REGULAR SUBSETS OF VALUED FIELDS AND BHARGAVA'S v-ORDERINGS

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Abstract. We generalize notions and results obtained by Amice for regular compact subsets S of a local field K and extended by Bhargava to every compact subsets of K. Considering any ultrametic valued field K and subsets S that are regular in a generalized sense (but not necessarily compact), we show that they still have strong properties like having v-orderings $\{a_n\}_{n\geq 0}$ which 1- satisfy a generalized Legendre formula, 2- are very well ordered and well distributed sequences in the sense of Helsmoortel and 3- remain v-ordering by troncation of the first terms: for every $k \geq 0$, $\{a_n\}_{\geq k}$ is still a v-ordering.

1. Introduction

In her study of continuous p-adic functions, Amice [1] introduced regular compact subsets of a local field K. This notion has been first extended to precompact subsets of discrete valuation domains [11], and then to fractional subsets of a valued field K as regular subsets of K [9]. This notion is interesting because it appears, for instance, in discrete dynamical systems in the following way:

If $\varphi : K \to K$ is an isometry, then for every element $x \in K$ the forward orbit $O_+^{\varphi(x)} = \{\varphi^n(x) \mid n \in \mathbb{N}\}$ of x under the action of φ is a preregular subset of K [9, Theorem 11.2].

On the other hand, Bhargava ([2] and [3]) introduced the notions of factorials and v-orderings associated to subsets of discrete valuation domains and these notion have been extended to subsets of rank-one valuation domains [8]. We are going to study here the strong links between the regular subsets of a valued field and various kinds of v-orderings.

The paradigmatic example is given by $\mathbb{Z}$ which is a regular subset of the valued field of p-adic numbers $\mathbb{Q}_p$ whatever the prime number p. The sequence $\{n\}_{n\geq 0}$ is a v_p-ordering of $\mathbb{Z}$ for every p with the following very strong properties: on the one hand, for every $k \geq 0$, the sequence $\{n\}_{n\geq k}$ is still a v_p-ordering and, on the other hand, for every $h \in \mathbb{N}$, the sequence formed by any p^h consecutive terms is a complete set of representatives of $\mathbb{Z}$ modulo p^h. Moreover, we have explicit formulas like Legendre's formula for $v_p(n!)$.

Recall the definition of the factorial ideals and the notion of v-ordering associated to a subset S of an ultrametric valued field K. Denote by v the rank-one valuation

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of K and by V its valuation domain. First consider the ring of integer-valued polynomials on S :

$$\mathrm{Int}(S,V)=\{f\in K[X]\mid f(S)\subseteq V\}. \tag{1.1}$$

Then, for every $n\in\mathbb{N}$, the set formed by the leading coefficients of the polynomials of $\mathrm{Int}(S,V)$ of degree $\leq n$ is a fractional ideal whose valuation is denoted by $w_S(n)$. The function $w_S:\mathbb{N}\to\mathbb{R}$ is called the *characteristic function* of S while the sequence $\{w_S(n)\}_{n\geq 0}$ is called the *characteristic sequence* of S. Bhargava [2] showed how to compute this sequence step by step by introducing the notion of v-ordering and proving the following proposition:

Proposition 1.1. *A* v-ordering *of* S *is a sequence* $\{a_n\}_{n\in\mathbb{N}}$ *of elements of* S *where,* a_0 *being any element of* S*, for every* $n\geq 1, a_n$ *is chosen in such a way that:*

$$v\left(\prod_{k=0}^{n-1}(a_n-a_k)\right)=\inf_{x\in S}\left(\prod_{k=0}^{n-1}(x-a_k)\right). \tag{1.2}$$

For any v*-ordering of* S*, we have:*

$$w_S(n)=v\left(\prod_{k=0}^{n-1}(a_n-a_k)\right). \tag{1.3}$$

This proposition, first proved for a discrete valuation v, still holds for any valued field K [8]. Note that, in the case a valued field in general, a subset S does not always admit v-orderings, nevertheless it always has a characteristic sequence.

Remark 1.2*.* For every $n\in\mathbb{N}$, the *n-th factorial ideal* $(n!)_S$ *of the subset* S is the ideal defined by:

$$(n!)_S=\{x\in K\mid xf\in V[X]\ \forall f\in\mathrm{Int}(S,V)\}=\{x\in K\mid v(x)\geq w_S(n)\}. \tag{1.4}$$

This notion of factorial ideal, which is locally defined, may be globalized in order to define the factorial ideals associated to a subset of a Dedekind domain (see [2]). Such a globalization generalizes the classical factorials $n!$. Nevertheless, we will not be interested here by this global point of view.

Let us now precise the hypotheses and notation that will be used in the whole paper. Then, we will prove that, when S is regular, some of its v-orderings have fine properties analogously to those of $\mathbb{Z}$. One of our main results is Theorem 6.11 below.

2. Hypotheses and notation

2.1. Let K be a valued field.

Recall that a valued field K is a field endowed with a rank-one valuation v with $v(K^*)\subseteq\mathbb{R}$. Denote by V the corresponding valuation domain.

For $\gamma\in\mathbb{R}$ and $x,y\in K$, $x\equiv y\pmod{\gamma}$ will mean $v(x-y)\geq\gamma$.

For $x\in K$ and $\gamma\in\mathbb{R}$, we denote respectively by $B(x,\gamma)$ and by $\overset{\bullet}{B}(x,\gamma)$ the ball and the sphere of center x and radius $e^{-\gamma}$, that is:

$$B(x,\gamma)=\{y\in K\mid v(x-y)\geq\gamma\}\quad\text{and}\quad\overset{\bullet}{B}(x,\gamma)=\{y\in K\mid v(x-y)=\gamma\}. \tag{2.1}$$

2.2. **Let S be an infinite fractional subset of K.**

Recall that S is a *fractional subset* of K when there exists $d \in K^*$ such that $dS = \{ds \mid s \in S\} \subseteq V$.

For every $\gamma \in \mathbb{R}$, denote by:

– $S(a, \gamma)$ the equivalence class modulo γ of the element $a \in S$:

$$S(a,\gamma) = S \cap B(a,\gamma), \tag{2.2}$$

– S_γ any set of representatives of the classes of S modulo γ, so that:

$$S = \cup_{a \in S_\gamma} S(a,\gamma), \tag{2.3}$$

– q_γ the cardinality (finite or infinite) of S_γ:

$$q_\gamma = q_\gamma(S) = Card(S \; mod \; \gamma) = Card(S_\gamma). \tag{2.4}$$

An important number for the subset S is the following:

$$\gamma_\infty = \gamma_\infty(S) = \sup\{\, \gamma \mid q_\gamma \text{ finite}\} \in \mathbb{R} \cup \{+\infty\}. \tag{2.5}$$

2.3. **The critical valuations $\{\gamma_k\}_{k\geq 0}$ of the fractional subset S.**

Recall that the study of the function:

$$\gamma \in \mathbb{R} \mapsto q_\gamma \in \mathbb{N} \cup \{\infty\} \tag{2.6}$$

shows that:

Proposition 2.1. [9, Prop. 5.1] *There is a strictly increasing sequence $\{\gamma_k\}_{k\geq 0}$ (finite or infinite) such that*

$$\gamma_0 = \gamma_0(S) = \inf\{v(x-y) \mid x, y \in S, x \neq y\} \tag{2.7}$$

and

$$\forall k \geq 1 \quad \gamma_{k-1} < \gamma \leq \gamma_k \;\Leftrightarrow\; q_\gamma = q_{\gamma_k}. \tag{2.8}$$

This sequence is called the sequence of critical valuations *of S. Moreover,*

1. *if the sequence is finite, say $\{\gamma_k\}_{0\leq k\leq l}$, then q_{γ_∞} is finite and $\gamma_l = \gamma_\infty$,*
2. *if the sequence is infinite, then q_{γ_∞} is infinite and*

$$\lim_{k\to+\infty} \gamma_k = \gamma_\infty. \tag{2.9}$$

Remarks 2.2.

1. The assertion 'S is a fractional subset of K' is equivalent to $\gamma_0 \neq -\infty$.
2. For every $\gamma < \gamma_\infty$, γ is a critical valuation if and only $\gamma = v(x-y)$ for some $x, y \in S$.
3. About γ_0 defined in (2.7): we always have $q_{\gamma_0} = 1$ while γ_0 is not necessarily a minimum. If γ_0 is not a minimum, then $\gamma_0 = \gamma_\infty$. [We know [9, Thm 4.3] that the equality $\gamma_0 = \gamma_\infty$ characterizes the subsets S whose polynomial closure $\overline{S}$ (see § 2.5 below) is a ball $B(x, \gamma_0)$.] To avoid this trivial case, we will also assume that $\gamma_0 \neq \gamma_\infty$, so that we have the following (finite or infinite) sequence of inequalities:

$$-\infty < \gamma_0 < \gamma_1 < \ldots < \gamma_k < \gamma_{k+1} < \ldots < \gamma_\infty \leq +\infty. \tag{2.10}$$

4. It follows from Proposition 2.1 that it is possible to choose the elements of the S_γ's, the sets of representatives of S modulo γ, in such a way that

$$\gamma_0 \leq \gamma < \delta \leq \gamma_\infty \Rightarrow S_{\gamma_0} \subset S_\gamma \subset S_\delta \subset S_{\gamma_\infty}. \tag{2.11}$$

(5) The containment $\cup_k S_{\gamma_k} \subseteq S_{\gamma_\infty}$ may be strict.

2.4. The tree $T(S)$ associated to the fractional subset S.

In the following, we always assume that the containments in (2.11) hold. So that, we may consider the sequence $\{x_n\}_{0\le n<q_{\gamma_\infty}}$ of elements of S such that:

$$\forall k \ge 0 \;\; S_{\gamma_k} = \{x_0, \ldots, x_{q_{\gamma_k}-1}\}, \text{ and } \cup_{k\ge 0} \; S_{\gamma_k} = \{x_n \mid n \ge 0\}. \tag{2.12}$$

The tree $T(S)$ associated to S may be described in the following way:
– the *vertices* at each level $k \ge 0$ are the elements $x_0, \ldots, x_{q_{\gamma_k}-1}$ of S_{γ_k},
– the *edges* between levels k and $k+1$ join x at level k to y at level $k+1$ if and only if $y \in S(x, \gamma_k)$.

Note that, when x appears at a level, then it remains at all the following levels. In particular, if $S_{\gamma_0} = \{x_0\}$, then the element x_0, which is the *root* of the tree, appears at each level. Note also that

$$\forall x \in S_{\gamma_k} \, \forall y \in S_{\gamma_{k+1}} \setminus S_{\gamma_k} \; [\, y \in S(x, \gamma_k) \;\Rightarrow\; v(x-y) = \gamma_k \,], \tag{2.13}$$

because $y \in S(x, \gamma_k)$ implies $v(x-y) \ge \gamma_k$, and $x, y \in S_{\gamma_{k+1}}$ implies $v(x-y) < \gamma_{k+1}$; so that, we may conclude with Remark 2.2(2).

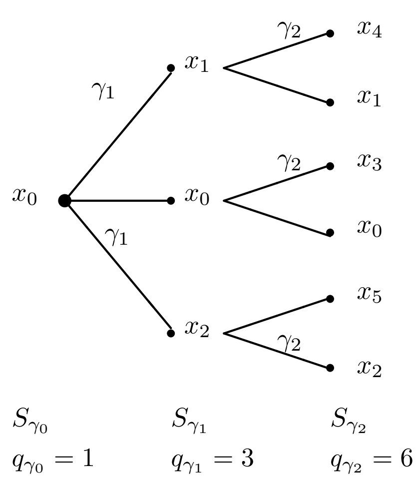
Thus, if we write $v(x-y)$ on the edge joining x and y, we see that, there will be γ_k on all the edges between levels k and $k+1$ except for the edges joining an x to itself. In other words:

$$\forall k \ge 0 \, \forall x \in S_k \quad S_{\gamma_{k+1}} \cap S(x, \gamma_k) \subseteq \; \{x\} \cup \overset{\bullet}{B}(x, \gamma_k). \tag{2.14}$$

As a consequence, for all $x, y \in \cup_k S_{\gamma_k}$:

$$v(y-z) = \gamma_k \Leftrightarrow k = \min\{l \mid \text{the unique path from } y \text{ to } z \text{ goes through level } l\}.$$

Here is an example of a finite regular tree $T(S)$:



Of course, if q_{γ_∞} is finite, the tree is finite. Assume that q_{γ_∞} is infinite. Then, each $z \in S_{\gamma_\infty}$ corresponds to one and only one infinite branch of the tree: the vertex of this branch at level k is the unique element $z_k \in S_{\gamma_k}$ such that $v(z - z_k) \ge \gamma_k$. An element $z \in S_{\gamma_\infty}$ belongs to $\cup_{k\ge 0} S_{\gamma_k}$ if and only if the sequence $\{z_k\}_{k\ge 0}$ formed by the vertices of the corresponding infinite branch is ultimately constant. Note that it could exist infinite branches which do not correspond to any element of S_{γ_∞}. This is always the case when $S_{\gamma_\infty} = \cup_k S_{\gamma_k}$.

