euler's
ii. Crank Nicolson
iii. Sanz-Serna
iv. Lie Splitting and Strang Splitting
v. RK34 (inverse problem)

Stability properties and accuracy (C.-Sadaka, CPAA 2013)

Time marching schemes

i. Backward Euler's

$$\frac{\widehat{u}_k^{n+1}-\widehat{u}_k^{n}}{\Delta t}+\left(i\frac{2\pi k}{L}\right)^3\cdot\widehat{u}_k^{n+1}+\gamma_k\cdot\widehat{u}_k^{n+1}+\frac{1}{2}\cdot\left(i\frac{2\pi k}{L}\right)\cdot\widehat{(u^{n+1})^2}_k=\widehat{f}_k$$

ii. Crank Nicolson

$$\frac{u^{(n+1)}-u^{(n)}}{\Delta t}+L_\gamma\frac{u^{(n+1)}+u^{(n)}}{2}+\mathcal{L}^3\frac{u^{(n+1)}+u^{(n)}}{2}+\mathcal{L}\left(\frac{(u^{(n+1)})^2+(u^{(n)})^2}{4}\right)=f$$

iii. Sanz Serna

$$\frac{u^{(n+1)}-u^{(n)}}{\Delta t}+L_\gamma\frac{u^{(n+1)}+u^{(n)}}{2}+\mathcal{L}^3\frac{u^{(n+1)}+u^{(n)}}{2}+\frac{1}{2}\mathcal{L}\left(\frac{u^{(n+1)}+u^{(n)}}{2}\right)^2=f$$

Time marching schemes (cont.)

iv. Strang Splitting 1

$$\frac{u^{(n+1/3)} - u^{(n)}}{\Delta t/2} + L_\gamma u^{(n+1/3)} = 0,$$

$$\frac{u^{(n+2/3)} - u^{(n+1/3)}}{\Delta t} + \mathcal{L}^3 u^{(n+2/3)} + \frac{1}{2}\mathcal{L}(u^{(n+2/3)})^2 = f,$$

$$\frac{u^{(n+1)} - u^{(n+2/3)}}{\Delta t/2} + L_\gamma u^{(n+1)} = 0,$$

v. Strang Splitting 2

$$u^{(n+1/3)} = e^{-\frac{\Delta t}{2}\mathcal{L}_\gamma} u^{(n)}$$

$$\frac{u^{(n+2/3)} - u^{(n+1/3)}}{\Delta t} + \frac{\mathcal{L}^3\left(u^{(n+1/3)} + u^{(n+2/3)}\right)}{2} + \frac{\mathcal{L}(u^{(n+1/3)} + u^{(n+2/3)})^2}{8} = f,$$

$$u^{(n+1)} = e^{-\frac{\Delta t}{2}\mathcal{L}_\gamma} u^{(n+2/3)}$$

Theorem

- *(Backward Euler)*

$$|u^{(n+1)}|_2^2 + |u^{(n+1)} - u^{(n)}|^2 + \Delta t(\sum_{k\in\mathbb{Z}} \gamma_k |\hat{u}_k^{(n+1)}|^2 \leq |u^{(n)}|_2^2 + \Delta t \sum_{k\in\mathbb{Z}} \frac{1}{\gamma_k}|\hat{f}_k|^2$$

In addition if $f = 0$, then $\lim_{n\to+\infty} |u^{(n)}|_{L^2} = 0$. We have

$$|u^{(n)}|_{L^2}^2 \leq \left(\prod_{j=1}^{n} \frac{1}{1+2\Delta t(G^{(j)})^2}\right)|u_0|_{L^2}^2.$$

- *(Sanz-Serna)* $|u^{(n+1)}|_{L^2} + \frac{\Delta t}{4}|u^{(n+1)} + u^{(n)}|_\gamma^2 \leq |u^{(n)}|_{L^2}^2 + \Delta t|f|_{\frac{1}{\gamma}}^2$.
- *(Splitting 2) Unconditional L^2 stability. Moreover, if $f = 0$,* $\lim_{n\to+\infty} |u^{(n)}|_{L^2} = 0$. *If $u^{(0)} \in L^2(\mathbb{T}) \cap H_{\frac{1}{\gamma}}$ then*

$$|u^{(n)}| \leq \frac{e^{-1}}{N\Delta t}|u^{(0)}|_{\frac{1}{\gamma}}.$$

Initial data

$u_0(x) = S1 = 3\text{csech}(\sqrt{\frac{c}{2}}(x - pL))^2$, with $c = 1, p = 0.4$ is the soliton
$u_0(x) = S2 = \chi_{0.4L<x<0.6L}$ corresponds to the crennel
$u_0(x) = S3 = \sin(\frac{2\pi x}{L})$is the sine data
$u_0(x) = S4 = 50\chi_{x>\pi}\sin(4x)$(this is the initial data to compute time-periodic solutions for $L = 2\pi$ (inspired by that used by Cabral-Rosa))

Unless specified, for all the numerical simulations, we work with $L = 100$, $N = 2^9$ and $\Delta t = 0.0005$.

Damping profiles

$$\gamma_k = 1,\ \gamma_k = \chi_{k_1\le|k|\le k_2},\ \gamma_k = \frac{1}{(1+|k|)^\alpha}, \alpha = 1/4, 1, 2\cdots$$

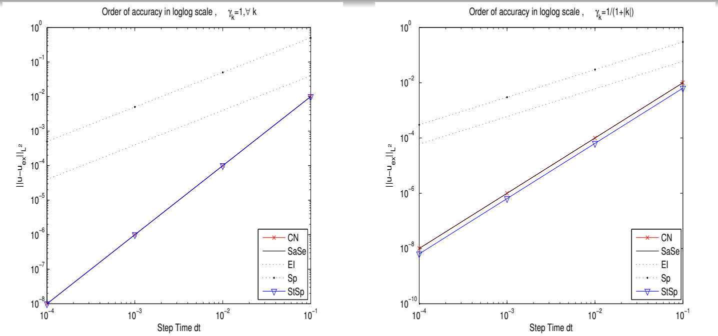


Figure : Comparison of the rate of the accuracy of the schemes for $\gamma_k = 1, \forall k$ (left) and for $\gamma_k = \dfrac{1}{1+|k|}$ (right).

