Integer-Valued Polynomials: Looking for Regular Bases (a Survey)

Jean-Luc Chabert

Abstract This paper reviews recent results about the additive structure of algebras of integer-valued polynomials, and particularly, the question of the existence and the construction of regular bases. Doing this, we will be led to consider questions of combinatorial, arithmetical, algebraic, ultrametric or dynamical nature.

2010 MSC. Primary 13F20; Secondary 11S05, 11R21, 11B65

Key words: Integer valued polynomials, generalized factorials, v-orderings, Kempner's formula, regular basis, Pólya fields, divided differences, Mahler's theorem

1 Introduction

The $\mathbb{Z}$-algebra

$$\mathrm{Int}(\mathbb{Z}) = \{f(X) \in \mathbb{Q}[X] \mid f(\mathbb{Z}) \subseteq \mathbb{Z}\}$$

is a paradigmatic example because several of its properties still hold for some general algebras of integer-valued polynomials. As a ring, $\mathrm{Int}(\mathbb{Z})$ has a lot of interesting properties, but here we focus our survey on its additive structure since it is a cornerstone for the study of quite all other properties. The $\mathbb{Z}$-module $\mathrm{Int}(\mathbb{Z})$ is free: it admits a basis formed by the binomial polynomials, and this basis turns out to be also an orthonormal basis of the ultrametric Banach space $\mathscr{C}(\mathbb{Z}_p, \mathbb{Q}_p)$ whatever the prime p.

Now replace $\mathbb{Z}$ by the ring of integers of a number field as Pólya [43] and Ostrowski [41] did, or more generally, on the one hand, replace $\mathbb{Z}$ by any Dedekind domain D with quotient field K and, on the other hand, consider a subset S of D and the D-algebra of polynomials whose values on S are in D :

Jean-Luc Chabert
LAMFA CNRS-UMR 7352, Université de Picardie, 33 rue Saint Leu, 80039 Amiens, France, e-mail: jean-luc.chabert@u-picardie.fr

$$\mathrm{Int}(S,D) = \{f \in K[X] \mid f(S) \subseteq D\} \ .$$

During the last decades of the last century, there were many works about this D-algebra from the point of view of commutative algebra. As said above, our aim here is to characterize the cases where the D-module $\mathrm{Int}(S,D)$ admits bases, and more precisely, regular bases, that is, with one and only one polynomial of each degree and, when there are such bases, we want to describe them. This was already the subject of [17, Chapter II] but, during the last 15 years, a lot of new results were obtained, especially thanks to several notions of v-ordering introduced by Bhargava from 1998 to 2009 ([9], [10], [11], [12]).

In §2, we recall a few properties of $\mathrm{Int}(\mathbb{Z})$ that will be generalized, in particular those concerning factorials. Then, in §3, we consider the factorial ideals and, in §4, the notion of v-ordering and its links with integer-valued polynomials. In §5 we study the existence and the construction of regular bases while §6 is devoted to effective computations. Then, in §7, we consider some particular sub-algebras of $\mathrm{Int}(S,D)$ and, in §8, we apply our knowledge of regular bases to obtain orthonormal bases of ultrametric Banach spaces. We end in §9 with the case of several indeterminates.

2 The Paradigmatic Example: $\mathrm{Int}(\mathbb{Z})$

In the ring of *integer-valued polynomial* $\mathrm{Int}(\mathbb{Z}) = \{f(X) \in \mathbb{Q}[X] \mid f(\mathbb{Z}) \subseteq \mathbb{Z}\}$, there are polynomials without any integral coefficients, for instance:

$$\binom{X}{n} = \frac{X(X-1)\dots(X-n+1)}{n!} \ \ (n \geq 2) \ \text{ or } \ F_p(X) = \frac{X^p - X}{p} \ \ (p \in \mathbb{P}) \ .$$

2.1 Some Algebraic Structures

As a subset of $\mathbb{Q}[X]$, $\mathrm{Int}(\mathbb{Z})$ is stable by addition, multiplication and composition.

Proposition 1. *The binomial polynomials form a basis of the $\mathbb{Z}$-module* $\mathrm{Int}(\mathbb{Z})$: *every $g(X) \in \mathrm{Int}(\mathbb{Z})$ may be uniquely written as*

$$g(X) = \sum_{k=0}^{\deg(g)} c_k \binom{X}{k} \quad \textit{with } c_k \in \mathbb{Z} \qquad \text{(Pólya [42], 1915)}. \tag{1}$$

Proposition 2. [17, §2.2] *The set $\{1, X\} \cup \{(X^p - X)/p \mid p \in \mathbb{P}\}$ is a minimal system of polynomials with which every element of* $\mathrm{Int}(\mathbb{Z})$ *may be constructed by means of sums, products and composition.* [See Example 3 below]

Proposition 3. [17] $\mathrm{Int}(\mathbb{Z})$ *is a two-dimensional non-Noetherian Prüfer domain.*

For instance, the ideal $\mathfrak{I} = \{f(X) \in \mathrm{Int}(\mathbb{Z}) \mid f(0) \text{ is even}\}$ is not finitely generated.

2.2 A Polynomial Approximation in Ultrametric Analysis

Let p be a fixed prime number. Recall that $|x| = e^{-v_p(x)}$ is an absolute value on $\mathbb{Q}_p$ where v_p denotes the p-adic valuation. For every compact subset S, the norm of a continuous function $\varphi : S \to \mathbb{Q}_p$ is then $\|\varphi\|_S = \sup_{x\in S} |\varphi(x)|$.

Proposition 4. (Mahler [39], 1958) *Every function $\varphi \in \mathscr{C}(\mathbb{Z}_p, \mathbb{Q}_p)$ may be written*

$$\varphi(x) = \sum_{n=0}^{\infty} c_n \binom{x}{n} \text{ with } c_n \in \mathbb{Q}_p \text{ and } \lim_{n\to+\infty} v_p(c_n) = +\infty. \tag{2}$$

Moreover, $\|\varphi\|_{\mathbb{Z}_p} = \|\{c_n\}_{n\in\mathbb{N}}\|$, that is, $\inf_{x\in\mathbb{Z}_p} v_p(\varphi(x)) = \inf_{n\in\mathbb{N}} v_p(c_n)$.

One says that the binomial fonctions $\binom{x}{n}$ form an *orthonormal basis* of the Banach space $\mathscr{C}(\mathbb{Z}_p, \mathbb{Q}_p)$. The coefficients c_n are unique and may be computed recursively:

$$c_n = \varphi(n) - \sum_{k=0}^{n-1} c_k \binom{n}{k} .$$

2.3 Some Properties of the Factorials

As denominators of the binomial polynomials, the factorials and their generalizations will play an important rôle in the description of bases. We recall some properties which will be preserved in a more general context.

Property A. *For all $k, l \in \mathbb{N}$,* $\binom{k+l}{k} = \frac{(k+l)!}{k!\times l!} \in \mathbb{N}$.
Equivalently, the product of l consecutive integers is divisible by $l!$:

$$\frac{(k+1)(k+2)\dots(k+l)}{l!} \in \mathbb{N} . \tag{3}$$

When considering integers which are not consecutive, we still have the following:
Property B. *For every sequence $x_0, x_1, \dots, x_n$ of $n+1$ integers, the product*

$$\prod_{0\le i<j\le n} (x_j - x_i) \quad \text{is divisible by} \quad 1!\times 2!\times\dots\times n! . \tag{4}$$

We now consider links between factorials and polynomials.
Property C. *For every monic polynomial $f \in \mathbb{Z}[X]$ of degree n,*

$$d(f) = \gcd\{f(k) \mid k \in \mathbb{Z}\} \quad \text{divides} \quad n! \qquad \text{(Pólya [42], 1915).} \tag{5}$$

Property D. *For every integer-valued polynomial g of degree n*

$$n! \times g(X) \in \mathbb{Z}[X] . \tag{6}$$

Property E. *The subset formed by the leading coefficients of the integer-valued polynomials of degree* $\leq n$ *is* $\frac{1}{n!}\mathbb{Z}$.

Property K. *The number of polynomial functions from* $\mathbb{Z}/n\mathbb{Z}$ *to* $\mathbb{Z}/n\mathbb{Z}$ *is:*

$$\prod_{k=0}^{n-1} \frac{n}{\gcd(n,k!)} \qquad \text{(Kempner [36], 1921)} \tag{7}$$

where functions are induced by polynomials of $\mathbb{Z}[X]$.