Finally, note that the name of the vertices of $T(S)$ depends on the choices made for the elements of the S_k, but the structure of the tree as well as the values written on the edges remain the same.

2.5. The polynomial closure $\overline{S}$ of the fractional subset S.

Definition 2.3. The *polynomial closure* $\overline{S}$ of S is the largest subset T of K containing S such that, for every $f \in K[X]$, $f(S) \subseteq V$ implies $f(T) \subseteq V$.

Equivalently, $\overline{S}$ of is the largest subset T such that $S \subseteq T \subseteq K$ and $w_T = w_S$ (see Remark [6, IX.3.1]). Let us recall some results about this polynomial closure.

Proposition 2.4. [10, Theorem 5.2] *S is polynomially closed if and only:*
– for every $x \in K$, if there is a sequence $\{x_n\}_{n\in\mathbb{N}}$ of elements of S such that $v(x - x_n)$ is strictly monotone with limit $\gamma \in \mathbb{R} \cup \{+\infty\}$, then $B(x, \gamma) \subseteq S$.
– for every $x_0 \in K$, if there is a sequence $\{x_n\}_{n\in\mathbb{N}^*}$ of elements of S such that $v(x_n - x_m) = \gamma$ for all $n > m \geq 0$, then $B(x_0, \gamma) \subseteq S$.

As an easy consequence:

Corollary 2.5. *For every fractional subset S with polynomial closure $\overline{S}$, one has:*

$$T(\overline{S}) = T(S) \tag{2.15}$$

We will need also the following:

Proposition 2.6. [10, Theorem 5.3] *The polynomially closed subsets of K are the closed subsets for some polynomial topology on K.*

As an obvious consequence, since every closed ball is polynomially closed,we have the following containments without any hypothesis on S:

$$\overline{S} \subseteq \cap_{k\geq 0} \left(\cup_{x\in S_{\gamma_k}} B(x, \gamma_k) \right). \tag{2.16}$$

3. Regular subsets

In order to extend Mahler's theorem on approximation of continuous p-adic functions on compact subsets, Amice [1] introduced the notion of a regular compact subset of a local field: a compact subset S of a local field is said to be regular if, for every $n \in \mathbb{Z}$, there exists $\alpha_n \in \mathbb{N}$ such that, for every $x \in S$, the S-ball $S(x, n)$ is a disjoint union of α_n S-balls of the form $S(z, n + 1)$ where $z \in S$.

It follows from this definition that:

$$\forall n \in \mathbb{Z} \quad q_{n+1} = \alpha_n \times q_n.$$

In [11], this notion is extended to the case where K is not necessarily complete, the residue field is not necessarily finite, and the subset S is only assumed to be precompact (that is, the completion of S is compact, and we know that this is equivalent to $\gamma_\infty = +\infty$, see for instance [7, Lemma 3.1]). Here, we generalize this notion to any fractional subset S of any valued field K and we define what we call preregular subsets.

Definition 3.1. The fractional subset S of the valued field K is said to be a *preregular subset* if, for every $\gamma < \delta$ such that q_γ is finite, $Card(S(x,\gamma) \bmod \delta)$ does not depend on $x \in S$ in the following sense:

(1) if q_δ is finite, then every non-empty S-ball $S(x,\gamma)$ is the disjoint union of $\frac{q_\delta}{q_\gamma}$ non-empty S-balls $S(y,\delta)$,
(2) if q_δ is infinite, then every non-empty S-ball $S(x,\gamma)$ is the disjoint union of infinitely many non empty S-balls $S(y,\delta)$.

Condition (1) is equivalent to both following assertions:

$$\forall k \geq 0,\ q_{\gamma_{k+1}} = \alpha_k q_{\gamma_k} \text{ (where } \alpha_k \in \mathbb{N} \text{ and } \alpha_k \geq 2) \tag{3.1}$$

$$\forall a \in S,\ Card\ S(a,\gamma_k) \bmod \gamma_{k+1} \ = \alpha_k. \tag{3.2}$$

– If q_{γ_∞} is infinite (that is, when $\gamma_\infty = \lim \gamma_k$), condition (2) is an obvious consequence of condition (1).
– If q_{γ_∞} is finite (that is, when $\gamma_\infty = \gamma_l$ for some l), condition (2) is equivalent to: for every $\delta > \gamma_\infty$, for every $x \in S_{\gamma_\infty}$, $S(x,\gamma_\infty)$ is the union of infinitely many $S(y,\delta)$ where $y \in S$.

The following link with trees terminology is straightforward.

Lemma 3.2. *The following assertions are equivalent:*

(1) *For all $k \geq 1$, every non-empty S-ball $S(x,\gamma_{k-1})$ is the disjoint union of $\frac{q_{\gamma_k}}{q_{\gamma_{k-1}}}$ non-empty S-balls $S(y,\gamma_k)$.*
(2) *$T(S)$ is regular in the following sense: for each level $k \geq 0$, the number of edges joining a fixed vertex x at level k to a vertex at level $k+1$ does not depend on x.*
(3) *For every level $k \geq 0$ of $T(S)$, for all vertices y,z at level k, the subtrees of $T(S)$ with roots y and z respectively are isomorphic (that is, there is a bijection between the vertices of the subtrees which preserves levels and edges).*
(4) *For every critical valuation γ, the following subsets are isometric when x runs over $S\gamma$:*
$$B(x,\gamma) \cap (\cup_{k\geq 0} S_{\gamma_k}).$$

Corollary 3.3. *When q_{γ_∞} is infinite, S is a preregular subset if and only the associated tree $T(S)$ is regular in the previous sense.*

Note also that, when S is a regular compact subset in Amice's sense, then:
– S is polynomially closed (since S is compact [6, Thm. IV.1.15]),
– for all $y,z \in S$, for every $n \in \mathbb{Z}$, the S-balls $S(y,n)$ and $S(z,n)$ are isometric.
So that, there is another way to extend the notion of regular subset :

Definition 3.4. The fractional subset S of K is said to be a *regular subset* if it satisfies the following properties:

(1) S is polynomially closed,
(2) for all $y,z \in S$, for every $\gamma \in \mathbb{R}$, the S-balls $S(y,\gamma)$ and $S(z,\gamma)$ are isometric.

Of course, there are links between preregular subsets and regular subsets.

Theorem 3.5. *The following assertions are equivalent:*

(1) *S is preregular,*

(2) $\overline{S}$ *is regular,*
(3) $T(S)$ *is regular and, for all* $x \in S$, $B(x, \gamma_\infty) \subseteq \overline{S}$,
(4) $T(S)$ *is regular and, when* q_{γ_∞} *is finite* ($\gamma_\infty = \gamma_l$), $\overline{S}$ *is a finite union of balls with radius* γ_l :

$$\overline{S} = \cup_{x \in S_{\gamma_l}} B(x, \gamma_l). \tag{3.3}$$

In particular:

Corollary 3.6. *Assume that* q_{γ_∞} *is infinite. Then,* $\overline{S}$ *is regular if and only if* $T(S)$ *is regular.*

Before proving Theorem 3.5, we have to prove some lemmas.

Lemma 3.7. *Assume that* q_{γ_∞} *is infinite. If there exists* $x \in S$ *such that, for every* $\gamma < \gamma_\infty$, *the* S*-ball* $S(x, \gamma)$ *contains at least two distinct classes modulo* γ_∞, *then* $B(x, \gamma_\infty) \subseteq \overline{S}$.

Proof. By considering any γ such that $\gamma_\infty - 1 < \gamma < \gamma_\infty$, we see that there exists some $x_0 \in S$ such that $\gamma_\infty - 1 < v(x - x_0) < \gamma_\infty$. Then, by considering another γ such that $\max\{\gamma_\infty - \frac{1}{2}, v(x - x_0)\} < \gamma < \gamma_\infty$, we see that there exists $x_1 \in S$ such that $v(x - x_0) < v(x - x_1)$ and $\gamma_\infty - \frac{1}{2} < v(x - x_1) < \gamma_\infty$. And so on... Thus, we may construct a sequence $\{x_n\}_{n \in \mathbb{N}}$ of elements of S such that the sequence $\{v(x - x_n)\}_{n \in \mathbb{N}}$ is strictly increasing with limit γ_∞. The conclusion follows from Proposition 2.4. $\square$

Analogously,

Lemma 3.8. *Assume that* q_{γ_∞} *is finite. If* $x \in S$ *is such that, for every* $\delta > \gamma_\infty$, *the* S*-ball* $S(x, \gamma_\infty)$ *contains infinitely many classes modulo* δ, *then* $B(x, \gamma_\infty) \subseteq \overline{S}$.

Proof. If there exist infinitely many $\{x_n\}_{n>0}$ in S such that, letting $x_0 = x$, we have $v(x_n - x_m) = \gamma_\infty$ for all $n > m \geq 0$ then, by Proposition 2.4, $B(x, \gamma_\infty) \subseteq \overline{S}$.

Assume now that there are at most finitely many such x_n. By considering any δ such that $\gamma_\infty < \delta < \gamma_\infty + 1$, we see that there exists some $y_0 \in S$ such that $\gamma_\infty < v(x - y_0) < \gamma_\infty + 1$. Then, by considering another δ such that $\gamma_\infty < \delta < \min\{\gamma_\infty + \frac{1}{2}, v(x - y_0)\}$, we see that there exists $y_1 \in S$ such that $v(x - y_1) < v(x - y_0)$ and $\gamma_\infty < v(x - y_1) < \gamma_\infty + \frac{1}{2}$. And so on... Thus, we may construct a sequence $\{y_n\}_{n \in \mathbb{N}}$ of elements of S such that the sequence $\{v(x - y_n)\}_{n \in \mathbb{N}}$ is strictly decreasing with limit γ_∞. The conclusion follows from Proposition 2.4. $\square$

Remark 3.9. If q_{γ_∞} is finite (as in Lemma 3.8), then there always exists at least one $x \in S$ such that, for every $\delta > \gamma_\infty$, the S-ball $S(x, \gamma_\infty)$ contains infinitely many classes modulo δ, and hence, such that $B(x, \gamma_\infty) \subseteq \overline{S}$. Otherwise, for every $x \in S_{\gamma_\infty}$, there would exist $\delta_x > \gamma_\infty$ such that $S(x, \gamma_\infty)$ contains at most finitely many distinct classes modulo δ_x. Since S_{γ_∞} is finite, we would have:

$$\delta_0 = \inf_{x \in S_{\gamma_\infty}} \delta_x = \min_{x \in S_{\gamma_\infty}} \delta_x > \gamma_\infty$$

and, for every $x \in S_{\gamma_\infty}$, $S(x, \gamma_\infty)$ would contain at most finitely many distinct classes modulo δ_0. Consequently, q_{δ_0} would be finite. This is a contradiction since $\delta_0 > \gamma_\infty$.