Damping for homogeneous equation

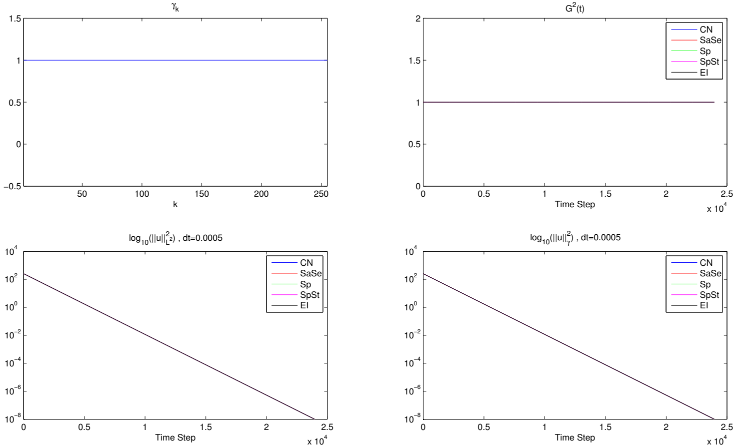


Figure : $u_0(x) = S3,\ \gamma_k = 1, \forall k,\ f(x) = 0$

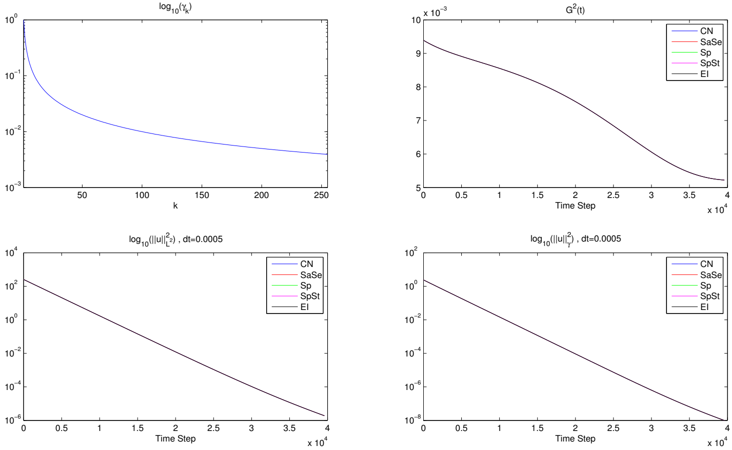


Figure : $u_0(x) = S3, \gamma_k = \frac{1}{(1+|k|)}, \ f(x) = 0$

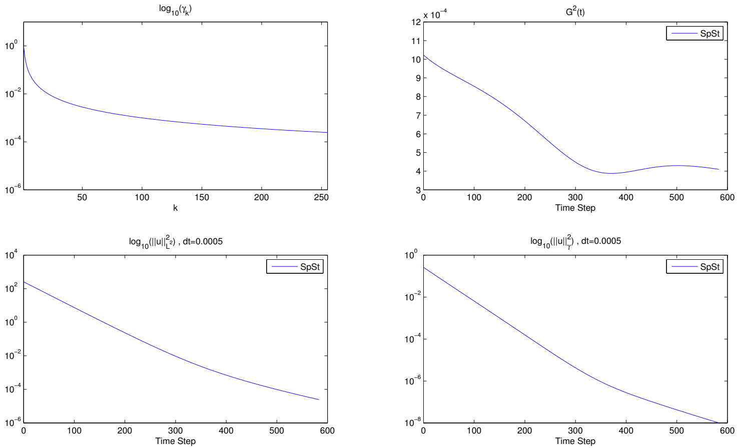


Figure : $u_0(x) = S3$, $\gamma_k = \frac{1}{(1+|k|)^2}$, $f(x) = 0$

NonHomogeneous equation : Steady states

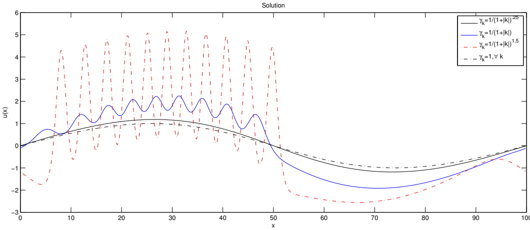


Figure : Computed steady states for $\gamma_k = \dfrac{1}{(1+|k|)^{1.5}}$, $f = \sin(2\pi x/L)$, various $u_0(x)$: evolution of $\| \frac{du}{dt} \|_2$

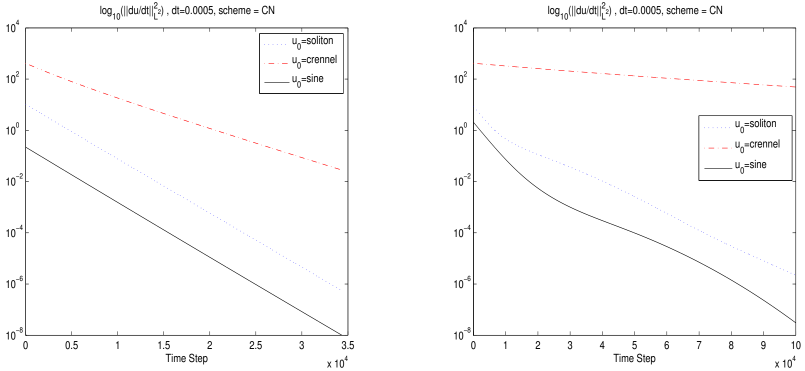


Figure : Computed steady states for $\gamma_k = 1, \forall k$ (left) and for $\gamma_k = \frac{1}{(1+|k|)^{.5}}$ (right), $f(x) = \sin(2\pi x/L)$, various $u_0(x)$: evolution of $\| \frac{du}{dt} \|^2_{L^2}$.