Finally, recall Legendre's formula:

$$n! = \prod_{p\in\mathbb{P}} p^{w_p(n)} \quad \text{where } w_p(n) = \sum_{k\geq 1}\left[\frac{n}{p^k}\right] \quad \text{(Legendre, 1808)}. \tag{8}$$

3 General Integer-Valued Polynomials and Generalized Factorials

Notation. *In the sequel, D always denotes a Dedekind domain with quotient field K and S a subset of D.*

The D-algebra of *integer-valued polynomials on D* is:

$$\mathrm{Int}(D) = \{f(X) \in K[X] \mid f(D) \subseteq D\} \tag{9}$$

and the D-algebra of *integer-valued polynomials on S* (with respect to D) is:

$$\mathrm{Int}(S,D) = \{f(X) \in K[X] \mid f(S) \subseteq D\}. \tag{10}$$

As $S \subseteq D$, we have $D[X] \subseteq \mathrm{Int}(D) \subseteq \mathrm{Int}(S,D) \subseteq K[X]$. Let us consider a more general situation by introducing a D-algebra $\mathbb{B}$ such that $D[X] \subseteq \mathbb{B} \subseteq K[X]$.

Definition 1. (Pólya [43]) A basis of the D-module $\mathbb{B}$ is said to be a *regular basis* if it is formed by one and only one polynomial of each degree.

3.1 Characteristic Ideals and Regular Bases

Definition 2. The *characteristic ideal* of index n of the D-algebra $\mathbb{B}$ is the set $\mathfrak{I}_n(\mathbb{B})$ formed by 0 and the leading coefficients of the polynomials in $\mathbb{B}$ of degree n.

If $\mathbb{B} = \mathrm{Int}(S,D)$, we write $\mathfrak{I}_n(S,B)$ instead of $\mathfrak{I}_n(\mathrm{Int}(S,D))$.

Clearly, $\{\mathfrak{I}_n(\mathbb{B})\}_{n\in\mathbb{N}}$ is an increasing sequence of D-modules such that:

$$\forall k,l \in \mathbb{N} \quad D \subseteq \mathfrak{I}_k(\mathbb{B}) \subseteq K \ \text{ and } \ \mathfrak{I}_k(\mathbb{B}) \cdot \mathfrak{I}_l(\mathbb{B}) \subseteq \mathfrak{I}_{k+l}(\mathbb{B}). \tag{11}$$

Recall that a *fractional ideal* of D is a sub-D-module $\mathfrak{J}$ of K for which there exists a nonzero element $d \in D$ such that $d\mathfrak{J} = \{dj \mid j \in \mathfrak{J}\} \subseteq D$. A very simple argument using Vandermonde's determinant leads to:

Lemma 1. [17, Prop. I.3.1] *Let f be a polynomial of $K[X]$ with degree n. Assume that $x_0, x_1, \ldots, x_n$ are distinct elements of K such that $f(x_i) \in D$ for $0 \leq i \leq n$, then df belongs to $D[X]$ where $d = \prod_{0 \leq i < j \leq n}(x_j - x_i)$.*

As a consequence,
– if $n < \mathrm{Card}(S)$, then $\mathfrak{J}_n(S,D)$ is a fractional ideal of D.
– if $n \geq \mathrm{Card}(S)$, then $\mathfrak{J}_n(S,D) = K$ because $\left(\prod_{s \in S}(X-s)\right) K[X] \subseteq \mathrm{Int}(S,D)$.

In particular, if $\mathrm{Card}(S)$ is infinite, all the $\mathfrak{J}_n(S,D)$'s are fractional ideals and, more generally, so are all the $\mathfrak{J}_n(\mathbb{B})$'s if $\mathbb{B} \subseteq \mathrm{Int}(S,D)$.

Clearly, by definition of the characteristic ideals, we have:

Proposition 5. [17, Prop. II.1.4] *A sequence of polynomials $\{f_n\}_{n \geq 0}$ where $\deg(f_n) = n$ is a regular basis of $\mathbb{B}$ if and only if, for every $n \geq 0$, the leading coefficient of f_n generates the ideal $\mathfrak{J}_n(\mathbb{B})$. In particular, The D-algebra $\mathrm{Int}(\mathbb{B})$ admits a regular basis as a D-module if and only if all the $\mathfrak{J}_n(\mathbb{B})$'s are principal.*

Thus, if S is finite, the D-module $\mathrm{Int}(S,D)$ cannot admit any regular basis since $\mathfrak{J}_n(S,D) = K$ for $n \geq \mathrm{Card}(S)$.

3.2 The Factorial Ideals of a Subset S

When $S = D = \mathbb{Z}$, $\mathfrak{J}_n(\mathbb{Z},\mathbb{Z}) = \frac{1}{n!}\mathbb{Z}$. Thus, it is natural to define new factorials as the inverses of the characteristic ideals. In general, D is not a principal ideal domain and we cannot define these new factorials as numbers, but as ideals.

Recall that the inverse of a non-zero fractional ideal $\mathfrak{J}$ of D is the fractional ideal $\mathfrak{J}^{-1} = \{x \in K \mid x\mathfrak{J} \subseteq D\}$. If $\mathfrak{J}$ contains 1, then $\mathfrak{J}^{-1}$ is an integral ideal. Since D is assumed to be a Dedekind domain, the non-zero fractional ideals of D form a multiplicative group (and $\mathfrak{J} \cdot \mathfrak{J}^{-1} = D$) and an ideal $\mathfrak{a}$ divides an ideal $\mathfrak{b}$ if and only if $\mathfrak{b} \subseteq \mathfrak{a}$. By convention, we write $K^{-1} = (0)$ and $(0)^{-1} = K$.

Definition 3. The *factorial ideal* $(n!)_S^D$ of index n of the subset S with respect to the domain D is the inverse of the fractional ideal $\mathfrak{J}_n(S,D)$:

$$n!_S^D = \mathfrak{J}_n(S,D)^{-1}. \tag{12}$$

The sequence $\{n!_S^D\}_{n \in \mathbb{N}}$ is a decreasing sequence of integral ideals of D and

$$\forall n \quad n!_S^D \mid (n+1)!_S^D \ , \ \ 0!_S^D = D \ , \ \ [n!_S^D = (0) \ \Leftrightarrow \ n \geq \mathrm{Card}(S)] \, , \tag{13}$$

3.3 First Generalized Properties of the Factorials Ideals

Proposition 6 (Generalized property A). *Let $k, l \in \mathbb{N}$ be any integers then*

$$k!_S^D \times l!_S^D \text{ divides } (k+l)!_S^D. \tag{14}$$

This is a straightforward consequence of (11).

Proposition 7 (Generalized property D). [17, Prop. II.1.7] *For every polynomial $g(X) \in \mathrm{Int}(S,D)$ of degree n, we have:*

$$n!_S^D \times g(X) \subseteq D[X]. \tag{15}$$

Recall that, for every polynomial $f \in K[X]$:
– the *content* of f is the ideal $c(f)$ of D generated by the coefficients of f,
– the *fixed divisor* of f over S is the ideal $d(S,f)$ of D generated by the values of f on S.

Proposition 8 (Generalized property C). [10, Th. 2] *With the previous notation, for every $f \in K[X]$ of degree n,*

$$d(S,f) \text{ divides } c(f) \times n!_S^D. \tag{16}$$

Proof. By definition, $d(S,f)^{-1} \times f \subseteq \mathrm{Int}(S,D)$. Then, $n!_S^D \times d(S,f)^{-1} \times f \subseteq D[X]$ by Proposition 7. Finally, $n!_S^D \times d(S,f)^{-1} \times c(f) \subseteq D$. □

Proposition 9. *1. For every $b \in D \setminus \{0\}$ and every $c \in D$, $n!_{bS+c}^D = b^n n!_S^D$.*
2. For every $T \subseteq S$, $n!_S^D$ divides $n!_T^D$. In particular, $n!$ divides $n!_S^{\mathbb{Z}}$.

Proof. 1. The equality follows from the isomorphism of D-algebras:
$$f(X) \in \mathrm{Int}(S,D) \mapsto f\left(\tfrac{X-c}{b}\right) \in \mathrm{Int}(bS+c,D).$$
2. The divisibility relation follows from the containment $\mathrm{Int}(S,D) \subseteq \mathrm{Int}(T,D)$. □

We generalized Properties A, C and D and, by definition, Property E is satisfied by the ideals $n!_S^D$. We will study Properties B and K in the next section by means of localization.