Lemma 3.10. *Assume that q_{γ_∞} is infinite. If, for every $x \in S$, $B(x, \gamma_\infty) \subseteq \overline{S}$, then*

$$\overline{S} = \cap_{k \geq 0} \left(\cup_{x \in S_{\gamma_k}} B(x, \gamma_k) \right). \tag{3.4}$$

Proof. Without any hypothesis, we have by (2.16): $\overline{S} \subseteq \cap_{k \geq 0} \left(\cup_{x \in S_{\gamma_k}} B(x, \gamma_k) \right)$. Let us prove the reverse containment. Let $y \in \cap_{k \geq 0} \left(\cup_{x \in S_{\gamma_k}} B(x, \gamma_k) \right)$; we have to prove that $y \in \overline{S}$. By hypothesis, $\cup_{x \in S} B(x, \gamma_\infty) \subseteq \overline{S}$, thus we may also assume that $y \notin \cup_{x \in S} B(x, \gamma_\infty)$. For each $k \geq 0$, there is an $x_k \in S$ such that $v(y - x_k) \geq \gamma_k$. Moreover, by the particular choice of y, $\gamma_k \leq v(y - x_k) < \gamma_\infty$. Consequently, we may extract from the sequence $\{x_k\}_{k \geq 0}$ a subsequence $\{x_{k_n}\}_{n \geq 0}$ such that $\gamma_{k_n} \leq v(y - x_k) < \gamma_{k_{n+1}} \leq v(y - x_{k_{n+1}}) < \gamma_\infty$. The sequence $\{v(y - x_{k_n})\}_{n \geq 0}$ is strictly increasing with limit γ_∞. It follows from Proposition 2.4 that y belongs to $\overline{S}$. □

Proof. of Theorem 3.5. (1) ⇒ (3) follows from Lemmas 3.2, 3.7 and 3.8.

(3) ⇒ (4): by hypothesis we have the containments

$$S \subseteq \cup_{x \in S_{\gamma_l}} B(x, \gamma_l) \subseteq \overline{S}$$

and, when q_{γ_∞} is finite ($\gamma_\infty = \gamma_l$), we obtain (3.3) since a finite union of closed balls is polynomially closed.

(4) ⇒ (2): if q_{γ_∞} is finite, $\overline{S}$ is clearly regular. If q_{γ_∞} is infinite, the fact that $T(S)$ is regular implies that we have the hypotheses of Lemma 3.7, and consequently, those of Lemma 3.10. Thus, we have (3.4) which shows that $\overline{S}$ is entirely characterized by $T(S)$, and the regularity of $T(S)$ implies the regularity of $\overline{S}$.

(2) ⇒ (1): Since $\overline{S}$ is regular, $T(S)$ is regular. When q_{γ_∞} is infinite, we may conclude by Corollary 3.3. Assume now that q_{γ_∞} is finite and that S is not preregular, that is, that there exist $x \in S$ and $\delta > \gamma_\infty$ such that $S(x, \gamma_\infty)$ contains only finitely many distinct classes modulo δ: there exist $s \in \mathbb{N}$ and $y_1, \ldots, y_s \in S$ such that $S(x, \gamma_\infty) \subseteq \cup_{i=1}^s B(y_j, \delta)$. On the other hand, $\overline{S}$ is regular, so that by the previous proof, $B(x, \gamma_\infty) \subseteq \overline{S}$. Then,

$$B(x, \gamma_\infty) = \overline{S} \cap B(x, \gamma_\infty) = \overline{S(x, \gamma_\infty)} \subseteq \cup_{i=1}^s B(y_j, \delta) \subset B(x, \gamma_\infty).$$

This is a contradiction because the last contaiment is strict. Indeed, since γ_∞ is finite, either v is not discrete or the residue field is infinite. □

In order to obtain results between partial properties, we also define:

Definition 3.11. The infinite fractional subset S of K is said to be a *preregular subset* of order $\gamma_h \leq \gamma_\infty$ if the tree $T(S)$ is regular for each level $k < h$: for each level $0 \leq k < h$, the number of edges joining a fixed vertex x at level k to a vertex at level $k + 1$ does not depend on x.

4. VWD sequences

4.1. Definitions.

In her study of regular compact subsets S of local fields, Amice [1] introduced the notion of a 'suite très bien répartie' of S. We consider such a notion for any fractional subset S of a valued field K:

Definition 4.1. A sequence $\{a_n\}_{n\in\mathbb{N}}$ of elements of S is a *very well distributed sequence* of S (shortly, a *VWD sequence* of S) if, for every $\gamma \in \mathbb{R}$ and every $s \in \mathbb{N}$, the following q_γ consecutive elements, $a_{sq_\gamma}, \ldots, a_{(s+1)q_\gamma-1}$, are in distinct classes modulo γ.

Another way to say that the sequence $\{a_n\}_{n\in\mathbb{N}}$ is a VWD sequence of S is:

$$\forall \gamma \in \mathbb{R} \ \ \forall n \neq m \in \mathbb{N} \quad \left(\left[\frac{n}{q_\gamma}\right] = \left[\frac{m}{q_\gamma}\right] \ \Rightarrow \ v(a_n - a_m) < \gamma \right). \tag{4.1}$$

Remark 4.2.

(1) If q_γ is finite, the condition in Definition 4.1 means that the q_γ elements form a complete system of representatives of S *mod* γ. Obviously for the γ's such that q_γ is finite, we just have to check the condition for the critical valuations γ_k.

(2) If q_γ is infinite, the condition means that all the elements of the sequence are in distinct classes modulo γ.

(3) If q_{γ_∞} is infinite, the condition for the γ's such that q_γ is infinite means that, for all $n \neq m$, $v(a_n - a_m) < \gamma_\infty$. Nevertheless, this inequality is a consequence of the condition for the γ's such that q_γ is finite: since $q_{\gamma_\infty} = \lim_k q_{\gamma_k}$, whatever $n < m$, there exists k such that $m - n < q_{\gamma_k}$, and hence, $v(a_m - a_n) < \gamma_k < \gamma_\infty$.

(4) If q_{γ_∞} is finite, the condition for the γ's such that q_γ is infinite implies that, for every $\delta > \gamma_\infty = \gamma_l$, one has $v(a_n - a_m) < \delta$. Since $\gamma_l = \inf\{\delta \mid \delta > \gamma_l\}$, the condition is equivalent to: $v(a_n - a_m) \leq \gamma_l$ for all $n \neq m$.

We are going to state links between VWD sequences and regular subsets. In order to understand clearly what happens, we still introduce a partial property:

Definition 4.3. Let $\delta \in \mathbb{R}$ and $N \in \mathbb{N}$. One says that a sequence $\{a_n\}_{0\leq n<N}$ of elements of S is *a VWD sequence of order* δ *and length* N if

$$\forall \gamma \leq \delta \ \ \forall n < m < N \quad \left\{ \left[\frac{n}{q_\gamma}\right] = \left[\frac{m}{q_\gamma}\right] \ \Rightarrow \ v(a_n - a_m) < \gamma \right\}. \tag{4.2}$$

We will speak of a *VWD sequence of order* δ when $N = q_\delta$.

If q_{γ_∞} is infinite, a sequence is a VWD sequence of S if and only if it is a VWD sequence of S of order δ for every $\delta < \gamma_\infty$.

If q_{γ_∞} is finite, a sequence is a VWD sequence of S if and only if it is a VWD sequence of S of order $\gamma_l = \gamma_\infty$ and, for all $n \neq m$, $v(a_n - a_m) \leq \gamma_l$.

4.2. VWD sequences and characteristic function.

Lemma 4.4. *Let* $\delta \leq \gamma_\infty$ *and* $N < q_\delta$. *If there exists a sequence* $\{a_n\}_{0\leq n<N}$ *of elements of* S *which is a VWD sequence of order* δ *and length* $N \leq q_\delta$, *then the characteristic function* w_S *of* S *satisfies:*

$$\forall n < N \quad w_S(n) = v\left(\prod_{m=0}^{n-1}(a_n - a_m)\right) = n\gamma_0 + \sum_{h\geq 1}\left[\frac{n}{q_{\gamma_h}}\right](\gamma_h - \gamma_{h-1}). \tag{4.3}$$

In particular, the sequence $\{a_n\}_{0\leq n<N}$ *is the beginning of a* v*-ordering of* S.

Note that the symbol $\sum_{h\geq 1}$ means that $h \in \mathbb{N}^*$ when the sequence of critical valuations is infinite and that $1 \leq h \leq l$ when the sequence is finite ($q_{\gamma_\infty} = q_{\gamma_l}$).

Proof. Let $\{a_n\}_{0\leq n<N}$ be a VWD sequence of order δ and length $N \leq q_\delta$ of S. Let $\gamma \leq \delta$ and fix an $n < N$. Let x be any element of S. It follows from (4.2) that there are at least $\left[\frac{n}{q_\gamma}\right]$ elements a_m with $m < n$ such that $v(x - a_m) \geq \gamma$. Consequently, $v\left(\prod_{m=0}^{n-1}(x - a_m)\right) \geq n\gamma_0 + \sum_{h\geq 1}\left[\frac{n}{q_{\gamma_h}}\right](\gamma_h - \gamma_{h-1})$. On the other hand, for every γ, there are exactly $\left[\frac{n}{q_\gamma}\right]$ elements a_m with $m < n$ such that $v(a_n - a_m) \geq \gamma$. Consequently,

$$v\left(\prod_{m=0}^{n-1}(a_n - a_m)\right) = n\gamma_0 + \sum_{h\geq 1}\left[\frac{n}{q_{\gamma_h}}\right](\gamma_h - \gamma_{h-1}) \leq v\left(\prod_{m=0}^{n-1}(x - a_m)\right). \tag{4.4}$$

Then (4.4) shows that the sequence $\{a_n\}_{0\leq n<N}$ is the beginning of a v-ordering of S, and hence, we have (4.3). □

This lemma has an obvious consequence:

Corollary 4.5. *If S admits a VWD sequence $\{a_n\}_{n\in\mathbb{N}}$, then*
1. *the sequence $\{a_n\}_{n\in\mathbb{N}}$ is a v-ordering of S,*
2. *the characteristic function w_S of S satisfies:*

$$\forall n \in \mathbb{N} \quad w_S(n) = v\left(\prod_{m=0}^{n-1}(a_n - a_m)\right) = n\gamma_0 + \sum_{h\geq 1}\left[\frac{n}{q_{\gamma_h}}\right](\gamma_h - \gamma_{h-1}). \tag{4.5}$$

4.3. **VWD sequences and regular subsets.**

We prove first that the existence of VWD sequences implies some divisibility properties for the q_γ.

Lemma 4.6. *If S admits a VWD sequence of order δ and length $2q_\delta$ then, for every $\gamma < \delta$, q_γ divides q_δ.*

Proof. We may assume that q_δ is finite. Let $\{a_n\}_{n\in\mathbb{N}}$ be a VWD sequence of order δ and length $2q_\delta$. We have to prove that, for all $\gamma < \delta$, q_γ divides q_δ. In each sequence $a_{iq_\delta}, \ldots, a_{(i+1)q_\delta - 1}$, there is a complete system of representatives of S *mod* δ. Thus, for every $x \in S$,

$$\begin{aligned} Card(S(x,\gamma)\ mod\ \delta) &= Card\{a_n \mid 0 \leq n < q_\delta,\ v(x - a_n) \geq \gamma\} \\ &= Card\{a_n \mid q_\delta \leq n < 2q_\delta,\ v(x - a_n) \geq \gamma\}. \end{aligned}$$

Consequently, $Card\{a_n \mid 0 \leq n < 2q_\delta,\ v(x - a_n) \geq \gamma\}$ is even.

Assume that q_γ does not divide q_δ and write $2q_\delta = sq_\gamma + r$ with $0 \leq r < q_\gamma$. Suppose that $r = 0$, then s has to be odd. So, for every $x \in S$, there are s representatives of x mod γ in $\{a_0, \ldots, a_{2q_\delta - 1}\}$. We have a contradiction.
Suppose now that $r \neq 0$. Once again, there is some $y \in S$ with an odd number of representatives mod γ in $\{a_0, \ldots, a_{2q_\delta - 1}\}$, as there is at least one element with s representatives, and another with $s + 1$ representatives. We still have a contradiction. We can conclude that q_γ divides q_δ. □

Corollary 4.7. *When S admits a VWD sequence of order γ_h and length $2q_{\gamma_h}$, then S is preregular of order γ_h.*

Proof. We just have to notice that, for every $k \leq h$, $a_0, \ldots, a_{q_{\gamma_k}-1}$ is a complete system of representatives of S *mod* γ_k and contains exactly $\frac{q_{\gamma_k}}{q_{\gamma_{k-1}}}$ complete systems of representatives of S *mod* γ_{k-1}. □

Remark 4.8. The following example shows that the condition on the length is sharp. Let $V=\mathbb{R}[X]_{(X)}$, then $\mathfrak{M}=XV$ and $V/\mathfrak{M}\simeq\mathbb{R}$ is infinite. Let S be the subset formed by the classes modulo X^4 of $0, X, X^2, X+X^3$ and X^2+X^3. Then, $\gamma_0=1, q_0=1\,;\gamma_1=2, q_1=2\,;\gamma_2=3, q_2=3\,;\gamma_3=4=\gamma_\infty, q_{\gamma_\infty}=5$. The sequence $(0, X, X^2, X+X^3, X^2+X^3)$ is VWD of order γ_∞, but we do not have the condition of divisibility for the q_i's. We can also see that this set is not regular.