Periodic solutions

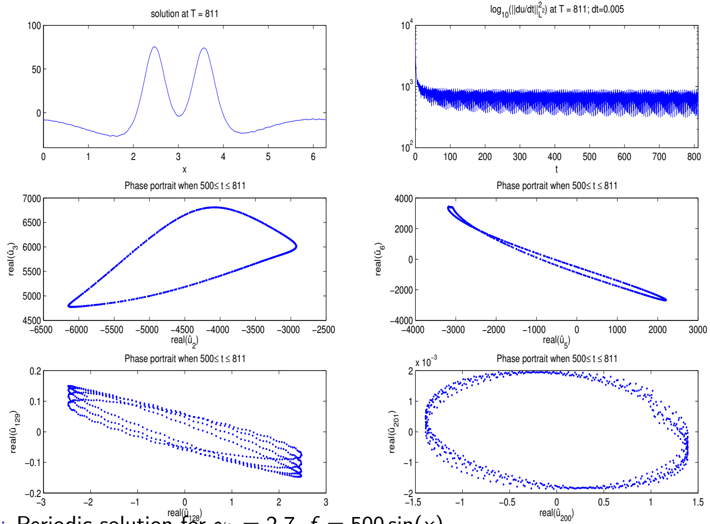


Figure : Periodic solution for $\gamma_k = 2.7,\ f = 500\sin(x)$,
$L = 2\pi,\ u_0(x) = 50\chi \ldots sin(4x)$

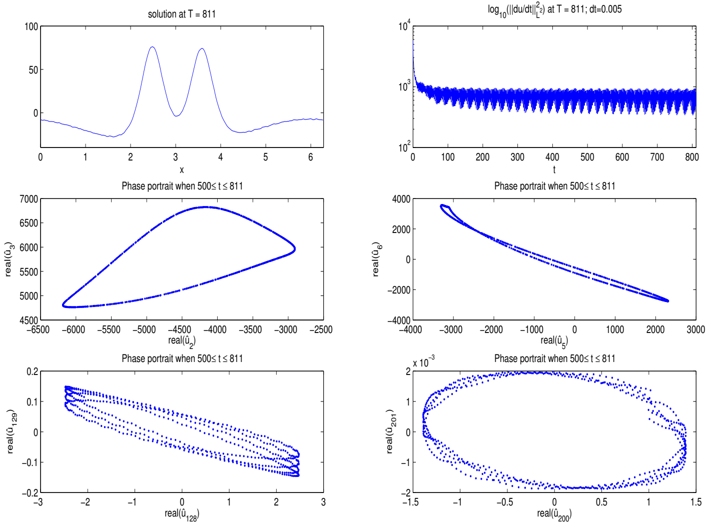


Figure : Periodic solution for $\gamma_k = 9.9/(1+|k|)^{1.2}$, $f = 500\sin(x)$, $L = 2\pi, u_0(x) = 50\chi_{x>\pi} sin(4x)$

NonHomogeneous equation : Sobolev regularization

(According to Mallat) $u \in L^2$ belongs to H^s, $s > 0$ iff

$$\sum_{N=1}^{+\infty} N^{2s-1} ||u - u_N||^2_{L^2} < +\infty, \;\; u_N = \sum_{|k|>N} \widehat{u}(k) e_k(x). \qquad (3)$$

$u - -- > U_N, \quad U_N - -- > U_{N/2} \quad : \sum_{k=1}^{N/2} k^{2s-1} \left(\sum_{\ell=k}^{N} |\hat{u}_\ell|^2 \right) < +\infty$

Numerical Sobolev exponent: computed by considering the tail of the spectral energy of the solution.

$$\sum_{\ell=k}^{N} |\hat{u}_\ell|^2 \simeq \frac{C}{k^{2s}}, \text{ for } k \gg 1,$$

$$v_k = \ln \sum_{\ell=k}^{N} |\hat{u}_\ell|^2 \simeq \ln(C) - 2s \ln k, \text{ for } k \gg 1.$$

compute s by a linear regression $\min_{s,\kappa} \sum_{k=N-m}^{N} (v_k - (\kappa - 2s\ln(k))^2$.

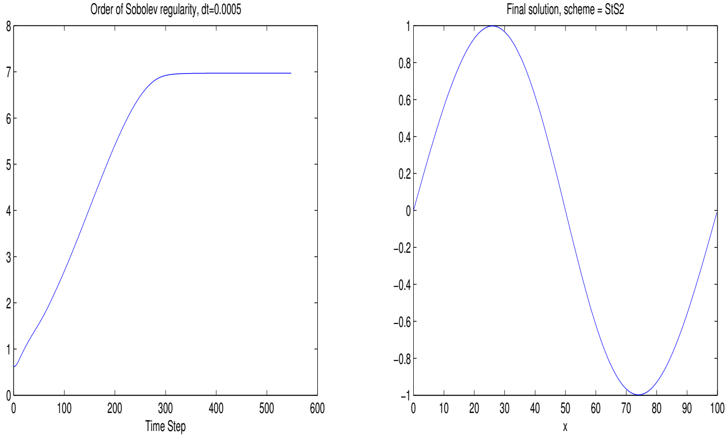
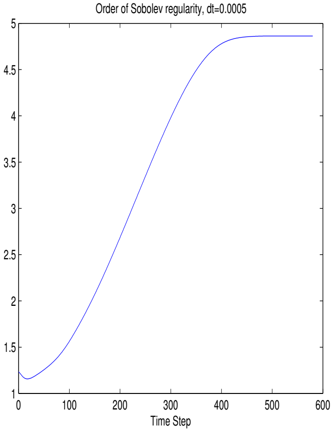


Figure : $u_0(x) = S2$,$\gamma_k = 1, \forall k \ f(x) = \sin(2\pi x/L)$



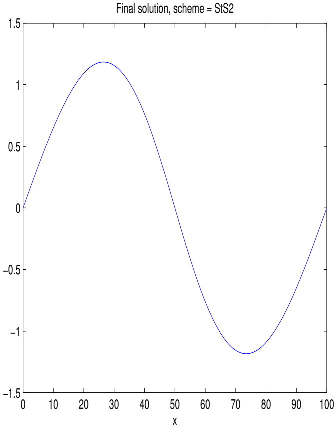


Figure : $u_0(x) = S1$,$\gamma_k = \frac{1}{(1+|k|)^{1/4}}$, $f(x) = \sin(2\pi x/L)$