3.4 Localization

Clearly,

$$\mathrm{Int}(S,D) = \cap_{\mathfrak{m} \in \mathrm{Max}(D)} \mathrm{Int}(S,D_{\mathfrak{m}}) \text{ and } \quad \forall \mathfrak{p} \in \mathrm{Spec}(D) \;\; \mathrm{Int}(S,D)_{\mathfrak{p}} \subseteq \mathrm{Int}(S,D_{\mathfrak{p}}).$$

Since D is Noetherian, we have the reverse containment [17, Prop. I.2.7]:

$$\forall \mathfrak{p} \in \mathrm{Spec}(D) \qquad \mathrm{Int}(S,D)_{\mathfrak{p}} = \mathrm{Int}(S,D_{\mathfrak{p}}). \tag{17}$$

We deduce localization formulas for the characteristic ideals and the factorial ideals:

$$\mathfrak{I}_n(S,D) = \cap_{\mathfrak{m}\in\mathrm{Max}(D)} \mathfrak{I}_n(S,D_\mathfrak{m}) \ , \ n!_S^D = \cap_{\mathfrak{m}\in\mathrm{Max}(D)} n!_S^{D_\mathfrak{m}}, \tag{18}$$

$$\mathfrak{I}_n(S,D)_\mathfrak{m} = \mathfrak{I}_n(S,D_\mathfrak{m}) \text{ and } (n!_S^D)_\mathfrak{m} = n!_S^{D_\mathfrak{m}}. \tag{19}$$

4 Local Studies (Bhargava's v-Orderings) and Globalizations

Is there an easy way to compute these factorials? Yes, by means of the notion of v-ordering introduced by Bhargava [9]. This is a local notion, thus we consider localizations, which are discrete valuation domains.

Notation for paragraphs 4.1 and 4.2. *We denote by v a discrete valuation on K, by V the corresponding valuation domain and by S any subset of V.*

4.1 Definitions and Examples

Before v-orderings, we used sequences introduced by Helsmoortel to study the case where $S = V$. The sequences called VWDWO sequences are the sequences which satisfy the equivalent statements of the following proposition.

Proposition 10. [17, §II.2] *Assume that the residue field of V is finite with cardinality q and denote by $\mathfrak{m}$ the maximal ideal. For a sequence $\{a_n\}_{n\geq 0}$ of elements of V, the three following assertions are equivalent:*

1. *Denoting by $v_q(x)$ the largest exponent k such that q^k divides x,*
$$\forall m,n \in \mathbb{N} \quad v(a_n - a_m) = v_q(m-n).$$
2. *For all $r,s \in \mathbb{N}$, $\{a_{r+1}, a_{r+2}, \ldots, a_{r+q^s}\}$ is a complete system of representatives of $V \pmod{\mathfrak{m}^s}$.*
3. *For all $n > m \geq 0$, $v\left(\prod_{k=m}^{n-1}(a_n - a_k)\right) = \sum_{k\geq 1}\left[\frac{n-m}{q^k}\right]$.*

Example 1. The following sequence $\{a_n\}_{n\geq 0}$ is a VWDWO sequence of V. Choose a generator π of $\mathfrak{m}$ and a system of representatives $\{a_0 = 0, a_1, \ldots, a_{q-1}\}$ of V modulo $\mathfrak{m}$.

$$\text{For } n = n_0 + n_1 q + n_2 q^2 + \ldots + n_r q^r \text{ with } 0 \leq a_i < q, \tag{20}$$

$$\text{let } a_n = a_{n_0} + a_{n_1}\pi + a_{n_2}\pi^2 + \ldots + a_{n_r}\pi^r. \tag{21}$$

We also had the VWD sequences introduced by Amice [8] for her regular compact subsets of local fields. But Bhargava's v-orderings are more general.

Definition 4. A *v-ordering* of a subset S of V is a sequence $\{a_n\}_{n\geq 0}$ of elements of S such that, for every $n \geq 1$:

$$v\left(\prod_{k=0}^{n-1}(a_n-a_k)\right)=\min_{x\in S}\ v\left(\prod_{k=0}^{n-1}(x-a_k)\right)\ . \tag{22}$$

Since v is discrete, there always exist v-orderings. Such sequences may be constructed inductively on n choosing any element of S for a_0.

Example 2.

1. For every prime p, the sequence $\{n\}_{n\geq 0}$ is a p-ordering of $\mathbb{Z}$.
2. For every integer $q\geq 2$ and every prime p, the sequence $\{q^n\}_{n\geq 0}$ is a p-ordering of the set $S_q=\{q^n\mid n\in\mathbb{N}\}$ (cf. [17, Exercise II.15]).
3. If $\mathrm{Card}(S)=s<+\infty$ and $\{a_n\}_{n\geq 0}$ is a v-ordering, then $S=\{a_i\mid 0\leq i<s\}$.
4. If $\{a_n\}_{n\geq 0}$ is a VWDWO sequence of V as defined in Proposition 10 then, for every $k\geq 0$, the sequence $\{a_n\}_{n\geq k}$ is a v-ordering of V.

4.2 *v*-Orderings and Integer-Valued Polynomials

There are strong links between v-orderings and integer-valued polynomials:

Proposition 11. [9] *Let $\{a_n\}_{n\geq 0}$ be a sequence of distinct elements of S. Consider the associated sequence of polynomials:*

$$f_0(X)=1 \quad \textit{and} \quad f_n(X)=\prod_{k=0}^{n-1}\frac{X-a_k}{a_n-a_k} \quad (\textit{for } n\geq 1). \tag{23}$$

Then the following assertions are equivalent:

1. *The sequence $\{a_n\}_{n\geq 0}$ is a v-ordering of S,*
2. *For every $n\geq 0$, $f_n(S)\subseteq V$,*
3. *The sequence $\{f_n\mid n\in\mathbb{N}\}$ is a basis of the V-module $\mathrm{Int}(S,V)$,*
4. *For every $d\in\mathbb{N}$, for every $g\in K[X]$ with degree d, one has:*

$$g(a_0),g(a_1),\ldots,g(a_d)\in V\ \Leftrightarrow\ g(S)\subseteq V. \tag{24}$$

Corollary 1. *If $\{a_n\}_{n\in\mathbb{N}}$ is a v-ordering of S, then the following numbers do not depend on the choice of the v-ordering $\{a_n\}_{n\in\mathbb{N}}$ of S :*

$$w_S(n)=v\left(\prod_{k=0}^{n-1}(a_n-a_k)\right). \tag{25}$$

Proof.

$$w_S(n)=\begin{cases}-v(\mathfrak{I}_n(S,V)) & \text{for } 0\leq n<\mathrm{Card}(S)\\ +\infty & \text{for } n\geq \mathrm{Card}(S)\end{cases} \tag{26}$$

□

For instance, by Proposition 10 (Pólya [43]):

$$w_V(n) = \begin{cases} w_q(n) = \sum_{k\geq 1}\left[\frac{n}{q^k}\right] & \text{if } q = \mathrm{Card}(V/\mathfrak{m}) < +\infty \\ w_q(n) = 0 & if\ q = \mathrm{Card}(V/\mathfrak{m}) = +\infty \end{cases} \tag{27}$$

4.3 Globalization: ν-Orderings and Factorials

Notations. Consider again a Dedekind domain D. For every maximal ideal $\mathfrak{m}$ of D, we denote by $v_{\mathfrak{m}}$ the corresponding valuation of K and by $w_{S,\mathfrak{m}}(n)$ the integers defined in Formula (25) (for the valuation $v = v_{\mathfrak{m}}$), finally we speak of $\mathfrak{m}$-orderings instead of $v_{\mathfrak{m}}$-orderings.

Proposition 12. *For every $n \in \mathbb{N}$ such that $n < \mathrm{Card}(S)$, we have:*

$$n!_S = \prod_{\mathfrak{m}\in\mathrm{Max}(D)} \mathfrak{m}^{w_{S,\mathfrak{m}}(n)}. \tag{28}$$

Proof. It follows from (19) and (26) that $\left(n!_S^D\right)_{\mathfrak{m}} = n!_S^{D_{\mathfrak{m}}} = \mathfrak{m}^{w_{S,\mathfrak{m}}(n)} D_{\mathfrak{m}}$. $\square$

Proposition 13. [40, Lemma 8] *Whatever the infinite subset S of $\mathbb{Z}$, there are no three equal consecutive terms in the sequence $\{n!_S^{\mathbb{Z}}\}_{n\geq 0}$.*

Question 1. [40] Does there exist an infinite subset S of $\mathbb{Z}$ such that there are infinitely many two equal consecutive terms in the sequence $\{n!_S^{\mathbb{Z}}\}_{n\geq 0}$?

Proposition 14 (Generalized property B). *For all $x_0, x_1, \ldots, x_n \in S$:*

$$\prod_{0\leq i<j\leq n} (x_j - x_i)D \quad \textit{is divisible by} \quad 1!_S^D \times 2!_S^D \times \ldots \times n!_S^D. \tag{29}$$

Bhargava's proof [11] is given for $\mathbb{Z}$, but it also works for D and it really deserves to be read because it shows how powerful is the notion of ν-ordering. There are many generalizations of Kempner's formula (7) (see for instance [29]), but Bhargava seems to be the first who considered functions defined on subsets.