Corollary 4.9. *Assume that S admits a VWD sequence of order γ_∞. Then,*

(1) *when $q_{\gamma_\infty}=+\infty$, S is a preregular subset,*
(2) *when $q_{\gamma_\infty}<+\infty$, S is a preregular subset of order $\gamma_\infty=\gamma_l$ if and only if q_{γ_l-1} divides q_{γ_l}.*

Proof. Thanks to Lemma 4.6, when q_{γ_∞} is infinite, for every $k\geq 0$, q_{γ_k} divides $q_{\gamma_{k+1}}$. Denote by $\{a_n\}$ a VWD sequence of order γ_∞ of S. For every $\delta\leq\gamma_\infty$, $\{a_n\}_{0\leq n<q_\delta}$ is a complete of representatives of S modulo δ and for every $\gamma<\delta$, it contains $\left[\frac{q_\delta}{q_\gamma}\right]$ complete sets of residues of S modulo γ : the tree $T(S)$ is regular. When q_{γ_∞} is finite, the proof is very similar. □

As a consequence, we have:

Proposition 4.10. *If S admits a VWD sequence then S is a preregular subset.*

Proof. We only have to see what happens when $q_{\gamma_\infty}=q_{\gamma_l}<+\infty$. By Corollary 4.9, we know that the tree $T(S)$ is regular. Let $\{a_n\}$ be a VWD sequence of S. Then, for every $s\in\mathbb{N}$, $a_{sq_{\gamma_l}},\ldots,a_{(s+1)q_{\gamma_l}-1}$, are in distinct classes modulo γ_l. Hence, for every $0\leq n<q_{\gamma_l}$, the infinite set $T_n=\{a_{n+sq_{\gamma_l}}\mid s\geq 1\}$ is in $S(a_n,\gamma_l)$. In fact, by Remark 4.2.(4),

$$T_n\subseteq S(a_n,\gamma_l)\cap\overset{\bullet}{B}(a_n,\gamma_l). \tag{4.6}$$

So that, for every $n<q_{\gamma_l}$ and every $\delta>\gamma_l$, the S-ball $S(a_n,\gamma_l)$ contains infinitely many non-empty S-balls $S(a_m,\delta)$ where $a_m\in T_n$. □

Conversely,

Proposition 4.11. *Every preregular subset S admits a VWD sequence of order γ_∞. In particular, a preregular subset such that q_{γ_∞} is infinite admits a VWD sequence.*

Proof. We start with the sets of representatives S_{γ_k} of S *mod* γ_k chosen in such a way that, for every $k\geq 0$, $S_{\gamma_k}\subset S_{\gamma_{k+1}}$ (see (2.11)). We construct a sequence $\{b_n\}_{0\leq n<q_{\gamma_\infty}}$ by induction on k. The b_n's with $n<q_{\gamma_k}$ are the elements of S_{γ_k} ordered in the following way. We begin with $S_{\gamma_0}=\{b_0\}$. Assuming that the elements of S_{γ_k} are ordered in a sequence $\{b_0,\ldots,b_{q_{\gamma_k}-1}\}$, at the $(k+1)$-th step, we have to define b_n for $q_k\leq n<q_{\gamma_{k+1}}$. We know that $S_{\gamma_{k+1}}$ is formed by $\alpha_k=\frac{q_{\gamma_{k+1}}}{q_{\gamma_k}}$ complete systems of representatives of S *mod* γ_k, one of them being S_{γ_k}. For each $s\in\{2,\ldots,\alpha_k\}$, we choose for $b_{sq_{\gamma_k}},\ldots,b_{(s+1)q_{\gamma_k}-1}$ one of these systems and each of these systems is ordered in the following way:

$$\text{for } 2\leq s\leq\alpha_k \text{ and } 0\leq j<q_{\gamma_k}\quad v\left(b_{sq_{\gamma_k}+j}-b_j\right)\geq\gamma_k. \tag{4.7}$$

Note that, if q_{γ_∞} is finite, the lemma is proved and, if q_{γ_∞} is infinite, the lemma is proved also, because, for all $n<m$, there exists $\gamma<\gamma_\infty$ such that $m<q_\gamma$, and hence, $v(a_n-a_m)<\gamma<\gamma_\infty$. □

5. Legendre subsets

5.1. **Definitions.**

In the case where $S = V = \mathbb{Z}_p$ for some prime number p, one has $\gamma_k = k$ and $q_{\gamma_k} = p^k$, and then, Formula (4.5) leads to Legendre's formula for the p-adic valuation of $n!$:

$$v_p(n!) = \sum_{k\geq 1} \left[\frac{n}{p^k}\right]. \tag{5.1}$$

This formula was extended by Pólya [14] to the case where $S = V$ is the ring of a discrete valuation v with finite residue field with cardinality q :

$$v(n!_V) = \sum_{k\geq 1} \left[\frac{n}{q^k}\right] \tag{5.2}$$

More recently, a subset S of a discrete valuation domain V with maximal ideal $\mathfrak{M}$ whose characteristic function satisfies the following formula:

$$w_S(n) = v(n!_S) = \sum_{k\geq 1} \left[\frac{n}{q_k}\right], \tag{5.3}$$

where q_k denotes the number of classes of S modulo $\mathfrak{M}^k$, was called a Legendre subset in [11]. We generalize this definition:

Definition 5.1. The subset S is called a *Legendre subset* of K if its characteristic function w_S satisfies the following generalized Legendre formula:

$$\forall n \in \mathbb{N} \quad w_S(n) = v(n!_S) = n\gamma_0 + \sum_{k\geq 1} \left[\frac{n}{q_{\gamma_k}}\right] (\gamma_k - \gamma_{k-1}). \tag{5.4}$$

With this definition, it follows from Lemma 4.4 that a subset that has a VWD sequence is a Legendre subset. Let us prove some partial converses of this assertion. At this point, we do not even know if a Legendre subset admits a v-ordering. Nevertheless, we can use v-orderings modulo ε which always exist:

Definition 5.2. [8]. Let $\varepsilon > 0$ and $N \in \mathbb{N} \cup \{\infty\}$. A *$v$-ordering of S modulo ε of length N* is a sequence $\{a_n\}_{0\leq n<N}$ of distinct elements of S such that, for each $0 \leq n < N$, one has:

$$v\left(\prod_{k=0}^{n-1}(a_n - a_k)\right) \leq \inf_{x\in S}\; v\left(\prod_{k=0}^{n-1}(x - a_k)\right) + \varepsilon. \tag{5.5}$$

Such sequences have the following property:

Lemma 5.3. [8] *Let $\{a_n\}_{0\leq n<N}$ be a v-ordering of S modulo ε of length N . Then for each $n < N$:*

$$v\left(\prod_{k=0}^{n-1}(a_n - a_k)\right) - \varepsilon \leq w_S(n) \leq v\left(\prod_{k=0}^{n-1}(a_n - a_k)\right) + n\varepsilon. \tag{5.6}$$

In order to see what happens step by step, we give a new definition:

Definition 5.4. The subset S is called a *Legendre subset of length* N if its characteristic function w_S satisfies partially the generalized Legendre formula:

$$\forall n < N, \quad w_S(n) = v(n!_S) = n\gamma_0 + \sum_{k\geq 1} \left[\frac{n}{q_{\gamma_k}}\right](\gamma_k - \gamma_{k-1}). \tag{5.7}$$

5.2. **Legendre subsets and VWD sequences.**

Lemma 5.5. *A Legendre subset of length q_{γ_m} admits a VWD sequence of order γ_m.*

Proof. Assume that S is a Legendre subset of length q_{γ_m}. Fix $\varepsilon > 0$ and consider a sequence $\{a_n\}_{0\leq n<N}$ that is a v-ordering of S modulo ε and length $N = q_{\gamma_m}$. Suppose that $\{a_n\}_{0\leq n<N}$ is not VWD of order γ_m and length N.

Since every sequence of length N of elements of S is a VWD sequence of order γ_0 and length N, there exists a least integer r such that $1 \leq r \leq m$ and $\{a_n\}_{0\leq n<N}$ is not a VWD sequence of order γ_r and length N. Then, $\{a_n\}_{0\leq n<N}$ is a VWD sequence of order γ_{r-1} and length N.

Now, there is a least integer $s \leq N$ such that there exists $j < s$ with:

$$\left[\frac{j}{q_{\gamma_r}}\right] = \left[\frac{s}{q_{\gamma_r}}\right] \quad \text{and} \quad v(a_s - a_j) \geq \gamma_r.$$

For every h such that $0 \leq h \leq m$, let c_h denote the following cardinality:

$$c_h = \mathrm{Card}\{i \mid 0 \leq i < s, v(a_s - a_i) \geq \gamma_h\}.$$

We then have

$$v\left(\prod_{k=0}^{s-1}(a_s - a_k)\right) = \sum_{0\leq h\leq m} \gamma_h(c_h - c_{h+1}).$$

As $\{a_n\}_{0\leq n<N}$ is a VWD sequence of order γ_{r-1} and length N, we have:

$$c_h = \left[\frac{s}{q_{\gamma_h}}\right] \quad \text{for} \quad 0 \leq h \leq r-1.$$

Moreover, by definition of s, we also have

$$c_r = 1 + \left[\frac{s}{q_{\gamma_r}}\right].$$

Let us prove now that:

$$c_h \geq \left[\frac{s}{q_{\gamma_h}}\right] \quad \text{for} \quad r+1 \leq h \leq m.$$

If $\left[\frac{s}{q_{\gamma_h}}\right] = 0$, there is nothing to prove. Thus, assume that $\left[\frac{s}{q_{\gamma_h}}\right] \geq 1$ and also that $c_h < \left[\frac{s}{q_{\gamma_h}}\right]$. So there exists $\lambda \leq \left[\frac{s}{q_{\gamma_h}}\right] - 1$ such that the sequence $a_{\lambda q_{\gamma_h}}, a_{\lambda q_{\gamma_h}+1}, \ldots, a_{(\lambda+1)q_{\gamma_h}-1}$ does not contain any term congruent to a_s modulo γ_h. As there are q_{γ_h} terms in this sequence, it does contain at least 2 terms a_{s_0} and a_{j_0} such that $j_0 < s_0 < s$ and $v(a_{s_0} - a_{j_0}) \geq \gamma_h$. This is a contradiction with the choice of s. We then have:

$$v\left(\prod_{k=0}^{s-1}(a_s - a_k)\right) \geq s\gamma_0 + \sum_{h=1}^{m} \left[\frac{s}{q_{\gamma_h}}\right](\gamma_h - \gamma_{h-1}) + (\gamma_r - \gamma_{r-1}).$$

And, as S is a Legendre set of length q_{γ_m},

$$w_S(s) = s\gamma_0 + \sum_{h=1}^{m} \left[\frac{s}{q_{\gamma_h}}\right] (\gamma_h - \gamma_{h-1}).$$

Moreover, by Lemma 5.3, we have:

$$v\left(\prod_{k=0}^{s-1}(a_s - a_k)\right) - \varepsilon \leq w_S(s).$$

Hence,

$$\varepsilon \geq \gamma_r - \gamma_{r-1}.$$

Choosing then

$$\varepsilon < \min_{1\leq k\leq m} (\gamma_k - \gamma_{k-1}),$$

would lead to a contradiction. And hence, $\{a_n\}$ is a VWD sequence of order γ_m. $\square$

Proposition 5.6. *Every Legendre subset of length q_{γ_∞} admits a VWD sequence of order γ_∞.*

Proof. If q_{γ_∞} is finite, $\gamma_\infty = \gamma_l$, and Lemma 5.5 gives the proposition. We assume that q_{γ_∞} is infinite, and we construct a VWD sequence $(a_n)_{0\leq n<q_{\gamma_\infty}}$ of order γ_∞ by constructing inductively sequences $\{a_n^m\}_{0\leq n<q_{\gamma_m}}$ that are VWD sequences of order γ_m such that $a_n^m = a_n^{m-1}$ for $0 \leq n < q_{\gamma_{m-1}}$.