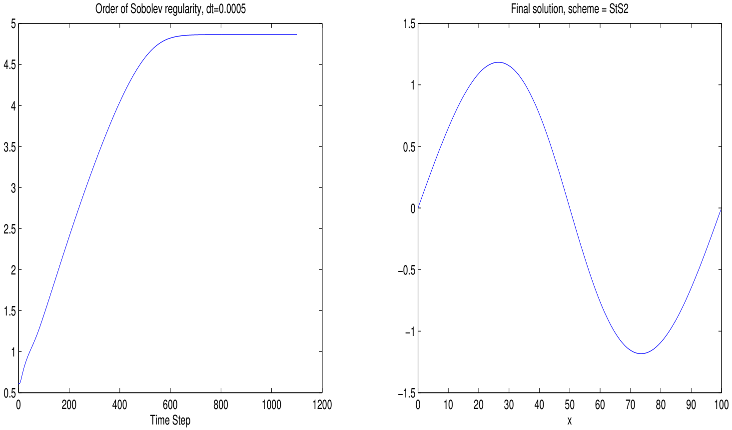


Figure : $u_0(x) = S2$,$\gamma_k = \frac{1}{(1+|k|)^{1/4}}$, $f(x) = \sin(2\pi x/L)$

Comparison of various dampings

- For $\mathcal{L}_\gamma u = \sum_{k\in\mathbb{Z}} \gamma_k \hat{u}_k(t) e^{\frac{2i\pi x}{L}}$, a Fourier-like approximation is used in space
- For $\mathcal{L}u = \nu \int_0^t \frac{u_t}{\sqrt{t-s}}$ (see Chen-Dumont-Dupaigne-Goubet) we use the approx $\nu\sqrt{\frac{3}{2\Delta t}} \sum_{j=0}^{n} g_{n+1-j} u^j$ where g_j are Gear's coefficients (Dubois 01), the space approximation is done by Fourier.
- For $\mathcal{L}u = \chi_{[a,b]} u$ (see Laurent, Rosier, Zhang ..), we use compact schemes in finite differences to reach a spectral-like accuracy (Lele 92)

To make an hierarchy between these dampings for the long time behavior, we compare the associated function $G(t)$.

we can range the operators in the following way, by decreasing damping rate

1. $\mathcal{L}(u) = \nu \int_0^t \frac{u_t}{\sqrt{t-s}} ds$
2. $\mathcal{L}(u) = -\mu \partial_x^2 u$
3. $\mathcal{L}(u) = \gamma u$
4. $\mathcal{L}(u) = \gamma u \chi_{[a,b]}$
5. $\mathcal{L}(u) = \mathcal{L}_\gamma(u) = \sum_{k \in Z} \gamma_k \hat{u}_k e^{\frac{2ik\pi x}{L}}$, with $\gamma_k = \frac{1}{1+|k|}$

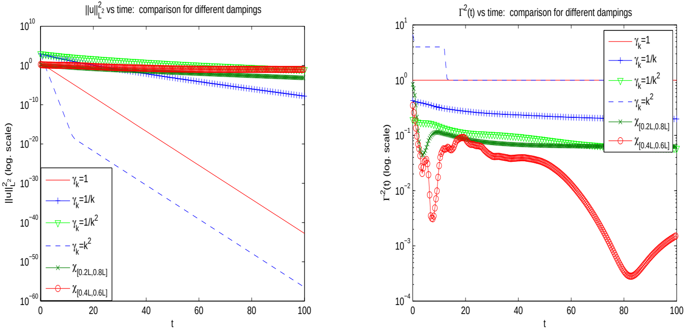


Figure : Comparison of the time evolution of L^2-norm (left) and of $\Gamma^2(t)$ (right) for the damping operators $\mathcal{L}_\gamma$ and $\chi_{[a,b]}$ on $[0, L]$ with $L = 20$.

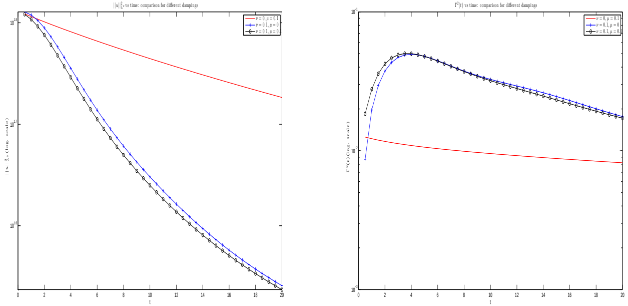


Figure : Comparison of the time evolution of L^2-norm (left) and of $\Gamma^2(t)$ (right) for the damping operators $\mathcal{L}(u) = -\mu u_{xx} + \sqrt{\nu}\int_0^t \frac{u_t}{\sqrt{t-s}}ds$ for different ν and μ with $L = 500$.

Damping reconstruction by solving an inverse problem

Integrate numerically $\frac{1}{2}\frac{d|u|^2}{dt} + G^2(t)|u|^2 = 0, on[\mathrm{t}_n, t_{n+1}]$ and get

$$G^2(t_{n+1}) \simeq \frac{|u(t_n)|^2 - |u(t_{n+1})|^2}{\frac{\Delta t}{2}(|u(t_{n+1})|^2 + |u(t_n)|^2)} \equiv \Gamma_n^2$$

Case of the local damping in space $\mathcal{L} = \chi_{[a,b]}$. Let x_i be the (regularly spaced) grid points and ℓ_i the vector defined by

$$\ell_i = \begin{cases} 1 & \text{if } x_i \in [a,b], \\ 0 & \text{otherwise.} \end{cases}$$

The computed values of $G^2(t_n)$ at discrete times t_n are the numbers

$$G^2(t_n) = \frac{\sum_{i=1}^{N} \ell_i (u_i^n)^2 h}{\sum_{i=1}^{N} (u_i^n)^2 h}$$

Damping reconstruction by solving an inverse problem

Integrate numerically $\frac{1}{2}\frac{d|u|^2}{dt} + G^2(t)|u|^2 = 0, on[\mathrm{t}_n, t_{n+1}]$ and get

$$G^2(t_{n+1}) \simeq \frac{|u(t_n)|^2 - |u(t_{n+1})|^2}{\frac{\Delta t}{2}(|u(t_{n+1})|^2 + |u(t_n)|^2)} \equiv \Gamma_n^2$$

Case of the local damping in space $\mathcal{L} = \chi_{[a,b]}$. Let x_i be the (regularly spaced) grid points and ℓ_i the vector defined by

$$\ell_i = \begin{cases} 1 & \text{if } x_i \in [a,b], \\ 0 & \text{otherwise.} \end{cases}$$

The computed values of $G^2(t_n)$ at discrete times t_n are the numbers

$$G^2(t_n) = \frac{\sum_{i=1}^{N} \ell_i (u_i^n)^2 h}{\sum_{i=1}^{N} (u_i^n)^2 h}$$ Idea: fitting of G with Γ.

real numbers γ_i solution of the constraint least square problem

$$\text{Inf}_{\ell_i \geq 0} \sum_{m=0}^{M} \left(\frac{\displaystyle\sum_{i=1}^{N} \ell_i (u_i^n)^2}{\sum_{i=1}^{N} (u_i^n)^2} - \Gamma^2(t_n) \right)^2 .$$

This minimization problem will be solved numerically using the Matlab® function `fmincon`.