Proposition 15 (Property K). [9, Theorem 5] *Let D be a Dedekind domain, let $\mathfrak{I}$ be a proper ideal of D with finite norm $N = \mathrm{Card}(D/\mathfrak{I})$ and let S be a subset of D whose elements are non-congruent modulo $\mathfrak{I}$. Then, the number of polynomial functions from S to $D/\mathfrak{I}$ (induced by a polynomial of $D[X]$) is equal to:*

$$\prod_{k=0}^{N-1} \frac{N}{\mathrm{Card}(D/(\mathfrak{I}, k!_S))} \tag{30}$$

where $(\mathfrak{I}, k!_S)$ denotes the ideal of D generated by $\mathfrak{I}$ and $k!_S$.

Note that $k!_S = (0)$ for k greater than the number of classes of S modulo $\mathfrak{I}$, and then, $(\mathfrak{I}, k!_S) = \mathfrak{I}$.

5 Regular Bases

From now on, S is suppose to be infinite. Recall that a regular basis is a basis with one and only one polynomial of each degree, and that $\mathrm{Int}(S,D)$ admits a regular basis if and only if the factorial ideals $n!_S^D$ are (non-zero) principal ideals. Yet, even when this is the case, it may be difficult to describe a regular basis.

5.1 The Local Case

In the local case, that is, in $D_{\mathfrak{m}}$, there always exist regular bases (as we suppose S to be infinite) constructed by means of an $\mathfrak{m}$-ordering of S as shown by Proposition 11. We recall here an example of regular basis constructed in a different way.

Example 3. [17, §II.2] Given a polynomial $g(X)$, we denote by g^{*k} the k-th iterate of g by composition and we let $g^{*0}(X) = X$. In particular, for a fixed prime number p, starting with $F_p(X) = \frac{X^p - X}{p}$, $F_p^{*0}(X) = X, F_p^{*}(X) = F_p(X), F_p^{*2}(X) = F_p(F_p(X))$, and by iteration, $F_p^{*k}(X) = F_p(F_p^{*k-1}(X))$. Finally, for every integer $n = n_0 + n_1 p + \ldots + n_s p^s$ where $0 \leq n_j < p$, we let $F_{p,n} = \prod_{k=0}^{s} (F_p^{*k})^{n_k}$. Note that $F_{p,0} = 1, F_{p,1} = X$ and $F_{p,p^k} = F_p^{*k}$.

Then, the polynomials $\{F_{p,n}(X)\}_{n\geq 0}$ form a basis of the $\mathbb{Z}_{(p)}$-module $\mathrm{Int}(\mathbb{Z}_{(p)})$. Moreover, as a $\mathbb{Z}_{(p)}$-algebra, $\mathrm{Int}(\mathbb{Z}_{(p)})$ is generated by the set $\left\{F_p^{\circ k}(X) \mid k \geq 0\right\}$ and this is a minimal set of generators. Finally, every polynomial of $\mathrm{Int}(\mathbb{Z}_{(p)})$ is obtained from $1, X$ and $\frac{X^p - X}{p}$ by means of sums, products and composition.

Remark 1. Analogously to Example 3, we may obtain in the local case minimal sets of generators of the V-algebra $\mathrm{Int}(S,V)$ by means of the sequence $\{w_S(n) = -v(\mathfrak{I}_n(S,V))\}_{n\geq 0}$. If g_n is a regular basis, we obtain a set of generators by considering only the g_n's where n satisfies the following:

$$w_S(n) > w_S(i) + w_S(j) \text{ for all } i,j > 0 \text{ such that } i + j = n \quad [33].$$

5.2 Simultaneous Orderings

Computation of factorials and description of bases of the ring of integer-valued polynomials is easy when there exist simultaneous orderings as for $\mathbb{Z}$.

Definition 5. A sequence $\{a_n\}_{n\in\mathbb{N}}$ of elements of S which is an $\mathfrak{m}$-ordering of S for every maximal ideal $\mathfrak{m}$ of D is called a *simultaneous ordering* of S.

The following proposition is the global version of Proposition 11.

Proposition 16. *Let $\{a_n\}_{n\geq 0}$ be a sequence of distinct elements of S. Consider the associated sequence of polynomials $f_n(X) = \prod_{k=0}^{n-1} \frac{X - a_k}{a_n - a_k}$ $(n \geq 0)$. Then the following assertions are equivalent:*

1. *The sequence* $\{a_n\}_{n\geq 0}$ *is a simultaneous ordering of* S*,*
2. *For every* $n \geq 0$*,* $f_n(S) \subseteq D$*,*
3. *The polynomials* $\{f_n \mid n \in \mathbb{N}\}$ *form a basis of the* D*-module* $\mathrm{Int}(S,D)$.
4. *For each* $g \in K[X]$ *with* $\deg(g) = d$*,* $g(S) \subseteq D$ *if and only if* $g(a_0), g(a_1), \ldots, g(a_d) \in D$.
5. *For every* $n \geq 1$*, we have:* $n!_S^D = \prod_{k=0}^{n-1}(a_n - a_k)D$.

Are there simultaneous orderings? In particular, which Dedekind domains admit simultaneous orderings [9, Question 3]?

Example 4. 1. $\mathbb{Z}$ admits the simultaneous ordering $\{n\}_{n\geq 0}$.
2. Every semi-local Dedekind domain admits simultaneous orderings (obtained by the Chinese remainder theorem, see Proposition 18 below)
3. $\mathbb{F}_q[T]$, the analog of $\mathbb{Z}$ for function fields, admits a simultaneous ordering $\{a_n\}_{n\geq 0}$, given by Formulas (20) and (21) where $\mathbb{F}_q^* = \{a_1, \ldots, a_{q-1}\}$ and π is replaced by T [11, §10], leading to *Carlitz factorials* [20].
4. Let K be a number field with ring of integers $\mathcal{O}_K$ and T be a multiplicative subset of $\mathcal{O}_K$. Then $\{n\}_{n\geq 0}$ is a simultaneous ordering of $D = T^{-1}\mathcal{O}_K$ if and only if every prime p is either invertible or completely split in D [17, Theorem IV.3.1].

Conjecture. *If* K *is a number field distinct from* $\mathbb{Q}$*, then its ring of integers* $\mathcal{O}_K$ *does not admit any simultaneous ordering.*

In 2003, Wood [46] proved this conjecture for imaginary quadratic number fields, while Adam [1] did an analogous study for 'imaginary' quadratic function fields in 2005. Adam and Cahen [5] proved in 2010 that there are at most finitely many real quadratic number fields whose ring of integers admits a simultaneous ordering.

Let us restrict our question on the existence of simultaneous orderings of subsets of $\mathbb{Z}$ [11, Question 30]. Let us say that a sequence $\{a_n\}_{n\geq 0}$ is *self simultaneously ordered* if it is a simultaneous ordering of the subset $S = \{a_n \mid n \in \mathbb{N}\}$ (formed by its own terms). We then have the following examples:

Example 5. ([6] and [11])

1. The sequence $\{q^n\}_{n\geq 0}$ where $|q| \geq 2$ is self simultaneously ordered. Denoting by S_q the subset $S_q = \{q^n \mid n \in \mathbb{N}\}$, it follows that:
   $$n!_{S_q} = q^{\frac{n(n-1)}{2}}(q^n - 1)(q^{n-1} - 1)\ldots(q-1) \text{ (Jackson's factorials).}$$
2. The sequence $\{n^2\}_{n\geq 0}$ is self simultaneously ordered. Denoting by $\mathbb{N}^{(2)}$ the subset $\mathbb{N}^{(2)} = \{n^2 \mid n \in \mathbb{N}\}$, it follows that $n!_{\mathbb{N}^{(2)}} = \frac{(2n)!}{2}$. Moreover, the subset $\mathbb{N}^{(k)} = \{n^k \mid n \in \mathbb{N}\}$ admits a simultaneous ordering if and only if $k = 1$ or 2 (recall that the sequence $\{n\}_{n\geq 0}$ of natural numbers is a simultaneous ordering of $\mathbb{N}$).
3. The sequence $\left\{\frac{n(n+1)}{2}\right\}_{n\geq 0}$ of triangular numbers is self simultaneously ordered. Denoting by S the subset $S = \left\{\frac{n(n+1)}{2} \mid n \geq 0\right\}$, it follows that $n!_S = \frac{(2n)!}{2^n}$.