Fix $a_0^0 \in S$. At the 0-th step, we have the sequence $\{a_0^0\}$ of length $q_0 = 1$. Suppose that at the m-th step we have a sequence $\{a_n^m\}_{0\leq n<q_{\gamma_m}}$. which is a VWD sequence of order γ_m extending the previous one. By Lemma 5.5, there exists a sequence $\{b_n\}_{0\leq n<q_{\gamma_{m+1}}}$ that is VWD of order γ_{m+1}. Then, $\{b_n \mid 0 \leq n < q_{\gamma_m}\}$ is a complete set of representatives of S modulo γ_{m+1} which contains $\alpha_m = \left[\frac{q_{\gamma_{m+1}}}{q_{\gamma_m}}\right]$ complete sets of residues of S modulo γ_m. Moreover,

$$\forall i \in \{0, \ldots, q_{\gamma_m} - 1\}\ \exists j \in \{0, q_{\gamma_{m+1}} - 1\} \text{ such that } v(a_i^m - b^j) \geq \gamma_{m+1}.$$

Replacing these b_j's by the corresponding a_i^m's ,we still have a complete set of residues of S mod γ_{m+1}. Then, by changing the places of this new sequence, we may obtain a sequence $\{a_n^{m+1}\}_{0\leq n<q_{\gamma_{m+1}}}$ which is VWD of order γ_{m+1} and which extends the sequence $\{a_n^m\}_{0\leq n<q_{\gamma_m}}$.

Finally, the sequence $\{a_n\}_{0\leq n<q_{\gamma_\infty}}$ is defined in the following way: for every $n < q_{\gamma_\infty}$ there is some m such that $n < q_{\gamma_m}$ and we put $a_n = a_n^m$. $\square$

5.3. **The equivalences.**

Putting together the previous results we obtain:

Theorem 5.7. *Let S be a fractional subset of V such that $q_{\gamma_\infty} = +\infty$, the following assertions are equivalent:*

1. *S is a Legendre subset,*
2. *S has a VWD sequence,*
3. *S is a preregular subset.*

Proof.
(1) $\Rightarrow$ (2) by Proposition 5.6,
(2) $\Rightarrow$ (1) by Lemma 4.5,
(3) $\Rightarrow$ (2) by Lemma 4.11,

(2) ⇒ (3) by Lemma 4.10.

□

Theorem 5.8. *Let S be a fractional subset of V such that $q_{\gamma_l} = q_{\gamma_\infty} < +\infty$ and assume that $q_{\gamma_{l-1}}$ divides q_{γ_l}. Then the following assertions are equivalent:*

1. *S is a Legendre subset of length q_{γ_∞},*
2. *S has a VWD sequence of order γ_∞,*
3. *S is a preregular subset of order γ_∞.*

Proof.
(1) ⇒ (2) by Proposition 5.6,
(2) ⇒ (1) by Lemma 4.4,
(3) ⇒ (2) by Lemma 4.11,
(2) ⇒ (3) by Lemma 4.9.

□

Remark 5.9. The previous theorem does not speak about the existence of infinite VWD sequences or v-orderings of S. Recall that, if q_{γ_∞} is infinite, then necessarily either v is not discrete, or $V/\mathfrak{M}$ is infinite. Let us consider two very simple examples in the case where v is not discrete: $S = \mathfrak{M} = B(0,1)$ and $T = V = B(0,0)$. Then, $\mathrm{Int}(S,V) = \mathrm{Int}(T,V) = V[X]$ and $\overline{S} = T$. A v-ordering $\{a_n\}_{n\in\mathbb{N}}$ of S (resp., T) is then a sequence of elements a_n of $\mathfrak{M}$ (resp., V) such that $v(a_n - a_m) = 0$ for all $n \neq m$. Consequently, T admits an infinite v-ordering if and only if $V/\mathfrak{M}$ is infinite, while S never admits a v-ordering. It is then more natural to consider the existence of v-orderings for $\overline{S}$ than for S. With respect to this question we have the following theorem:

Theorem 5.10. *Let S be a fractional subset of V such that $q_{\gamma_l} = q_{\gamma_\infty} < +\infty$. The following assertions are equivalent:*

1. *$\overline{S}$ admits a VWD sequence,*
2. *$\overline{S}$ is regular and admits a v-ordering,*
3. *S is preregular and $V/\mathfrak{M}$ is infinite.*

Proof.
(1) ⇒ (2) : Assume that $\overline{S}$ has a VWD sequence. Then, by Proposition 4.10, $\overline{S}$ is preregular, and hence, regular by Theorem 3.5. The existence of a v-ordering of $\overline{S}$ follows from Corollary 4.5.
(2) ⇒ (3) : If $\overline{S}$ is regular then, by Theorem 3.5, S is preregular and $\overline{S}$ is a finite union of balls of the form $B(x,\gamma_l)$. By [5, Lemma 3.4], the sequence (finite or infinite) formed by the elements of a v-ordering of $\overline{S}$ which are in $B(x,\gamma_l)$ form (the beginning of) a v-ordering of $B(x,\gamma_l)$. At least, one of these subsequences is infinite. Consequently, some $B(x,\gamma_l)$, and hence, $B(0,0) = V$ itself admits an infinite v-ordering. By Remark 5.9, $V/\mathfrak{M}$ is necessarily infinite.
(3) ⇒ (1) : By Lemma 4.11, S admits a VWD sequence of order γ_∞. Let $\{a_n\}_{0\leq n<q_{\gamma_\infty}}$ be such a sequence. We want to extend this finite sequence in an infinite VWD sequence of $\overline{S}$.

We know also that $\overline{S}$ is a finite union of balls of the form $B(x,\gamma_\infty)$. Since $V/\mathfrak{M}$ is infinite, there exists an infinite sequence $\{u_m\}_{m\geq 0}$ of elements of V such that $u_0 = 0$ and, for all $m \neq m', v(u_m - u_{m'}) = 0$. Let $t \in V$ be such that , $v(t) = \gamma_\infty$. Now, we define a_n for $n \geq q_{\gamma_\infty}$ in the following way:

$$\text{for } n = mq_{\gamma_\infty} + r \text{ with } r < q_{\gamma_\infty}, \text{ let } a_n = a_r + tu_m. \tag{5.8}$$

Clearly, the a_n's are in $\overline{S}$ since the balls $B(a_r, \gamma_\infty)$ are contained in $\overline{S}$. Let us verify that the sequence $\{a_n\}_{n\geq 0}$ is a VWD sequence. Let $n, n' \in \mathbb{N}$ with $n \neq n'$ and write $n = mq_{\gamma_\infty} + r$ and $n' = m'q_{\gamma_\infty} + r'$ with $r, r' < q_{\gamma_\infty}$. Then,

$$a_n - a_{n'} = (u_m - u_{m'})t + a_r - a_{r'}\ ,\ v((u_m - u_{m'})t) = \gamma_\infty \text{ and } v(a_r - a_{r'}) < \gamma_\infty \text{ if } r \neq r'.$$

Consequently,

$$v(a_n - a_{n'}) \begin{cases} = \gamma_\infty & \text{if } r = r' \\ < \gamma_\infty & \text{if } r \neq r'. \end{cases}$$

Thus, for every $\delta > \gamma_\infty$, $v(a_n - a_{n'}) < \delta$. For every $\gamma \leq \gamma_\infty$, if $\left[\frac{n}{q_\gamma}\right] = \left[\frac{n'}{q_\gamma}\right]$, then $\left[\frac{r}{q_\gamma}\right] = \left[\frac{r'}{q_\gamma}\right]$ since q_γ divides q_{γ_∞}, and hence, $v(a_n - a'_n) = v(a_r - a'_r) < \gamma$ by hypothesis on the first terms of the sequence. The sequence is then a VWD sequence of $\overline{S}$. □

6. VWDWO-sequences and strong v-orderings

In this section, we show that, perhaps after reordering the elements, the VWD sequences may have stronger properties.

6.1. **VWDWO sequences.**

We consider first the following generalization of a property introduced by E. Helsmoortel [12] for the valuation domain of a local field:

Definition 6.1. A sequence $\{a_n\}_{n\in\mathbb{N}}$ of elements of S is a *VWDWO sequence* of S if, for every $\gamma \in \mathbb{R}$, any subsequence formed by q_γ consecutive elements are in distinct classes modulo γ.

Clearly, a sequence $\{a_n\}_{\in\mathbb{N}}$ is a VWDWO sequence of S if and only if, for every $s \geq 0$, $\{a_n\}_{n\geq s}$ is a VWD sequence of S. If $S = V$ is a discrete valuation domain with finite residue field of cardinality q, then $q_k(V) = q^k$ and we know the q-adic pseudo-valuation ν_q on $\mathbb{Z}$ defined by:

$$\nu_q(n) = \max\{k \in \mathbb{N} : q^k | n\}.$$

We generalize this pseudo-valuation in the following way:

Definition 6.2. The *pseudo-valuation on $\mathbb{Z}$ associated to S* is the function ν_S defined, for $n \neq 0$, by:

$$\nu_S(n) = \max\{\gamma \in \mathbb{R} :\ q_\gamma | n\}.$$

The value $\nu_S(n)$ is of the form γ_k for some $k \in \mathbb{N}$. By convention, let $\nu_S(0) = +\infty$. Note that, in general, ν_S is not be a valuation since for the sum we have

$$\nu_S(n+m) \geq \min(\nu_S(n), \nu_S(m))$$

while, for the product, we only have an inequality:

$$\nu_S(nm) \geq \max(\nu_S(n), \nu_S(m))$$

Proposition 6.3. *Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of elements of S. The following assertions are equivalent:*

(1) $\{a_n\}_{n\in\mathbb{N}}$ *is a* VWDWO sequence *of S,*
(2) $\forall n \neq m \in \mathbb{N},\ \ q_\gamma | n - m \ \Leftrightarrow\ v(a_n - a_m) \geq \gamma,$
(3) $\forall n \neq m \in \mathbb{N},\ \ v(a_n - a_m) = \nu_S(n - m).$

Proof. (1) $\Rightarrow$ (2) : Assume (1) and consider $n \neq m \in \mathbb{N}$ and $\gamma \in \mathbb{R}$. Suppose first that $q_\gamma \mid n-m$. The subsequences $a_n, a_{n+1}, \ldots, a_{n+q_\gamma-1}$ and $a_{n+1}, \ldots, a_{n+q_\gamma}$ form two complete sets of representatives of S *mod* γ, and hence, $v(a_n - a_{n+q_\gamma}) \geq \gamma$. Consequently, by iteration, for every $\lambda \in \mathbb{N}^*$, one has $v(a_n - a_{n+\lambda q_\gamma}) \geq \gamma$, so

$$q_\gamma | n-m \quad \Rightarrow \quad v(a_n - a_m) \geq \gamma.$$

Now, suppose that $q_\gamma \not| n-m$. If q_γ is finite, then $m = n + kq_\gamma + r$ where $k \in \mathbb{N}$ and $0 < r < q_\gamma$. We have $v(a_n - a_{n+kq_\gamma}) \geq \gamma$ and $a_{n+kq_\gamma}, \ldots, a_m, \ldots, a_{n+(k+1)q_\gamma-1}$ is a complete set of representatives of S *mod* γ. Necessarily, $v(a_n - a_m) < \gamma$. If q_γ is infinite then, by definition, (1) implies that $v(a_n - a_m) < \gamma$.