$\mathcal{L} = \chi_{[0.4L, 0.6L]}$.

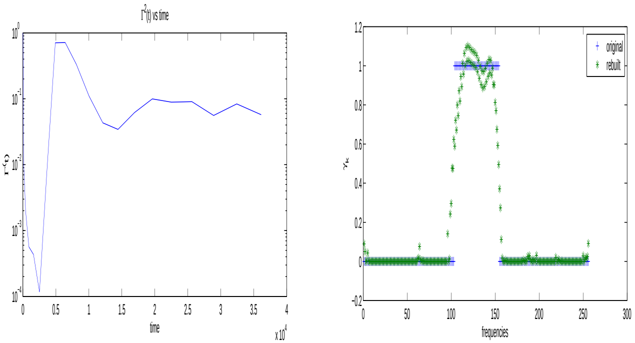


Figure : $\Gamma^2(t)$ (left), comparison of the original and the rebuilt coefficients $\mathcal{L} = \chi_{[0.4L, 0.6L]}$ (right)

Prevent the Blow up

Facts : For GKdV and $p \geq 4$, one can find inital data u_0 which blows up in finite time (in H^1 norm):

Property

Blow-up of $\|u_x\|_{L^2}$ in finite time for $p \geq 4$ and γ a small constant (Bona et al., 96; Martel and Merle, 02).

For a given blowing data u_0, it is possible to find γ^* such that the solution of the damped GKdV equation

$$u_t - u_{xxx} + \gamma u + u^p u_x = 0, \qquad u(x,0) = u_0(x)$$

doesn't bow up for any $\gamma \geq \gamma^*$.

Goal

- find such γ^*
- more generally find very weak dampings defined with a sequence γ_k

Scheme

Sanz-Serna scheme

$$\frac{\hat{u}_{n+1}(k) - \hat{u}_n(k)}{\Delta t} + \left(ik - ik^3 + \gamma_k\right) \frac{\hat{u}_{n+1}(k) + \hat{u}_n(k)}{2} + \frac{ik}{p+1} F\left[\left(\frac{\hat{u}_{n+1}(k) + \hat{u}_n(k)}{2}\right)^{p+1}\right] = 0.$$

Initial datum

We take a perturbed soliton with $c = 1.5$ and $d = 0.2L$

$$u(x,t) = 1.01 \left(\frac{(p+1)(p+2)(c-1)}{2}\right)^{\frac{1}{p}} \cosh^{-\frac{2}{p}}\left(\pm\sqrt{\frac{p(c-1)}{4}}(x-d)\right).$$

The optimal damping

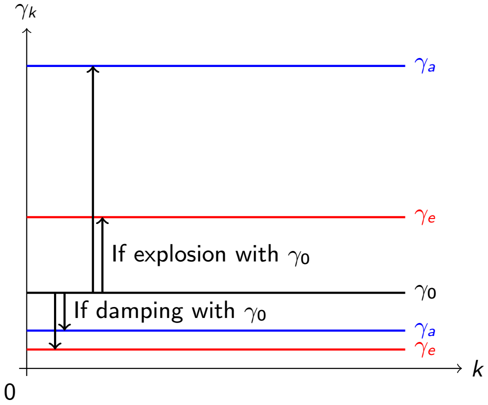


Figure : Initialization

The optimal damping

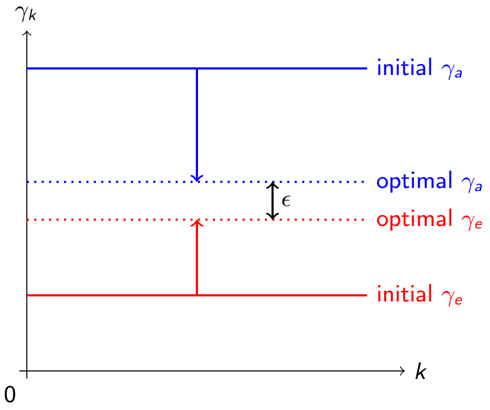


Figure : Dichotomy

Simulations

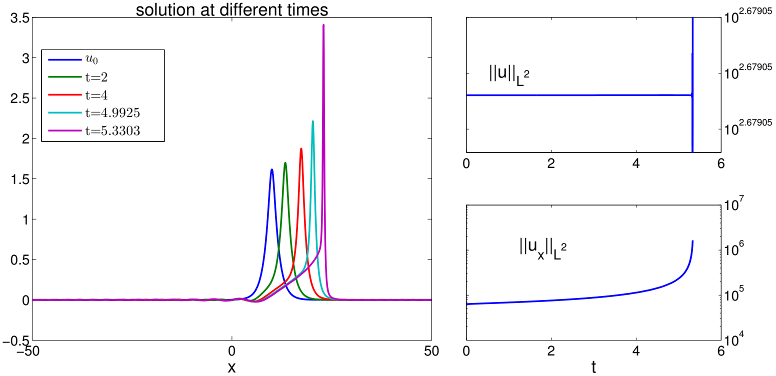


Figure : At left, solution at different times $t = 0,\ 2,\ 4,\ 4.9925$ and 5.3303. At right, H^1-norm and L^2-norm evolution without damping and a perturbed soliton as initial datum. Here $p = 5$.

Simulations

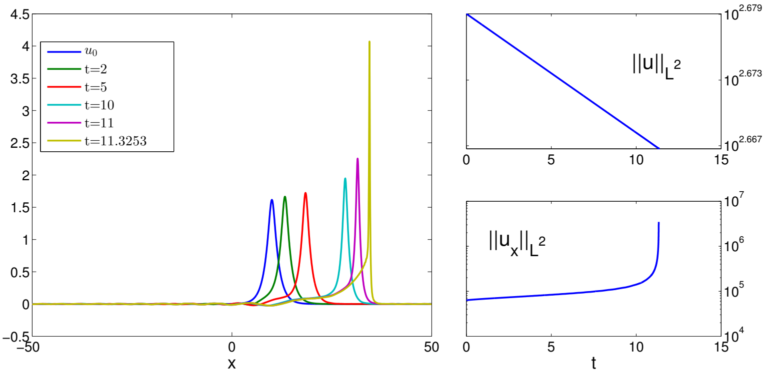


Figure : At left, solution at different times $t = 0$, 2, 5, 10, 11 and 11.3253. At right, H^1-norm and L^2-norm evolution with $\gamma_k = 0.0025$ and a perturbed soliton as initial datum. Here $p = 5$.