Noticing that if S admits a simultaneous ordering $\{a_n\}_{n\geq 0}$, then $T = bS+c$ where $b,c \in \mathbb{Z}, b \neq 0$, admits also a simultaneous ordering, namely $\{ba_n + c\}_{n\geq 0}$, we thus have many other simultaneously ordered subsets of $\mathbb{Z}$.

We also have the following for discrete dynamical systems.

Proposition 17. [6, Prop. 18] *Consider the dynamical system $(\mathbb{Z}, f)$ formed by the set $\mathbb{Z}$ and a non-constant polynomial $f \in \mathbb{Z}[X]$ distinct from $\pm X$. Then, for every $x \in \mathbb{Z}$, the sequence $\{f^n(x)\}_{n\geq 0}$, where f^n denotes the n-th iterate of f, is self simultaneously ordered. In other words, each orbit admits a simultaneous ordering.*

Equivalently, for every $x \in \mathbb{Z}$ and for all $m,n \in \mathbb{N}$ with $m \geq n \geq 1$:

$$\prod_{j=0}^{n-1}(f^n(x) - f^j(x)) \text{ divides } \prod_{j=0}^{n-1}(f^m(x) - f^j(x)).$$

Note that the sequence $\{q^n\}_{n\geq 0}$ in Example 5.1 stems from a dynamical system with $f(X) = qX$ and $x = 1$. By considering the orbit of 3 under the iteration of $X^2 - 2X + 2$, we obtain:

Corollary 2. *The sequence formed by the Fermat numbers $\{F_n = 2^{2^n} + 1\}_{n\geq 0}$ is self simultaneously ordered.*

Question 2. Are there other natural examples of subsets of $\mathbb{Z}$ admitting simultaneous orderings?

5.3 The General Case

To obtain regular bases, if any, we use the Chinese remainder theorem. Analogously to [17, Lemma II.3.4] or following [9, Theorem 11], we have:

Proposition 18. *For each $\mathfrak{m} \in \mathrm{Max}(D)$, let $\{a_{\mathfrak{m},n}\}_{n\geq 0}$ be an $\mathfrak{m}$-ordering of S. For $n > 0$, let $\{b_{n,k}\}_{0\leq k<n}$ be elements of D such that*

$$v_{\mathfrak{m}}(b_{n,k} - a_{\mathfrak{m},k}) > w_{S,\mathfrak{m}}(n) \text{ for all } \mathfrak{m} \text{ such that } w_{S,\mathfrak{m}}(n) \neq 0. \tag{31}$$

Finally, let $g_n(X) = \prod_{k=0}^{n-1}(X - b_{n,k})$. Then, the fixed divisor $d(g_n, S)$ of g on S (as defined before Proposition 8) is equal to the n-th factorial $n!_S^D$.

We derive a few corollaries with the g_n's as defined in Proposition 18.

Corollary 3. *We have the following isomorphism of D-modules:*

$$\mathrm{Int}(S,D) = \oplus_{n\geq 0}\mathfrak{I}_n(S,D)\, g_n(X). \tag{32}$$

We just have to verify that $c_0 g_0(X) + c_1 g_1(X) + \ldots + c_n g_n(X)$ with $c_j \in K$, belongs to $\mathrm{Int}(S,D)$ if and only if, for $0 \leq j \leq n$, $c_j \in \mathfrak{I}_j(S,D)$.

Corollary 4. *If the factorial ideals are principal, writing $n!_S^D = d_n D$, the polynomials $\frac{1}{d_n} g_n(X)$ then form a regular basis of the D-module $\mathrm{Int}(S,D)$.*

But at any rate, since a non-finitely generated projective module over a Dedekind domain is free:

Corollary 5. *The D-module* $\mathrm{Int}(S,D)$ *is free.*

Yet, if there is no regular basis, that is, if the factorial ideals are not principal, it may be difficult to describe a basis.

5.4 Pólya Groups and Pólya Fields

In this paragraph, K denotes an algebraic number field and $\mathcal{O}_K$ its ring of integers. We restrict here our study to the case where $S = D = \mathcal{O}_K$. The following group is a measure of the obstruction for $\mathrm{Int}(\mathcal{O}_K)$ to have a regular basis.

Definition 6. [17, Def. II.3.8] The *Pólya group* of K is the subgroup $\mathcal{P}o(K)$ of the class group of K generated by the classes of the factorial ideals of $\mathcal{O}_K$.

One knows [17, II.3.9] that $\mathcal{P}o(D)$ is also generated by the classes of the ideals:

$$\Pi_q(D) = \prod_{\mathfrak{m}\in\mathrm{Max}(D),\, N(\mathfrak{m})=q} \mathfrak{m} \quad (q \geq 2).$$

Proposition 19. *A Pólya field is a number field K which satisfies the following equivalent assertions*:

1. $\mathrm{Int}(\mathcal{O}_K)$ *admits a regular basis,*
2. *the fractional ideals $\mathfrak{I}_n(\mathcal{O}_K)$ are principal,*
3. *the integral ideals $(n!)_{\mathcal{O}_K}$ are principal,*
4. *the ideals $\Pi_q(\mathcal{O}_K)$ are principal,*
5. *$Po(K) \simeq \{1\}$.*

If $K/\mathbb{Q}$ is a galoisian extension, for every prime p, one has $p\mathcal{O}_K = \Pi_{p^f}(\mathcal{O}_K)^e$, and hence, to know whether K is a Pólya field, we just have to consider the ideals $\Pi_q(\mathcal{O}_K)$ such that the maximal ideals $\mathfrak{m}$ with $N(\mathfrak{m}) = q$ lye over the primes p that are ramified in K [41]. We also have:

Proposition 20. [22, Prop. 3.6] *If $K_1/\mathbb{Q}$ and $K_2/\mathbb{Q}$ are two galoisian extensions whose degrees are relatively prime, then :*

$$Po(K_1K_2) \simeq Po(K_1) \times Po(K_2). \tag{33}$$

For quadratic fields, the Pólya group corresponds to the group of ambiguous classes whose description was done by Hilbert (see [17, Proposition II.4.4]) and the characterization of the quadratic Pólya fields was done by Zantema [47]. Every cyclotomic field is a Pólya field [47]. A systematic study of the galoisian Pólya fields of degree ≤ 6 was recently undertaken by Leriche. For instance,

Proposition 21. [37, Prop. 3.2] *The cyclic cubic Pólya fields are the fields* $\mathbb{Q}[t]$ *where* t *is a root of* X^3-3X+1 *or of* $X^3-3pX-pu$ *where* p *is a prime of the form* $\frac{1}{4}(u^2+27w^2)$, $u \equiv 2 \pmod 3$ *and* $w \neq 0$.

She also characterized the galoisian Pólya fields in the cases of cyclic quartic or sextic fields (the latter, compositum of a cyclic cubic Pólya field and a quadratic Pólya field) as well as in the cases of cyclic fields of the form $\mathbb{Q}[j,\sqrt[3]{m}]$, and of biquadratic fields [37].

Adam [3] undertook a similar study for functions fields. He proved analogously that every cyclotomic function field in the sense of Carlitz is a Pólya field, and characterized the Kummer extensions and the 'totally imaginary' Artin-Scheier extensions of $\mathbb{F}_q(T)$ which are Pólya function fields.

Another interesting notion is the notion of Pólya extension : L/K is a *Pólya extension* if the $\mathcal{O}_L$-module $\mathrm{Int}(\mathcal{O}_K,\mathcal{O}_L)$ admits a regular basis. By the capitulation theorem, the Hilbert class field H_K of every number field K is a Pólya extension of K. Moreover, H_K gives an answer to the following embedding problem: is every number field contained in a Pólya number field? The answer is yes because it turns out that every Hilbert class field is a Pólya field [38, Corollary 3.2]. An open question is to determine the minimal degree of a Pólya field containing K.

6 Computation and Explicit Formulas

In this section, we restrict our study to the local case and consider a slightly more general situation because the notion of v-ordering may be defined for rank-one valuations v of K, that is, valuations v such that $v(K^*) \subseteq \mathbb{R}$. But since there do not always exist v-orderings, we have to assume conditions on S, for instance that S is *precompact*, that is, that its completion is compact (cf. [19, Cor. 1.6]).

Notation for Section 6. *K is a valued field endowed with a rank-one valuation v, the valuation domain is denoted by V, its maximal ideal by $\mathfrak{m}$, and S is an infinite precompact subset of V.*

In this general framework, all results of Paragraph 4.2 hold.