(2) $\Rightarrow$ (3) : Assume (2), then we have the equivalences:

$$[\nu_S(n-m) = \gamma_k] \Leftrightarrow [q_{\gamma_k} \mid n-m \text{ and } q_{\gamma_{k+1}} \not| n-m] \Leftrightarrow$$

$$[v(a_n - a_m) \geq \gamma_k \text{ and } v(a_n - a_m) < \gamma_{k+1}] \Leftrightarrow [v(a_n - a_m) = \gamma_k].$$

(3) $\Rightarrow$ (1) : Assume (2) and fix $\gamma \in \mathbb{R}$. If q_γ is infinite, then the a_n's are non-congruent modulo γ. If q_γ is finite, consider q_γ consecutive elements in the sequence: $a_n, \ldots, a_{n+q_\gamma-1}$. Then, for $n \leq i < j < n+q_\gamma$, $q_\gamma \not| j-i$, $\nu_S(j-i) < \gamma$, $v(a_j - a_i) < \gamma$, and hence, we have q_γ elements which are not congruent modulo γ, they form a complete set of representatives of S *mod* γ. □

Once more, for the convenience of the proofs, we introduce a partial property:

Definition 6.4. A sequence $\{a_n\}_{0 \leq n < N}$ of elements of S is said to be a *VWDWO sequence of S of order γ and length N* if the following equivalent assertions are satisfied:
(1) for every $\gamma \in \mathbb{R}$, any subsequence formed by q_γ (or less) consecutive elements are in distinct classes modulo γ.
(2)

$$\forall \gamma \in \mathbb{R} \quad \forall 0 \leq n < m < N \quad q_\gamma \text{ divides } n-m \;\Leftrightarrow\; v(a_n - a_m) \geq \gamma.$$

The fact that these two assertions are equivalent follows clearly from the proof of Proposition 6.3. The proposition below says that every VWD sequence may be reordered in such a way that it becomes a VWDWO sequence.

Proposition 6.5. *Let $\{b_n\}_{n\in\mathbb{N}}$ be a VWD sequence of S. Then, there exists a bijection $\varphi = \mathbb{N} \to \mathbb{N}$ such that the sequence $\{b_{\varphi(n)}\}_{n\in\mathbb{N}}$ is a VWDWO sequence of S. Moreover, we may assume that, for every $k \in \mathbb{N}$, φ induces a bijection from $\{b_{q_{\gamma_k}}, \ldots, b_{q_{\gamma_{k+1}-1}}\}$ onto itself.*

This will be a consequence of the following lemma.

Lemma 6.6. *For every VWD sequence $\{b_n\}_{0\leq n<q_{\gamma_k}}$ of S of order γ_k (and length q_{γ_k}), there exists a bijection φ_k from $\{0, 1, \ldots, q_{\gamma_k} - 1\}$ onto itself such that the sequence $\{b_{\varphi(n)}\}_{0\leq n<q_{\gamma_k}}$ is a VWDWO sequence of S of order γ_k (and length q_{γ_k}). Moreover, we may assume that, for every $h < k, \varphi_k$ induces a bijection from $\{b_{q_{\gamma_h}}, \ldots, b_{q_{\gamma_{h+1}-1}}\}$ onto itself.*

Proof. We define φ_k by induction on k. Let $\varphi_0(0) = 0$. Then, assume that φ_h is defined for some $h < k$. We want to define φ_{h+1}. For $0 \leq n < q_{\gamma_h}$, let $\varphi_{h+1}(n) = \varphi_h(n)$. We have to define $\varphi_{h+1}(n)$ for $q_{\gamma_h} \leq n < q_{\gamma_{h+1}}$. From now on, we write φ instead of φ_{h+1}.

Such an n is of the form:

$$n = sq_{\gamma_h} + r \text{ where } 0 \leq r < q_{\gamma_h} \text{ and } 1 \leq s \leq \alpha_h$$

(recall that $q_{\gamma_{h+1}} = \alpha_h q_{\gamma_h}$). Clearly, there exists one, and only one, j such that:

$$0 \leq j < q_{\gamma_h} \text{ and } v(b_{\varphi_h(j)} - b_n) = \gamma_h.$$

Then, let

$$\varphi^{-1}(n) = sq_{\gamma_h} + j, \text{ that is, } \varphi(sq_{\gamma_h} + j) = sq_{\gamma_h} + r.$$

Obviously, for every s such that $1 \leq s \leq \alpha_{h+1}$, φ induces a bijection from $\{sq_{\gamma_k}, \ldots, (s+1)q_{\gamma_k}-1\}$ onto itself. Let us prove now that the sequence $\{c_n\}_{0\leq n<q_{\gamma_{h+1}}}$ defined by:

$$c_n = b_{\varphi(n)}$$

is a VWDWO sequence of order γ_{h+1}. We have to compute $v(b_{\varphi(n_1)} - b_{\varphi(n_2)})$ for $0 \leq n_1 < n_2 < q_{\gamma_{k+1}}$.

If $n_2 < q_{\gamma_h}$, by induction hypothesis, $v(b_{\varphi(n_1)} - b_{\varphi(n_2)}) = \nu_S(n_1 - n_2)$. Suppose now that $n_2 \geq q_{\gamma_h}$, and write : $n_i = s_i q_{\gamma_h} + j_i$ and $\varphi(n_i) = s_i q_{\gamma_h} + r_i$. Then,

$$n_2 - n_1 = q_{\gamma_h}(s_2 - s_1) + (j_2 - j_1),$$

$$\nu_S(n_2 - n_1) = \nu_S(j_2 - j_1) < \gamma_h \text{ if } j_1 \neq j_2 \text{ and } \nu_S(n_2 - n_1) = q_{\gamma_h} \text{ otherwise.}$$

By construction,

$$v(b_{\varphi(n_i)} - b_{\varphi(j_i)}) = \gamma_h,$$

and, by induction hypothesis,

$$v(b_{\varphi(j_1)} - b_{\varphi(j_2)}) = \nu_S(j_2 - j_1) < \gamma_h \text{ for } j_1 \neq j_2.$$

Consequently,

— if $j_1 \neq j_2$, then $v(b_{\varphi(n_1)} - b_{\varphi(n_2)}) = v(b_{\varphi(j_1)} - b_{\varphi(j_2)}) = \nu_S(j_1 - j_2) = \nu_S(n_1 - n_2)$,

— if $j_1 = j_2$, then $v(b_{\varphi(n_1)} - b_{\varphi(n_2)}) \geq \gamma_h$ and $b_{\varphi(n_1)}, b_{\varphi(n_2)}$ are non-congruent mod γ_{h+1}, thus $v(b_{\varphi(n_1)} - b_{\varphi(n-2)}) = \gamma_h = \nu_S(n_1 - n_2)$.

∎

Proof. of Proposition 6.5. If $q_{\gamma_\infty} = +\infty$, the construction in Lemma 6.6 provides by induction an infinite sequence $(b_{\varphi(n)})$ which is a VWDWO sequence. When $q_{\gamma_\infty} = q_{\gamma_l} < +\infty$, the previous construction provides only a VWDWO sequence $\{b_{\varphi(n)}\}_{0\leq n<q_{\gamma_\infty}}$ of order γ_∞ and length q_{γ_∞}. Analogously to the proof of the lemma, we reorder the elements of the infinite VWD sequence $\{b_n\}_{n\geq 0}$ for $n \geq q_{\gamma_l}$: if $n = sq_{\gamma_l} + r$ with $s \geq 1$ and $0 \leq r < q_{\gamma_l}$, then there is one, and only one, $j < q_{\gamma_l}$ such that $v(b_n - b_{\varphi(j)}) \geq \gamma_l$. Then, let $\varphi^{-1}(n) = sq_{\gamma_l} + j$. The new sequences $\{b_{\varphi(n)}\}_{sq_{\gamma_l}\leq s<(s+1)q_{\gamma_l}}$ are VWDWO sequences of order γ_l and length q_{γ_l}. Then, it is easy to check that the whole sequence is a VWDWO sequence. ∎

6.2. **Strong v-orderings.**

Obviously, it follows from the fact that every VWD sequence of S is a v-ordering of S that, if $\{a_n\}_{n\geq 0}$ is a VWDWO sequence of S, then $\{a_n\}_{n\geq k}$ is still a v-ordering of S whatever the integer k. This leads us to the following definition:

Definition 6.7. A sequence $\{a_n\}_{n\in\mathbb{N}}$ of elements of S is called a *strong v-ordering* of S if, for every $k \geq 0$, the sequence $\{a_{k+n}\}_{n\in\mathbb{N}}$ is a v-ordering of S.

Obviously, there does not always exist such strong sequences. Here are the easiest examples.

Example 6.8. 1- The sequence $\{n\}_{n\in\mathbb{N}}$ in $\mathbb{Z}_p$. More generally:
2- Let V be a discrete valuation domain with finite residue field of cardinality q. Let $a_0 = 0, a_1, \ldots, a_{q-1}$ be a complete set of representatives of V modulo $\mathfrak{m}$ and let t be such that $v(t) = 1$. For $n = n_0 + n_1 q + \ldots + n_k q^k$ with $0 \leq n_i < q$, let $a_n = a_{n_0} + a_{n_1} t + \ldots + a_{n_k} t^k$. Then, the sequence $\{a_n\}_{n\geq 0}$ is a VWDWO sequence of V.
3- A valuation domain V with infinite residue field. Any infinite sequence of elements of V in distinct classes modulo the maximal ideal is a strong v-ordering.

Proposition 6.9. *Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of elements of S. The following assertions are equivalent:*

(1) *The sequence $\{a_n\}_{n\in\mathbb{N}}$ is a strong v-ordering of S.*
(2) *For every $k \geq 0$ and every $n > 0$,*
$$v(a_{k+n} - a_k) = w_S(n) - w_S(n-1).$$
(3) *$\{a_n\}_{n\in\mathbb{N}}$ is a v-ordering and, for every $k \geq 0$ and every $n > 0$,*
$$v(a_{k+n} - a_k) = v(a_n - a_0).$$

Proof. (1) $\Rightarrow$ (2) : For $k \geq 0$ and $n \geq 1$, one has:
$$a_{k+n} - a_k = \frac{\prod_{h=0}^{n-1}(a_{k+n} - a_{k+h})}{\prod_{h=1}^{n-1}(a_{k+n} - a_{k+h})}.$$
The valuation of the numerator is $w_S(n)$ since the sequence $a_k, a_{k+1}, \ldots, a_{k+n}$ is the beginning of a v-ordering of S and the valuation of the denominator is $w_S(n-1)$ since the sequence $a_{k+1}, \ldots, a_{k+n}$ is also the beginning of a v-ordering of S. Thus, $v(a_{k+n} - a_k) = w_S(n) - w_S(n-1)$.

(2) $\Rightarrow$ (3) is obvious.

(3) $\Rightarrow$ (1) : if $\{a_n\}_{n\geq 0}$ is a v-ordering of S, then the equalities in assertion (3) imply that, for each $k \geq 0$, $\{a_n\}_{n\geq k}$ is also a v-ordering of S. $\square$

Proposition 6.10. *A sequence of elements of S is a strong v-ordering of S if and only if it is a VWDWO sequence of S.*

Proof. We already know that a VWDWO sequence of S is a strong v-ordering of S. Conversely, let $\{a_n\}_{n\in\mathbb{N}}$ be a strong v-ordering of S. Since $v(a_{k+n} - a_k) = v(a_n - a_0)$, we just have to check that, for every $\gamma \in \mathbb{R}$, the following assertion holds:

(6.1) $$\forall\, n < q_\gamma \quad a_0, \ldots, a_n \text{ are in distinct classes modulo } \gamma.$$

Assume that (6.1) does not hold for some γ such that q_γ is finite and let $h > 0$ be the least k such that this is not true for γ_k: there exist i and j such that $0 \leq i < j < q_{\gamma_h}$ and $v(a_j - a_i) \geq \gamma_h$. Since $v(a_j - a_i) = v(a_{j-i} - a_0)$, there exists a least integer s such that $0 < s < q_{\gamma_h}$ and $v(a_s - a_0) \geq \gamma_h$. We are going to obtain a contradiction by proving that the subsequence $a_0, \ldots, a_s$ does not satisfy the inequalities of v-orderings.