Simulations

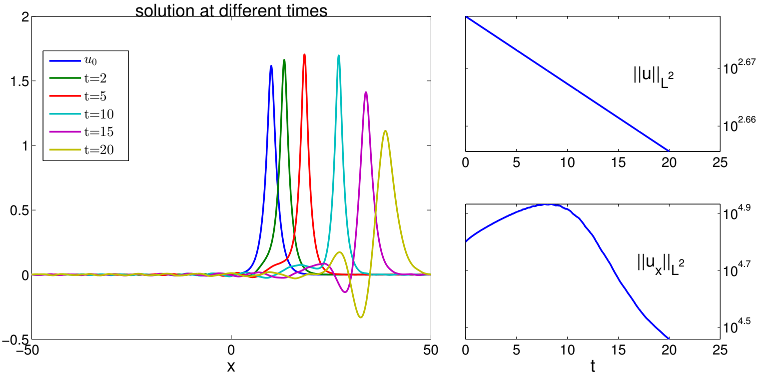
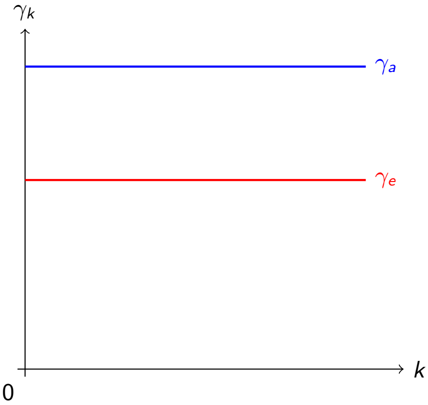
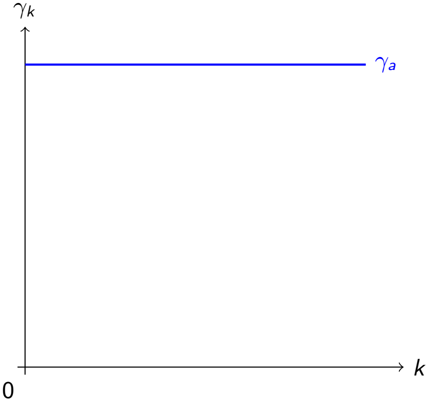


Figure : At left, solution at different times $t = 0, 2, 5, 10, 15$ and 20. At right, H^1-norm and L^2-norm evolution with $\gamma_k = 0.0027$ and a perturbed soliton as initial datum. Here $p = 5$.