For $a \in V$ and $\gamma \in \mathbb{R}$, we denote by $B(a,\gamma)$ the ball of center a and radius $e^{-\gamma}$:

$$B(a,\gamma) = \{x \in V \mid v(a-x) \geq \gamma\}.$$

6.1 How Can we Compute the Function $w_S(n)$?

We are interested here in the function $w_S(n) = -v(\mathfrak{I}_n(S,D)) = v(n!_S^D)$. The sequence $\{w_S(n)\}_{n\geq 0}$ is called the *characteristic sequence* of S.

Lemma 2. *If $\{a_n\}_{n\geq 0}$ is an $\mathfrak{m}$-ordering of S then, for every nonzero $b \in V$ and every $c \in V$, $\{ba_n + c\}_{n\geq 0}$ is an $\mathfrak{m}$-ordering of $bS+c = \{bs+c \mid s \in S\}$. Thus,*

$$\forall b, c \in V \quad w_{bS+c}(n) = nv(b) + w_S(n).$$

Lemma 3. [16, Lemma 3.4] *If $\{a_n\}_{n\geq 0}$ is a v-ordering of S then the subsequence formed by the a_n's which are in a ball $B(a,\gamma)$ is a v-ordering of $S \cap B(a,\gamma)$.*

Proposition 22. [31, Lemma 2] *Let $\{s_i \mid 1 \leq i \leq r\}$ be a system of representatives of S modulo $\mathfrak{m}$, that is, $S = \cup_{i=1}^r (S \cap (s_i + \mathfrak{m}))$ (where $s_i \not\equiv s_j \pmod{\mathfrak{m}}$ $\forall i \neq j$). If, for each i, $\{a_{i,n}\}_{n\geq 0}$ is a v-ordering of $S \cap (s_i + \mathfrak{m})$, we obtain a v-ordering of S by shuffling these v-orderings in such a way that the shuffling of the corresponding characteristic sequences leads to a non-decreasing sequence of integers.*

In particular, the characteristic sequence of S is the disjoint union of the characteristic sequences of the $S \cap (s_i + \mathfrak{m})$ sorted into a non-decreasing order.

Example 6. Assume that $\mathfrak{m} = \pi V$ and $\mathrm{Card}(V/\mathfrak{m}) = q < +\infty$. Consider

$$S = V \setminus \mathfrak{m} = \cup_{i=1}^{q-1} (a_i + \mathfrak{m}). \tag{34}$$

Let $\{a_n\}_{n\geq 0}$ be the v-ordering of V given by (21). For $1 \leq i \leq q-1$, $\{a_i + \pi a_n\}_{n\geq 0}$ is a v-ordering of $a_i + \mathfrak{m}$, and $w_{a_i+\mathfrak{m}}(n) = w_{\pi V}(n) = n + w_V(n) = n + w_q(n)$. We construct a v-ordering of S by taking successively one element of each of the $q-1$ partial orderings since the characteristic sequences of the sets $a_i + \mathfrak{m}$ are equal, we obtain the subsequence formed by the a_n's such that $v(a_n) = 0$, and we have:

$$w_{V\setminus\mathfrak{m}} = \left[\frac{n}{q-1}\right] + w_q\left(\left[\frac{n}{q-1}\right]\right) = \sum_{k\geq 0} \left[\frac{n}{(q-1)q^k}\right]. \tag{35}$$

6.2 Toward Symmetry : Homogeneous Subsets

As seen in Example 6, symmetry may help for the shuffling. We have a kind of symmetry when we consider *homogeneous subsets*, that is, subsets S for which there exists some $\gamma \in \mathbb{R}$ such that $S = \cup_{s\in S} B(s,\gamma)$.

Proposition 23. [16, Th. 3.6] *Assume that*
$S = \cup_{i=1}^r B(b_i,\gamma)$ *where* $v(b_i - b_j) < \gamma$ $(1 \leq i \neq j \leq r)$. *Then,*
$w_S(n) = \max_{\delta_1+\ldots+\delta_r=n} \left(\min_{1\leq i\leq r} w_S^i(\delta_1,\ldots,\delta_r)\right) \quad (\delta_1,\ldots,\delta_r \in \mathbb{N})$ *where*
$w_S^i(\delta_1,\ldots,\delta_r) = w_q(\delta_i) + \gamma\delta_i + \sum_{j\neq i} v(b_j - b_i)\delta_j$ *and* $w_q(m) = \sum_{k\geq 1} \left[\frac{m}{q^k}\right]$.

We may introduce more symmetry by assuming that $q = +\infty$ $(w_q(\delta_i) = 0)$:

Proposition 24. [15, Th. 4.4] *Assume that* $\mathrm{Card}(V/\mathfrak{m}) = +\infty$ *and that*

$$S = \cup_{i=1}^r B(b_i,\gamma) \quad \textit{where} \quad v(b_i - b_j) < \gamma \; (1 \leq i \neq j \leq r). \tag{36}$$

Consider the symmetric matrix $B = (\beta_{i,j}) \in \mathscr{M}_r(\mathbb{R})$ defined by

$$\beta_{i,j} = v(b_i - b_j) \text{ for } 1 \leq i \neq j \leq r \text{ , and } \beta_{i,i} = \gamma \text{ for } 1 \leq i \leq r. \tag{37}$$

Denote by B_i the matrix deduced from B by replacing every coefficients in i-th column by 1 and let $\nu(B) = \sum_{i=1}^r \det(B_i)$. If $n = m\nu(B) + n_0$ where $0 \leq n_0 < \nu(B)$, then $w_S(n) = m\det(B) + w_S(n_0)$.

6.3 Preregular Subsets: Generalized Legendre's Formula

The best way to obtain symmetry is by considering the notion of pre-regular subset which extends Amice's notion of regular compact subset of a local field [8].

To explain this notion, we introduce the following equivalence relations on V :

$$\forall \gamma \in \mathbb{R} \ \forall x,y \in V \quad x \equiv y \pmod{\gamma} := v(x-y) \geq \gamma.$$

We denote by $S \bmod \gamma$ the set of equivalence classes of the elements of S, and let:

$$q_\gamma = \mathrm{Card}(S \bmod \gamma)$$

The fact that S is an infinite precompact subset of V is equivalent to:

all the q_γ's are finite and $\lim_{\gamma \to +\infty} q_\gamma = +\infty$.

Definition 7. The precompact subset S is *preregular* if, for all $\gamma < \delta$, for every $x \in S$, $S \cap B(x,\gamma)$ contains exactly $\frac{q_\delta}{q_\gamma}$ non-empty subsets of the form $S \cap B(y,\delta)$.

This notion will allow us to generalize the following well known formulas obtained for v discrete, $\mathrm{Card}(V/\mathfrak{m}) = q$ and S regular:

$$v_p(n!) = \sum_{k\geq 1} \left[\frac{n}{p^k}\right] \ \text{Legendre 1808} \qquad v(n!_V) = \sum_{k\geq 1} \left[\frac{n}{q^k}\right] \ \text{Pólya 1909} \qquad v(n!_S) = \sum_{k\geq 1} \left[\frac{n}{q_k}\right] \ \text{Amice 1964}$$

Proposition 25. [25, Th. 1.5] *The precompact subset S is preregular if and only if, denoting by γ_k the critical valuations of S,*

$$v(n!_S) = n\gamma_0 + \sum_{k\geq 1} \left[\frac{n}{q_{\gamma_k}}\right](\gamma_k - \gamma_{k-1}) \qquad \text{[25] 2013} \tag{38}$$

Recall that the sequence $\{\gamma_k\}_{k\geq 0}$ of *critical valuation* of S is characterized by [23, Prop. 5.1] $\gamma_0 = \inf_{x\in S} v(x)$ and, for $k \geq 1 : \gamma_{k-1} < \gamma \leq \gamma_k \Leftrightarrow q_\gamma = q_{\gamma_k}$.

Let us mention an application of regularity to dynamical systems:

Proposition 26. [26, Cor. 4] *Assume that S is a regular compact subset of V and let $\varphi : S \to S$ be an isometry. Then, the discrete dynamical system (S,φ) is* minimal *(that is, for every $x \in S$, the orbit $\Omega(x) = \{\varphi^n(x) \mid n \in \mathbb{N}\}$ is dense in S) if and only if, for every $x \in S$, the sequence $\{\varphi^n(x)\}_{n\geq 0}$ is a v-ordering of S.*

6.4 Valuative Capacity

Since the function w_S is *super-additive* (that is, $w_S(n+m) \geq w_S(n) + w_S(m)$) the following limit, finite or infinite, called the *valuative capacity* of S, exists:

$$\delta_S = \lim_{n\to+\infty} \frac{w_S(n)}{n}. \tag{39}$$

The larger S, the smaller is δ_S. It is also equal to the limit [21, Th. 4.2]:

$$\lim_{n\to+\infty} \frac{2}{n(n+1)} \min_{x_0,\dots,x_n\in S} v(\prod_{0\leq i<j\leq n}(x_j - x_i)) = \delta_S. \tag{40}$$

In some sense its generalizes the notion of transfinite diameter in archimedean metric. For instance, $\delta_{\mathbb{Z}_{(p)}} = \frac{1}{p-1}$, $\delta_{p\mathbb{Z}_{(p)}} = \frac{p}{p-1}$, $\delta_{\mathbb{Z}_{(p)}\setminus p\mathbb{Z}_{(p)}} = \frac{p}{(p-1)^2}$.