We first prove that:

(6.2) $$\forall k < h\ \forall n \neq m \quad [v(a_n - a_m) \geq \gamma_k \ \Leftrightarrow\ q_{\gamma_k} | n - m]\,.$$

Let $k < h$. Then, by hypothesis, $0 \leq n < m < q_{\gamma_k}$ implies $v(a_m - a_n) < \gamma_k$. Moreover, there exists $j < q_{\gamma_k}$ such that $v(a_{q_{\gamma_k}} - a_j) \geq \gamma_k$, and hence, $v(a_{q_{\gamma_k}-j} - a_0) \geq \gamma_k$, thus $j = 0$, that is, $v(a_{q_{\gamma_k}} - a_0) \geq \gamma_k$.

Let $0 \leq n < m$ be such that $q_{\gamma_k} | (m-n)$. Then, $m - n = rq_{\gamma_k}$ and

$$v(a_m - a_n) = v(a_{rq_{\gamma_k}} - a_0) \geq \inf_{1 \leq s \leq r} v(a_{sq_{\gamma_k}} - a_{(s-1)q_{\gamma_k}}) = v(a_{q_{\gamma_k}} - a_0) \geq \gamma_k.$$

Now, let $0 \leq n < m$ be such that $q_{\gamma_k} \not| m - n$. Then, $m - n = rq_{\gamma_k} + t$ with $1 \leq t < q_{\gamma_k}$, $v(a_m - a_n) = v(a_{rq_{\gamma_k}+t} - a_0) < \gamma_k$ because $v(a_{rq_{\gamma_k}+t} - a_t) = v(a_{rq_{\gamma_k}} - a_0) \geq \gamma_k$ and $v(a_t - a_0) < \gamma_k$. Thus, the equivalence (6.2) is proved.

By hypothesis on s, $v(a_s - a_0) \geq \gamma_h > \gamma_{h-1}$, and hence, $q_{\gamma_{h-1}} | s$. Consequently, for every $k < h$, $a_0, \ldots, a_{s-1}$ contains exactly $\frac{s}{q_{\gamma_k}}$ complete sets of representatives of S *mod* γ_k. Let $x \in S$ be such that $a_0, a_1, \ldots, a_{s-1}, x$ are non-congruent mod γ_h. Then,

$$v\left(\prod_{n=0}^{s-1}(x - a_n)\right) = s\gamma_0 + \sum_{k=1}^{h-1}\left[\frac{s}{q_{\gamma_k}}\right](\gamma_k - \gamma_{k-1})$$

while

$$v\left(\prod_{n=0}^{s-1}(a_s - a_n)\right) \geq s\gamma_0 + \sum_{k=1}^{h-1}\left[\frac{s}{q_{\gamma_k}}\right](\gamma_k - \gamma_{k-1}) + (\gamma_h - \gamma_{h-1}).$$

This is a contradiction. Consequently, assertion (6.1) holds for every critical valuation γ_k. By Remark 4.2, the proof is complete if q_{γ_∞} is infinite.

Assume now that q_{γ_∞} is finite ($\gamma_\infty = \gamma_l$) and that assertion (6.1) does not hold for some γ such that q_γ is infinite. Then, there exist i and j such that $0 \leq i < j$ and $v(a_j - a_i) > \gamma_\infty$. Since $v(a_j - a_i) = v(a_{j-i} - a_0)$, there exists a least integer $s > 0$ such that $v(a_s - a_0) > \gamma_\infty$ and, by the first part of the proof, $q_{\gamma_\infty} | s$, so that, $a_0, \ldots, a_{s-1}$ contains exactly $\frac{s}{q_{\gamma_\infty}}$ complete systems of representatives of S *mod* γ_∞. Let $\delta = v(a_s - a_0)$ and, as in the first part of the proof, let $x \in S$ be such that $a_0, \ldots, a_{s-1}, x$ are in distinct classes mod δ and let $\varepsilon = \max_{0 \leq n < s} v(x - a_n)$. Then,

$$v\left(\prod_{n=0}^{s-1}(x - a_n)\right) = s\gamma_0 + \sum_{k=1}^{l}\left[\frac{s}{q_{\gamma_k}}\right](\gamma_k - \gamma_{k-1}) + (\varepsilon - \gamma_\infty)$$

while

$$v\left(\prod_{n=0}^{s-1}(a_s - a_n)\right) \geq s\gamma_0 + \sum_{k=1}^{l}\left[\frac{s}{q_{\gamma_k}}\right](\gamma_k - \gamma_{k-1}) + (\delta - \gamma_\infty).$$

This is a contradiction since $\varepsilon < \delta$. $\square$

6.3. **Toward a conclusion.**

Putting together Theorem 5.7 and Propositions 6.3, 6.5 and 6.9, we obtain in particular:

Theorem 6.11. *Let S be an infinite fractional subset of K such that $q_{\gamma_\infty}(S)$ is infinite. The following assertions are equivalent:*

(1) *S is a preregular subset or, equivalently, the polynomial closure $\overline{S}$ of S is regular.*

(2) *There exists a sequence $\{a_n\}_{n\in\mathbb{N}}$ of elements of S such that:*

$$\forall \gamma \leq \gamma_\infty \;\; \forall k \in \mathbb{N} \quad a_{kq_\gamma}, a_{kq_\gamma+1}, \ldots, a_{(k+1)q_\gamma - 1} \tag{6.3}$$

is a complete set of representatives of S mod γ (for γ_∞, this condition just means that the a_n's are non-congruent modulo γ_∞.)

(3) *There exists a sequence* $\{b_n\}_{n\in\mathbb{N}}$ *of elements of* S *such that:*

$$\forall\gamma\in\mathbb{R}\quad v(b_n-b_m)\geq\gamma\;\Leftrightarrow\;q_\gamma|(n-m)\,. \tag{6.4}$$

(4) *There exists a sequence* $\{c_n\}_{n\in\mathbb{N}}$ *of elements of* S *such that, for every* $k\in\mathbb{N}$, $\{c_n\}_{n\geq k}$ *is a* v*-ordering of* S.

(5) *The characteristic function* w_S *of* S *satisfies the generalized Legendre formula:*

$$w_S(n)=v(n!_S)=n\gamma_0+\sum_{k=0}^{+\infty}\left[\frac{n}{q_{\gamma_k}(S)}\right](\gamma_k-\gamma_{k-1})\,. \tag{6.5}$$

Let us recall the following example that we gave in [9, §12] and which turns out to be a regular subset.

Example 6.12. Let F be a field and let Γ be a subgroup of $\mathbb{R}$. Consider the integral domain

$$F[\Gamma]=F[\{X^\gamma\mid\gamma\geq0\};X^\gamma X^\delta=X^{\gamma+\delta}] \tag{6.6}$$

endowed with the valuation v defined by:

$$\forall k\in\mathbb{N}\;\forall a_k\in F\;\forall\delta_k\in\Gamma\quad v\left(\sum_{k=0}^{n}a_kX^{\delta_k}\right)=\min\{\delta_k\mid a_k\neq0\}. \tag{6.7}$$

Fix a strictly increasing sequence $\{\gamma_n\}_{n\in\mathbb{N}}$ of elements of Γ. For every $n\geq0$, choose a finite subset A_n of F containing 0 with cardinality $\alpha_n>1$:

$$A_n=\{a_{n,0}=0,a_{n,1},\ldots,a_{n,\alpha_n-1}\}\subseteq F.$$

Now consider the following subset T of $F[\Gamma]$:

$$T=\{a_0X^{\gamma_0}+a_1X^{\gamma_1}+\cdots+a_lX^{\gamma_l}\mid l\in\mathbb{N},\,a_h\in A_h,0\leq h\leq l\}. \tag{6.8}$$

Then, the q_{γ_k}'s defined by $q_{\gamma_k}=q_{\gamma_k}(T)=Card(T \bmod \gamma_k)$ satisfy

$$q_{\gamma_0}=1\text{ and }q_{\gamma_{k+1}}=\alpha_kq_{\gamma_k}. \tag{6.9}$$

If the sequence $\{\gamma_k\}_{k\geq0}$ is bounded, T is a preregular discrete subset and, if the sequence is unbounded, T is a regular precompact subset.

Now we describe a VWDWO sequence of T : for every $n\geq0$, write

$$n=n_0+n_1q_{r_1}+n_2q_{r_2}+\cdots+n_lq_{r_l}\quad\text{with }0\leq n_k<\alpha_k \tag{6.10}$$

and let

$$a_n=\sum_{k=0}^{l}a_{k,n_k}X^{\gamma_k}. \tag{6.11}$$

Then, the sequence $\{a_n\}_{n\in\mathbb{N}}$ defined by (6.10) and (6.11) is a v-ordering of the subset T defined by (6.8).

7. An application: polynomials with integer-valued divided differences on a regular subset

As previously said, Amice [1] introduced the regular compact subsets of local fields to extend Mahler's theorem on approximation of continuous p-adic functions by integer-valued polynomials. In the same vein, Bhargava [4, Theorem 22] proved that the r-times continuously differentiable functions on a compact subset of a local field may be approximated by polynomials with integer-valued divided differences up to the order r.

Let us recall what are these polynomials. For every $f \in L[x]$ where L denotes a field, define $\Phi^k(f) \in L[x_0, \ldots, x_n]$ by induction on k with $\Phi^0(f) = f(x_0)$ and, for $k \geq 1$,

$$\Phi^k(f)(x_0, x_1, \ldots, x_k) = \frac{\Phi^{k-1}(f)(x_0, \ldots, x_{k-1}) - \phi^{k-1}(f)(x_0, \ldots, x_{k-2}, x_k)}{x_{k-1} - x_k}.$$

Note that $\Phi^k(f)$ is a symmetric polynomial in the variables $x_0, \ldots, x_k$. If D is an integral domain with quotient field L and T is a subset of D, then the set formed by the polynomials whose k-th divided differences for $k = 0, \ldots, r$ are all D-valued on T is a ring that, following Bhargava [4, Definition 5], we denote by:

$$\mathrm{Int}^{\{r\}}(T, D) = \{f \in L[X] \mid \Phi^k(f) \in \mathrm{Int}(S, D) \text{ for } 0 \leq k \leq r\}. \tag{7.1}$$

Generalizing Bhargava's definition [4, §2.1] of the r-removed sequences to any subset S of a valued field K, we introduce:

Definition 7.1. Let K be a valued field with valuation domain V and let S be a subset of K. For every fixed integer $r \geq 0$ and for each $n \geq 0$, the set formed by the leading coefficients of the polynomials of $\mathrm{Int}^{\{r\}}(S, V)$ of degree $\leq n$ is a fractional ideal whose valuation is denoted by $w_S^{\{r\}}(n)$. The sequence of integers $\{w_S^{\{r\}}(n)\}_{n\geq 0}$ is called the *r-removed v-sequence* of S.

The name of the sequence comes from the following definition/proposition given for local fields but which may easily be extended to any valued field:

Proposition 7.2. [4, Theorem 7] *An* r-removed v-ordering *of S is a sequence $\{a_n\}_{n\in\mathbb{N}}$ of elements of S such that, for every $n \geq r$, a_n minimizes $\sum_{j\in J} v(s - a_j)$ over all $s \in S$ and over all subsets J of $\{0, \ldots, n-1\}$ of cardinality $n - r$.*

For any such r-removed v-ordering, we have:

$$w_S^{\{r\}}(n) = \sum_{j \in J_n} v(a_n - a_j) \tag{7.2}$$

where J_n denotes any subset J that provides the previous minimum.

Clearly,

$$w_S^{\{r\}}(n) = 0 \text{ for } n < r. \tag{7.3}$$

Note that, in the case of a valued field in general, a subset S does not always admit a r-removed v-ordering (as it does not always admit a v-ordering), nevertheless it always has a r-removed v-sequence. If S admits an r-removed v-ordering, it is then easy to compute the sequence $\{w_S^{\{r\}}(n)\}_{n\geq 0}$ step by step. But here we are going to see that, when S is a regular subset, there are explicit formulas for the function $w_S^{\{r\}}$ because the symmetry of the regular subsets is suited to the symmetry of the polynomials $\Phi^k(f)$. For instance, K. Johnson established the first formula for the r-removed p-sequence of $\mathbb{Z}$:

Proposition 7.3. [13, Thm 1] *Let p be a prime and consider $S = \mathbb{Z} \subset K = \mathbb{Q}_p$. Then,*

$$\begin{cases} \text{for } n < rp, & w_{\mathbb{Z}}^{\{r\}}(n) = 0 \\ \text{for } n \geq rp, & w_{\mathbb{Z}}^{\{r\}}(n) = \sum_{i=0}^{k} \left[\frac{n}{p^i}\right] - kr \text{ where } k = \left[\log(\frac{n}{r})/\log(p)\right]. \end{cases}$$

We establish below an analogous formula for regular subsets using the following lemmas that are very similar to those given in the case of discrete valuations, and whose proofs are also similar.