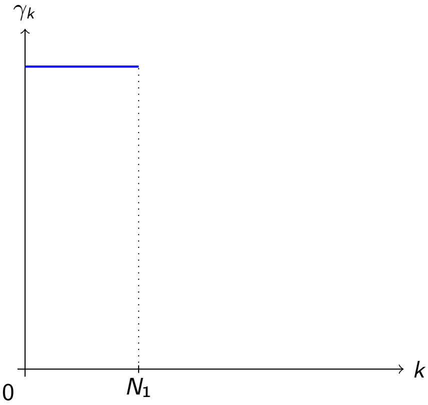
Band limited damping



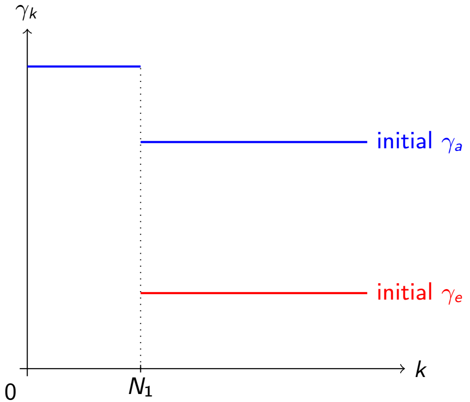
Band limited damping



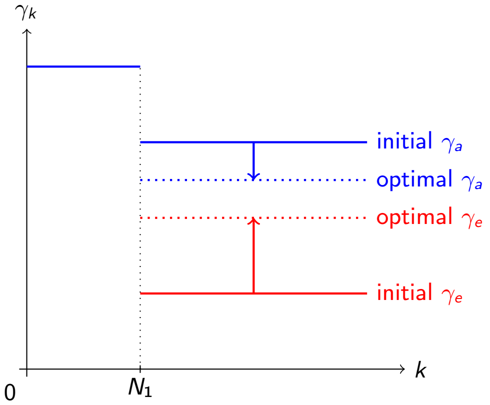
Band limited damping



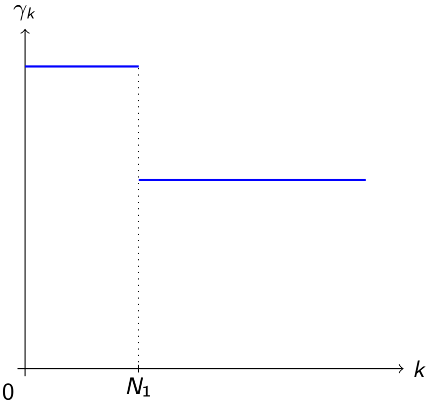
Band limited damping



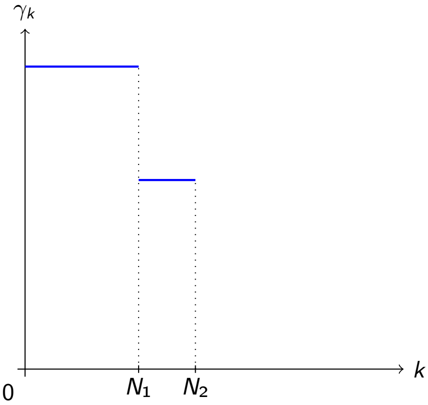
Band limited damping



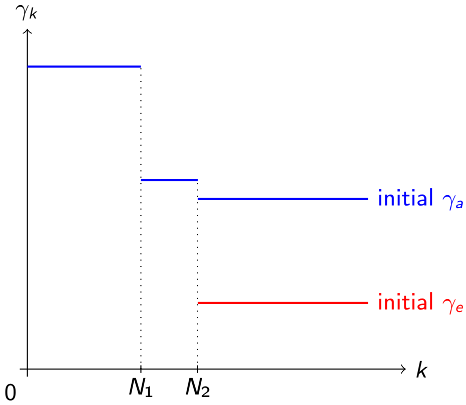
Band limited damping



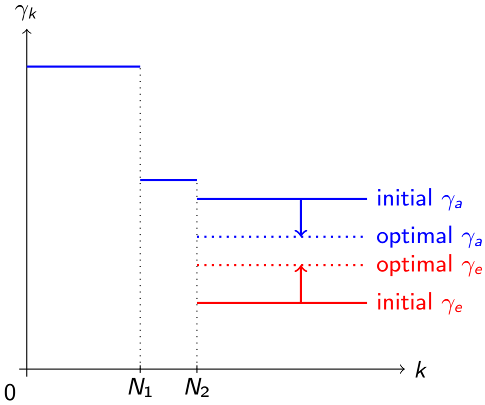
Band limited damping



Band limited damping



Band limited damping



Simulations

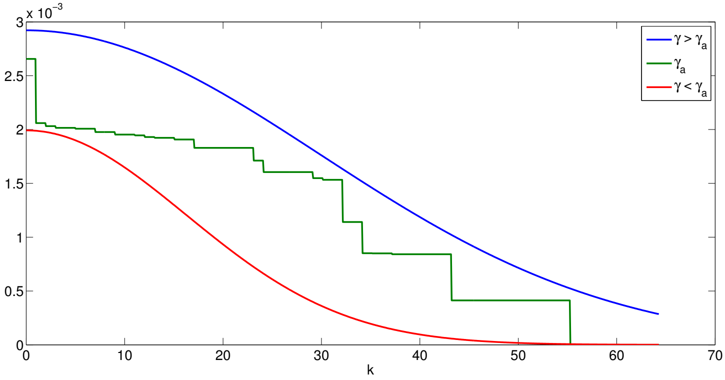


Figure : Example of a build damping. Here the initial datum is the perturbed soliton. Here $p = 5$.

Simulations

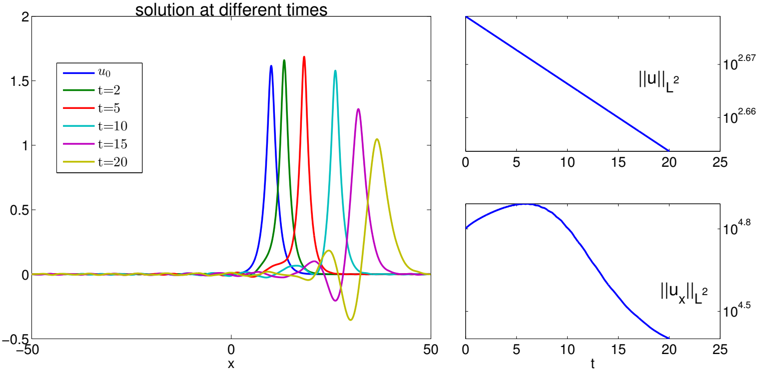


Figure : At left, solution at different times $t = 0$, 2, 5, 10, 15 and 20. At right, H^1-norm and L^2-norm evolution with $\gamma = \gamma_1$ and a perturbed soliton as initial datum. Here $p = 5$.

Simulations

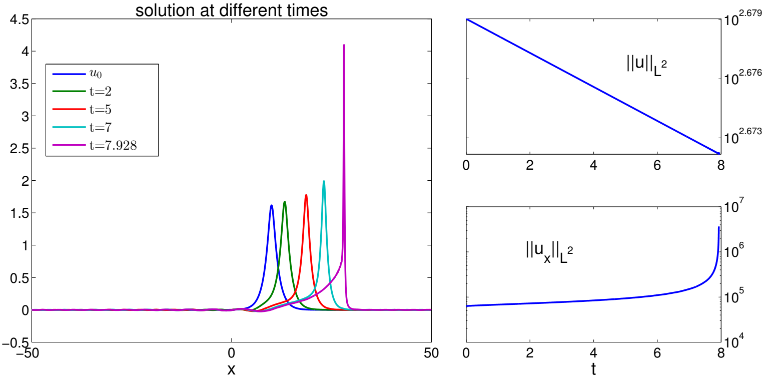


Figure : At left, solution at different times $t = 0, 2, 5, 7$ and 7.928. At right, H^1-norm and L^2-norm evolution with $\gamma = \gamma_2$ and a perturbed soliton as initial datum. Here $p = 5$.

This work

- Damping in energy norms
- Sobolev regularization effect even when $\gamma_k \to 0$
- Nontrivial dynamics for large t (as in the case $\gamma_k = \gamma$)
- Hierarchy of damping models and *a posteriori* reconstruction of the damping operator
  (same can be done with BBM equations)
- Frequential approach
  - Good for band limited frequencies problems.
  - Allows to build cheap and efficient dampings.

Next

- Damping with L_γ in orthogonal polynomial (or Hilbert) basis

$$L_\gamma(u) = \sum_{k=0}^{\infty} \gamma_k \hat{u}_k p_k(x)$$

and $u(x,t) = \sum_{k=0}^{\infty} \hat{u}_k p_k(x)$ with $\hat{u}_k = \dfrac{(u, p_k)_\omega}{(p_k, p_k)_\omega}$

- Equations as $\dfrac{\partial u}{\partial t} + L_\gamma u + F(u) = 0$ with $Re(F(u), u)_\omega = 0$ and L_γ defined as above. Example NLS : $i(u_t + \alpha u) + u_{xx} + |u|^2 u = f$ (asymptotic regularization Ghidaglia (88), Goubet (96))
- Coupling Least square constraint evolution with Regularization for solving ill-conditioned inverse problems
- Use a similar approach for other equations (NLS, KP, ...).

References I

J.L. Bona, V.A. Dougalis, O.A. Karakashian, and W.R. McKinney.
The effect of dissipation on solutions of the generalized Korteweg-de Vries equation.
J. Comput. Appl. Math., 74(1-2):127–154, 1996.
TICAM Symposium (Austin, TX, 1995).