Proposition 27. [31, Cor. 10] *Assume that* $S = \cup_{i=1}^r (S \cap (s_i + \mathfrak{m}))$ (*where* $s_i \not\equiv s_j \pmod{\mathfrak{m}}$ $\forall i \neq j$). *If, for* $i = 1,\dots,r$, $\delta_{S\cap(s_i+\mathfrak{m})} \neq 0$*, then* $\frac{1}{\delta_S} = \sum_{i=1}^r \frac{1}{\delta_{S\cap(s_i+\mathfrak{m})}}$.

This last result allows to compute easily some valuative capacities. For instance,

Corollary 6. [31, Prop. 11] *Assume v is discrete and* $\mathrm{Card}(V/\mathfrak{m}) = q < +\infty$:

$$\delta_{V\setminus\mathfrak{m}^k} = \frac{1}{(q-1)^2}\left(q - \frac{q^{2k}-q^2}{q^{2k}-1}\right).$$

More generally, in the spirit of Proposition 24:

Proposition 28. [15, Th 5.3] *Without particular hypothesis, denote by q the cardinality, finite or infinite, of the residue field. Assume that*

$$S = \cup_{i=1}^r B(b_i,\gamma) \quad \textit{where} \quad v(b_i - b_j) < \gamma \ (1 \leq i \neq j \leq r).$$

Consider the matrix $B^* = B + \frac{1}{q-1} I_r$ *and the number* $\nu(B^*)$ *with* B *and* $\nu(B^*)$ *as defined in Proposition 24. Then,* $\delta_S = \frac{\det(B^*)}{\nu(B^*)}$.

6.5 A Generalized Exponential Function

Returning to $\mathbb{Z}$, by analogy with the classical factorials and following [11, Question 33], we introduce an *exponential function* associated to any subset S of $\mathbb{Z}$:

$$\exp_S(x) = \sum_{n\geq 0} \frac{x^n}{n!_S} \tag{41}$$

where $n!_S$ denotes here the positive generator of the corresponding factorial ideal. By Proposition 9, $n!$ divides $n!_S$, and hence, the power series converges for all x.

We have the obvious formula

$$\exp_{bS+c}(x) = \sum_{n\geq 0} \frac{x^n}{b^n n!_S} = \exp_S\left(\frac{x}{b}\right) \tag{42}$$

When there exists a simultaneous ordering, it is sometimes easy to compute this exponential function. For instance, Examples 4 lead to:

$$\exp_{\mathbb{N}^{(2)}}(x) = \sum_{n\geq 0} \frac{x^n}{\frac{(2n)!}{2}} = 2\cosh\sqrt{|x|}\ ,\ \exp_{\left\{\frac{n(n+1)}{2}\mid n\geq 0\right\}}(x) = \sum_{n\geq 0} \frac{x^n}{\frac{(2n)!}{2^n}} = \cosh\sqrt{2|x|}.$$

In particular, let us consider the value for $x=1$ and introduce the number

$$e_S = \sum_{n\geq 0} \frac{1}{n!_S}. \tag{43}$$

For instance, for $S=\mathbb{N}^{(2)}$, $e_{\mathbb{N}^{(2)}} = \mathrm{e} + \frac{1}{\mathrm{e}}$.

Proposition 29. Mingarelli [40, Th. 28] *The number e_S is irrational.*

Question 3. [40] For which subsets S is e_S a transcendental number?

7 Sub-Algebras of $\mathrm{Int}(S,D)$

7.1 Derivatives and Finite Differences

Among the interesting sub-D-algebras of $\mathrm{Int}(S,D)$, one may consider the algebras $\mathrm{Int}^{(k)}(S,D)$ formed by the polynomials that are integer-valued on S together with their *derivatives* up to the order k :

$$\mathrm{Int}^{(k)}(S,D) = \{f(X)\in K[X] \mid f^{(h)}\in \mathrm{Int}(S,D)\ \ 0\leq h\leq r\}, \tag{44}$$

and the algebras $\mathrm{Int}^{[k]}(D)$ formed by the polynomials that are integer-valued on D together with their *finite differences* up to the order k defined inductively by $\mathrm{Int}^{[0]}(D)=\mathrm{Int}(D)$ and, for $k\geq 1$:

$$\mathrm{Int}^{[k]}(D) = \left\{ f\in K[X] \mid \forall h\in D, h\neq 0\ (f(X+h)-f(X))/h \in \mathrm{Int}^{[k-1]}(D)\right\}.$$

A review on these algebras is given in [17, Chap. IX]. Yet, new results appeared in characteristic $p>0$. For instance:

Proposition 30. [4, Th. 2.11] *Let* $\mathrm{Int}^{(\infty)}(S,D) = \cap_{k\geq 0}\mathrm{Int}^{(k)}(S,D)$. *If* $\mathrm{char}(D)=p>0$*, then*

$$\mathfrak{I}_n(\mathrm{Int}^{(\infty)}(S,D))^{-1} = \textstyle\prod_{\mathfrak{m}\in \mathrm{Max}(D)} \mathfrak{m}^{w_{S,\mathfrak{m}}([\frac{n}{p}])}.$$

Moreover, if $\{f_n(X)\}_{n\geq 0}$ is a regular basis of $\mathrm{Int}(S,D)$*, then the polynomials $F_{m,j} = f_m(X)^p X^j \quad (m\in\mathbb{N}, j\in\{0,\ldots,p-1\})$ form a regular basis of* $\mathrm{Int}^{(\infty)}(S,D)$.

In particular, we have an explicit basis for $\mathrm{Int}^{(\infty)}(\mathbb{F}_q[T])$ thanks to Example 4.3.

Proposition 31. [2, Th. 16] *If* $\mathrm{char}(D)=p>0$*, then*

$$\mathfrak{I}_n(\mathrm{Int}^{[k]}(D))^{-1} = \prod_{q\leq n}\left(\prod_{\mathfrak{m}\in\mathrm{Max}(D),N(\mathfrak{m})=q}\mathfrak{m}\right)^{\delta_q^{[k]}(n)}$$

where $\delta_q^{[k]}(n) = w_q(n) - \lambda_q^{[k]}(n)$ *and*

$\lambda_q^{[k]}(n) = \sup\left\{v_q(j_1)+\ldots+v_q(j_r) \mid r\leq k, j_1+\ldots+j_r\leq n, j_i\geq 1, p\nmid\binom{n}{j_1,\ldots,j_r}\right\}.$

There are also results about a multiplicative analog of finite differences, namely, the *Euler-Jackson differences* (see [7]): let $S_q = \{q^n \mid n\geq 0\}$ where q denotes a non-zero element of D which is not a root of unity, and, for $h\in\mathbb{N}^*$, let

$$\delta_{q^h} f(X) = \frac{f(q^hX)-f(X)}{(q^h-1)X}. \tag{45}$$

Then, $\mathrm{Int}_J^{[k]}(S_q,D)$ is defined inductively by

$$\mathrm{Int}_J^{[k]}(S_q,D) = \{f(X)\in K[X] \mid \forall h\in\mathbb{N}^*\ \delta_{q^h} f(X)\in\mathrm{Int}_J^{[k-1]}(S,D)\}. \tag{46}$$

7.2 Divided Differences

Contrarily to finite differences, divided differences make sense on subsets.

Definition 8. The *divided difference of order k* of a function $f: S\to K$ of one variable is defined inductively on k by: $\Phi^0(f) = f$ and, for $k\geq 1$:

$$\Phi^k(f)(x_0,\ldots,x_{k-1},x_k)\mapsto\frac{\Phi^{k-1}(f)(\ldots,x_{k-2},x_{k-1})-\Phi^{k-1}(f)(\ldots,x_{k-2},x_k)}{x_{k-1}-x_k},$$

defined on $S^{k+1}\setminus\Delta_k$ where $\Delta_k = \{(x_0,\ldots,x_k)\in S^{k+1} \mid x_i = x_j \text{ for some } i\neq j\}$.

The function $\Phi^k(f)$ is symmetric with respect to the $k+1$ variables $x_0,\ldots,x_k$.