Lemma 7.4. [13, Prop. 2] *Let S be a subset of K and a, b two elements of K. If S admits a r-removed v-ordering, then*

(1) *the r-removed v-sequences of S and $S + a$ are equal.*
(2) *the r-removed v-sequence of bS satisfies:*

$$\text{for } n \geq r \quad w_{bS}^{\{r\}}(n) = w_S^{\{r\}}(n) + v(b)(n - r) \tag{7.4}$$

The useful tool that allows us to compute these sequences is given in [13] with the notion of nondecreasing shuffle of finitely many nondecreasing sequences:

Lemma 7.5. [13, Prop. 3] *If the subset S is of the form $S = \cup_{1\leq j\leq q} S_j$ where the S_j's are contained in distinct classes modulo $\mathfrak{m}$, that is, for all $x \in S_i, y \in S_j$ ($i \neq j$) $v(x - y) = 0$, then $w_S^{\{r\}}$ is the shuffle of the $w_{S_j}^{\{r\}}$ for $1 \leq j \leq q$.*

When the functions $w_{S_j}^{\{r\}}$ are equal to each other, the previous lemma leads to:

Proposition 7.6. *If the subset S is of the form:*

$$S = \cup_{1\leq j\leq q} S_j$$

where for all $1 \leq i \neq j \leq q$,

$$w_{S_i}^{\{r\}} = w_{S_j}^{\{r\}} \text{ and } v(y_i - y_j) = 0 \text{ for } y_i \in S_i, y_j \in S_j,$$

then

$$w_S^{\{r\}}(n) = w_{S_1}^{\{r\}}\left(\left[\frac{n}{q}\right]\right). \tag{7.5}$$

In particular, with the hypothesis of the previous proposition, $w_S^{\{r\}}(n) = 0$ for $n < rq$. Using iteratively Formulas (7.4) and (7.5), we obtain:

Theorem 7.7. *If S is a preregular subset, then*

$$w_S^{\{r\}}(n) = \begin{cases} 0 & \text{for } n < r = rq_{\gamma_0} \\ n\gamma_0 + \sum_{j=1}^{k} \left[\frac{n}{q_{\gamma_j}}\right](\gamma_j - \gamma_{j-1}) - r\gamma_k & \text{for } rq_{\gamma_k} \leq n < rq_{\gamma_{k+1}} \\ n\gamma_0 + \sum_{j=1}^{l} \left[\frac{n}{q_{\gamma_j}}\right](\gamma_j - \gamma_{j-1}) - r\gamma_l & \text{for } n \geq rq_{\gamma_\infty}\ (\gamma_\infty = \gamma_l). \end{cases}$$

Proof. By definition, to compute $w_S^{\{r\}}$ we may replace S by its polynomial closure $\overline{S}$ and, by Theorem 3.5, we may assume that S is regular. By Lemma 7.4, we may also assume that $0 \in S$. Then $S = S(0, \gamma_0)$. By Lemma 4.11, S admits a VWD sequence $\{y_n\}_{0\leq n<q_{\gamma_\infty}}$ of order γ_∞, and by Proposition 6.5, we may assume that this sequence is a VWDWO sequence of order γ_∞ (with $y_0 = 0$).

If $\gamma_0 = \gamma_\infty$, one knows that $\overline{S} = B(0,\gamma_0)$ and $w_S(n) = n\gamma_0$, so that $w_S^{\{r\}}(n) = (n-r)\gamma_0$, and the formula is satisfied. Now, we assume that $\gamma_0 \neq \gamma_\infty$, then γ_0 is a minimum and $v(y_1) = v(y_{q_{\gamma_0}}) = \gamma_0$. More generally, we have $v(y_{q_{\gamma_k}}) = \gamma_k$. We will denote $y_{q_{\gamma_k}}$ by t_k, so that $v(t_k) = \gamma_k$.

The first case ($n < r$) is obvious. Let us prove the second case by induction on k. From

$$S = S(0,\gamma_0) = \cup_{x\in S_{\gamma_1}} S(x,\gamma_1), \tag{7.6}$$

we deduce

$$\frac{1}{t_0} S = \cup_{x\in S_{\gamma_1}} \frac{1}{t_0} S(x,\gamma_1), \tag{7.7}$$

and from Lemma 7.4 and Proposition 7.6, we have:

$$w_S^{\{r\}}(n) = w_{\frac{1}{t_0}S}^{\{r\}}(n) + (n-r)\gamma_0 \text{ for } n \geq r \tag{7.8}$$

and

$$w_{\frac{1}{t_0}S}^{\{r\}}(n) = w_{\frac{1}{t_0}S(0,\gamma_1)}^{\{r\}}\left(\left[\frac{n}{q_{\gamma_1}}\right]\right) \text{ for } n \geq 0. \tag{7.9}$$

So that,

$$w_S^{\{r\}}(n) = w_{\frac{1}{t_0}S(0,\gamma_1)}^{\{r\}}\left(\left[\frac{n}{q_{\gamma_1}}\right]\right) + (n-r)\gamma_0 \quad \text{for } n \geq r. \tag{7.10}$$

More generally, for every $k \geq 1$, from

$$S(0,\gamma_k) = \cup_{0\leq j<\alpha_k} S(y_{jq_{\gamma_k}},\gamma_{k+1}) \tag{7.11}$$

where

$$q_{\gamma_{k+1}} = \alpha_k \times q_{\gamma_k},$$

we deduce that:

$$w_{\frac{1}{t_k}S(0,\gamma_k)}^{\{r\}}(n) = w_{\frac{1}{t_k}S(0,\gamma_{k+1})}^{\{r\}}\left(\left[\frac{n}{\alpha_k}\right]\right) \qquad \text{for } n \geq 0. \tag{7.12}$$

Noticing that

$$w_{\frac{1}{t_{k-1}}S(0,\gamma_k)}^{\{r\}}(n) = w_{\frac{1}{t_k}S(0,\gamma_k)}^{\{r\}}(n) + (n-r)(\gamma_k - \gamma_{k-1}) \text{ for } n \geq r, \tag{7.13}$$

we obtain:

$$\begin{aligned} w_S^{\{r\}}(n) &= w_{\frac{1}{t_0}S(0,\gamma_1)}^{\{r\}}\left(\left[\frac{n}{q_{\gamma_1}}\right]\right) + (n-r)\gamma_0 \text{ for } n \geq r \\ &= w_{\frac{1}{t_1}S(0,\gamma_2)}^{\{r\}}\left(\left[\frac{\left[\frac{n}{q_{\gamma_1}}\right]}{\alpha_1}\right]\right) + \left(\left[\frac{n}{q_{\gamma_1}}\right]\right)(\gamma_1-\gamma_0) + (n-r)\gamma_0 \text{ for } \left[\frac{n}{q_{\gamma_1}}\right] \geq r \\ &= w_{\frac{1}{t_1}S(0,\gamma_2)}^{\{r\}}\left(\left[\frac{n}{q_{\gamma_2}}\right]\right) + n\gamma_0 + \left[\frac{n}{q_{\gamma_1}}\right](\gamma_1-\gamma_0) - r\gamma_1 \text{ for } n \geq rq_{\gamma_1} \end{aligned}$$

and so on,

$$w_S^{\{r\}}(n) = w_{\frac{1}{t_k}S(0,\gamma_{k+1})}^{\{r\}}\left(\left[\frac{n}{q_{\gamma_{k+1}}}\right]\right) + n\gamma_0 + \sum_{j=1}^{k}\left[\frac{n}{q_{\gamma_j}}\right](\gamma_j-\gamma_{j-1}) - r\gamma_k \text{ for } n \geq rq_{\gamma_k}$$

If $n < rq_{\gamma_{k+1}}$, then $w_{\frac{1}{t_k}S(0,\gamma_{k+1})}^{\{r\}}\left(\left[\frac{n}{q_{\gamma_{k+1}}}\right]\right) = 0$ and the formula is proved in the second case.

If q_{γ_∞} is finite (and $\gamma_\infty = \gamma_l$), it may be that $n \geq rq_{\gamma_\infty}$. In this case we have:

$$w_S^{\{r\}}(n) = w^{\{r\}}_{\frac{1}{t_{l-1}}S(0,\gamma_l)}\left(\left[\frac{n}{q_{\gamma_l}}\right]\right)+n\gamma_0+\sum_{j=1}^{l-1}\left[\frac{n}{q_{\gamma_j}}\right](\gamma_j-\gamma_{j-1})-r\gamma_{l-1} \text{ for } n \geq rq_{\gamma_{l-1}}$$

It follows from Theorem 3.5 that $\overline{S}(0,\gamma_l) = B(0,\gamma_l)$ and $w_{\frac{1}{t_l}S(0,\gamma_l)}(n) = 0$. Consequently,

$$w^{\{r\}}_{\frac{1}{t_{l-1}}S(0,\gamma_l)}(n) = (n-r)(\gamma_l - \gamma_{l-1}),$$

and the formula is proved in the third case. □

When S is a regular subset, it is easy to describe r-removed v-orderings:

Proposition 7.8. *If $\{a_n\}_{n\geq 0}$ is a VWDWO sequence of S then, for every $r \geq 0$, the sequence $\{a_n\}_{n\geq 0}$ is also a r-removed v-ordering of S.*

Proof. Since a WDWO sequence of S is a v-ordering of S, one has:

$$v\left(\prod_{k=0}^{n-1}(a_n - a_k)\right) = w_S(n).$$

Consequently, it is enough to prove that, for every $n \geq r$, it is possible to choose a set I of r indices i such that $0 \leq i < n$ and:

$$\sum_{i\in I} v(a_n - a_i) = w_S(n) - w_S^{\{r\}}(n).$$

Fix an $n \geq r$ and let k be such that $rq_{\gamma_k} \leq n < rq_{\gamma_{k+1}}$. Clearly, we may choose r indices i such that $n \equiv i \pmod{q_{\gamma_k}}$, and hence, such that $v(a_n - a_i) \geq \gamma_k$. For such a choice, we have: $\sum_{i\in I} v(a_n - a_i) \geq r\gamma_k$. But, among these indices, we may choose $\left[\frac{n}{q_{\gamma_{k+1}}}\right]$ indices i such that $n \equiv i \pmod{q_{\gamma_{k+1}}}$, and hence, such that $v(a_n - a_i) \geq \gamma_{k+1}$. For such a choice, we have:

$$\sum_{i\in I} v(a_n - a_i) \geq r\gamma_k + \left[\frac{n}{q_{\gamma_{k+1}}}\right](\gamma_{k+1} - \gamma_k).$$

And so on... Consequently, we may choose the i's such that:

$$\sum_{i\in I} v(a_n - a_i) = r\gamma_k + \sum_{j=k+1}^{\infty}\left[\frac{n}{q_{\gamma_j}}\right](\gamma_j - \gamma_{j-1}).$$

This is an equality because the sum on the right side is necessarily finite. And, this right side is exactly $w_S(n) - w_S^{\{r\}}(n)$. □

As a consequence of Propositions 7.2 and 7.8 and Theorem [4, Theorem 7], we have:

Proposition 7.9. *If $\{a_n\}_{n\geq 0}$ is a VWDWO sequence of S, then the following r-removed generalized binomial polynomials:*

$$f_0^{\{r\}}(x) = 1 \text{ and, for } n \geq 1,\ f_n^{\{r\}}(x) = \frac{1}{\prod_{0\leq k<n-r}(a_n - a_k)} \prod_{0\leq k<n}(x - a_k) \tag{7.14}$$

form a basis of the V-module $\mathrm{Int}^{\{r\}}(S,V)$ of polynomials whose k-th divided differences for $k = 0,\dots,r$ are all integer-valued on S.

Remark 7.10. Bhargava [4] proves also that the locally analytic functions of order h may be approximated by some polynomials and, in order to do this, he introduces the notion of v-ordering of order h. We could do analogous computations for this other kind of v-orderings in the case of regular subsets (see for instance [13]).

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