J. L. Bona and R. Smith.
The initial-value problem for the Korteweg-de Vries equation.
Philos. Trans. Roy. Soc. London Ser. A, 278(1287):555–601, 1975.

J.-P. Chehab, P. Garnier, and Y. Mammeri.
Long-time behavior of solutions of a BBM equation with generalized damping.
Discr. Cont. Dyn. Syst. B, 7:1897–1915, 2015.

J.-P. Chehab and G. Sadaka.
Numerical Study of a family of dissipative KdV equations.
CPAA, 12(1):519–546, January 2013.

References II

J.-P. Chehab and G. Sadaka.
On Damping Rates of dissipative KdV equations.
Commun. Pure Appl. Anal. 12 (2013), no. 1, 519–546., 6(1), December 2013.

P. Garnier.
Damping to prevent the blow-up of the Korteweg-de Vries equation.
ArXiv e-prints, Submitted, 2015.

J.-M. Ghidaglia.
Weakly damped forced Korteweg-de Vries equations behave as a finite-dimensional dynamical system in the long time.
J. Differential Equations, 74(2):369–390, 1988.

Y. Martel and F. Merle.
Stability of blow-up profile and lower bounds for blow-up rate for the critical generalized KdV equation.
Ann. of Math. (2), 155(1):235–280, 2002.

References III

Stéphane Vento.
Global well-posedness for dissipative Korteweg-de Vries equations.
Funkcial. Ekvac., 54(1):119–138, 2011.

Well-posedness

damped KdV equation (dKdV)

$$u_t + u_x + u_{xxx} + u^p u_x + L_\gamma(u) = 0,\ \forall x \in \mathbb{R} \text{ and } \forall t > 0,\ \ p \geq 1,$$
$$u(x,0) = u_0(x),\ \forall x \in \mathbb{R}.$$

Damping

- L_γ is defined by :
$$\widehat{L_\gamma(u)}(\xi) := \gamma(\xi)\hat{u}(\xi).$$
- $\hat{u}$ is the Fourier transform of u.
- $\gamma(\xi) > 0$, $\forall \xi \in \mathbb{R}$.
- The following condition is verified
$$\int_{\mathbb{R}} u(x) L_\gamma(u) d\mu(x) = \int_{\mathbb{R}} \gamma(\xi) |\hat{u}(\xi)|^2 \geq 0.$$

Notations

Multiply dKdV in L^2 by u, we obtain $\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 + |u|_\gamma^2 = 0$.

Space of study

$$H_{\gamma^s}(\mathbb{R}) := \left\{ u \in L^2(\mathbb{R}); \int_{\mathbb{R}} \gamma(\xi)^s |\hat{u}(\xi)|^2 d\xi < +\infty \right\},$$

equipped with the norm

$$|u|_{\gamma^s} := \sqrt{\int_{\mathbb{R}} \gamma(\xi)^s |\hat{u}(\xi)|^2}.$$

Reminder

$$u(x) = \int_{\mathbb{R}} \hat{u}(\xi) e^{i\xi x} d\xi.$$

$$\|u\|_{L^2}^2 = \int_{\mathbb{R}} |\hat{u}(\xi)|^2 d\xi.$$

Local and global well-posedness of the KdV equation

Cauchy problem considered

$$u_t + u_x + u_{xxx} + u^p u_x + L_\gamma(u) = 0, \quad x \in \mathbb{R}, \ t \in [0, T] \tag{4}$$

$$u(x, t = 0) = u_0(x). \tag{5}$$

Lemma

Assume that $s, \ r \in \mathbb{R}^+$. Then there exists a constant $C_r > 0$, depending only on r, such that $\forall u \in H_{\gamma^s}(\mathbb{R})$ and $\forall t > 0$ we have

$$|S_t u|^2_{\gamma^{s+r}} \leq \frac{C_r}{t^r} |u|^2_{\gamma^s} .$$

Theorem

Assume that there exists $r \in]0,2[$ *and for all* $\xi \in \mathbb{R}$, $\gamma(\xi) \geq \xi^{\frac{2}{r}}$. *We also assume that* $\int_{\mathbb{R}} \frac{1}{\gamma(\xi)^s} < +\infty$ *and there exists a constant* $C > 0$ *such that* $\forall \xi,\ \eta \in \mathbb{R}$ *and* $s \in \mathbb{R}^+$ *we have*

$$\sqrt{\gamma(\xi)^s} \leq C\left(\sqrt{\gamma(\xi-\eta)^s} + \sqrt{\gamma(\eta)^s}\right).$$

Then there exists a unique solution in $C\left([-T,T], H_{\gamma^s}(\mathbb{R})\right)$ *of the Cauchy problem* (4)-(5).

remark

Actually we can prove the local well-posedness in $H^s(\mathbb{R})$ for every γ using a parabolic regularisation

$$u_t + u_x + u_{xxx} + u^p u_x + L_\gamma(u) - \epsilon u_{xx} = 0.$$

Using the lemma with $\gamma(\xi) = \xi^2$, the same computations as the previous theorem and the limit $\epsilon \to 0$ give the result (Bona and Smith, 75).

Theorem

If $p < 4$, for all γ, the unique solution is global in time, valued in $H^1(\mathbb{R})$.
Else ($p \geq 4$), there exists a constant $\theta > 0$ such that if $\gamma(\xi) \geq \theta$, $\forall \xi \in \mathbb{R}$ then the unique solution is global in time, valued in $H^2(\mathbb{R})$.
(Bona et al., 96).

Theorem

*If $p < 4$, for all γ, the unique solution is global in time, valued in $H^1(\mathbb{R})$.
Else ($p \geq 4$), there exists a constant $\theta > 0$ such that if $\gamma(\xi) \geq \theta$, $\forall \xi \in \mathbb{R}$ then the unique solution is global in time, valued in $H^2(\mathbb{R})$.
(Bona et al., 96).*

Theorem

If $p < 4$, for all γ, the unique solution is global in time, valued in $H^1(\mathbb{R})$.
Else ($p \geq 4$), there exists a constant $\theta > 0$ such that if $\gamma(\xi) \geq \theta, \ \forall \xi \in \mathbb{R}$ then the unique solution is global in time, valued in $H^2(\mathbb{R})$.
(Bona et al., 96).

Remark

This result is also true on the torus $\mathbb{T}(0, L)$ on which we switch now.