Definition 9. The ring $\mathrm{Int}^{\{k\}}(S,D)$ of polynomials *integer-valued on S together with their divided differences* up to the order k is

$$\mathrm{Int}^{\{k\}}(S,D) = \{f\in K[X] \mid \Phi^h(f)(S^{h+1})\subseteq D\ \ 0\leq h\leq k\}. \tag{47}$$

The algebraic properties of this ring are studied in [13]. One has the containments:

$$\mathrm{Int}^{\{k\}}(D)\subseteq\mathrm{Int}^{[k]}(D)\subseteq\mathrm{Int}^{(k)}(D) \tag{48}$$

with the equality $\mathrm{Int}^{\{1\}}(D) = \mathrm{Int}^{[1]}(D)$ and, for every subset S, $\mathrm{Int}^{\{k\}}(S,D)\subseteq\mathrm{Int}^{(k)}(S,D)$. The construction of bases of $\mathrm{Int}^{\{k\}}(S,D)$ is described in [12].

Let us focus on the local case. As in §6, V denotes a rank-one valuation domain and S a precompact subset of V.

Definition 10. (Bhargava [12]) A *r-removed v-ordering* of S is a sequence $\{a_n\}_{n\geq 0}$ of elements of S where $a_0,a_1,\ldots,a_r$ are chosen arbitrarily and, for $n>r$, there exist r distinct integers $i_1,\ldots,i_r\in\{0,1,\ldots,n-1\}$ such that

$$v\left(\prod_{\substack{0\le k<n\\ k\ne i_1,\dots,i_r}}(a_n-a_k)\right)=\inf_{\substack{x\in S\\ 0\le j_1<\ldots<j_r<n}} v\left(\prod_{\substack{0\le k<n\\ k\ne j_1,\dots,j_r}}(x-a_k)\right)$$

Let $\alpha_n=\{1,\ldots,n\}\setminus\{i_1,\ldots,i_r\}$ be the set formed by the remaining indices.

Proposition 32. (Bhargava [12], 2009) *If $\{a_n\}_{n\in\mathbb{N}}$ is a r-removed v-ordering of S, then the following polynomials form a basis of the V-module* $\mathrm{Int}^{\{r\}}(S,V)$*:*

$$\binom{x}{n}^{\{r\}}_{\{a_k\}}=\frac{(x-a_0)(x-a_1)\cdots(x-a_{n-1})}{\prod_{k\in\alpha_n}(a_n-a_k)}. \tag{49}$$

In general, it is not so easy to construct a r-removed v-ordering of S, nor to compute the valuation of the denominator, that is, the number:

$$w_S^{\{r\}}(n)=v\left(\prod_{k\in\alpha_n}(a_n-a_k)\right). \tag{50}$$

Johnson [32] was the first one to give an explicit formula for $w_S^{\{r\}}(n)$ in case $S=V=\mathbb{Z}_{(p)}$. Given n, there is a unique integer l such that $rp^l\le n<rp^{l+1}$, and with this l, one has the formula:

$$w_{\mathbb{Z}_{(p)}}^{\{r\}}(n)=\sum_{k=1}^{l}\left[\frac{n}{p^k}\right]-r\times l. \tag{51}$$

This formula can be generalized to preregular precompact subsets as defined in § 6.3. For such a subset, the q_γ's (for $\gamma\in\mathbb{R}$) and the critical valuations γ_k's are also defined in § 6.3. Note that, given n, there is a unique integer l such that $rq_{\gamma_l}\le n<rq_{\gamma_{l+1}}$ and one has:

Proposition 33. [25] *If S is a preregular precompact subset of a rank-one valuation domain then, with the previous notation, for $r\,q_{\gamma_l}\le n<r\,q_{\gamma_{l+1}}$ we have:*

$$w_S^{\{r\}}(n)=n\gamma_0+\sum_{k=1}^{l}\left[\frac{n}{q_{\gamma_k}}\right](\gamma_k-\gamma_{k-1})-r\times\gamma_l\;. \tag{52}$$

Proposition 34. [25] *If S is a preregular precompact subset, then every VWDWO sequence of S is a r-removed v-ordering of S whatever r.*

Recall that a VWDWO sequence of a preregular subset generalizes the VWDWO sequences defined in Proposition 10 and is characterized by:

$$\forall n\ne m\;[v(a_n-a_m)>\gamma\;\Leftrightarrow\;q_\gamma|n-m] \tag{53}$$

Such sequences are easy to describe: they are the 'most regular sequences' in S!

7.3 Integer-Valued Polynomials of a Given Modulus

Definition 11. For every non-zero element a of D, the ring of *integer-valued polynomials on S of modulus a* is the ring

$$\mathrm{Int}_a(S,D) = \{f(X) \in K[X] \mid \forall s \in S \;\; f(aX+s) \in D[X]\}. \tag{54}$$

The algebraic properties of this ring are studied in [13]. We have the following containments: for all a, $\mathrm{Int}_a(S,D) \subseteq \mathrm{Int}(S,D)$, and if a divides b in D, then $\mathrm{Int}_a(S,D) \subseteq \mathrm{Int}_b(S,D)$. On the other hand, $\mathrm{Int}(S,D) = \cup_{a\in D\setminus\{0\}} \mathrm{Int}_a(S,D)$.

Once more, let us focus on a local study: D is assumed to be a rank-one valuation domain V.

Definition 12. (Bhargava [12]) Let $\alpha \in \mathbb{R}_+$. A *v-ordering of order α* of S is a sequence $\{a_n\}_{n\geq 0}$ of elements of S where a_0 is arbitrarily chosen and, for $n \geq 1$, a_n is chosen such that:

$$\sum_{k=0}^{n-1} \inf(\alpha, v(a_n - a_k)) = \inf_{s\in S} \left(\sum_{k=0}^{n-1} \inf(\alpha, v(s - a_k)) \right). \tag{55}$$

For such a v-ordering of order α, let

$$w_S^{(\alpha)}(n) = \sum_{k=0}^{n-1} \inf(\alpha, v(a_n - a_k)). \tag{56}$$

Proposition 35. (Bhargava [12], 2009) *If $v(a) = \alpha$, if $\{a_n\}_{n\in\mathbb{N}}$ is a v-ordering of order α of S, and if $v(t_n) = w_S^{(\alpha)}(n)$, then the following polynomials form a basis of the V-module* $\mathrm{Int}_a(S,V)$*:*

$$\binom{x}{n}^{(\alpha)}_{\{a_k\}} = \frac{(x-a_0)(x-a_1)\cdots(x-a_{n-1})}{t_n}. \tag{57}$$

For instance, for $\mathrm{Int}_{p^h}(\mathbb{Z}_{(p)})$, we have: $w^{(h)}_{\mathbb{Z}_{(p)}}(n) = \sum_{k=1}^{h} \left[\frac{n}{p^k}\right]$ (Johnson [32]). This formula may generalized:

Proposition 36. *If S is a preregular precompact subset, then*

$$w_S^{(\alpha)}(n) = n\gamma_0 + \sum_{k=1}^{l} \left[\frac{n}{q_{\gamma_k}}\right](\gamma_k - \gamma_{k-1}) \;\; \textit{where} \;\; \gamma_l \leq \alpha < \gamma_{l+1} \tag{58}$$

where the q_γ's and the γ_k's are defined in § 6.3.

As for r-removed v-orderings [Proposition 34], we still have:

Proposition 37. *If S is a preregular precompact subset, then every VWDWO sequence of S is a v-ordering of S of order α whatever α.*

8 Ultrametric Analysis: Extensions of Mahler's Theorem

Mahler's approximation theorem for the Banach ultrametric space $\mathscr{C}(\mathbb{Z}_p,\mathbb{Q}_p)$ (Prop. 4 above) may be generalized by replacing $\mathbb{Q}_p$ by a complete valued field and $\mathbb{Z}_p$ by a precompact subset.

Hypotheses: *K is a valued field endowed with a rank-one valuation v (V denotes the valuation domain) and S an infinite precompact subset of K.*

8.1 Polynomial Approximation in $\mathscr{C}(\widehat{S},\widehat{K})$

We denote by $\mathscr{C}(\widehat{S},\widehat{K})$ the ultrametric Banach space of continuous functions from the completion $\widehat{S}$ of S to the completion $\widehat{K}$ of K endowed with the uniform convergence topology.

Proposition 38. ([14] and [18, Theorem 2.4]) *Let K be a valued field and S be a precompact subset of K. Let $\{a_n\}_{n\geq n}$ be a v-ordering of S. Then, every function $\varphi\in\mathscr{C}(\widehat{S},\widehat{K})$ can be developed in series as follows:*

$$\varphi(x)=\sum_{n=0}^{\infty}c_n\prod_{k=0}^{n-1}\frac{X-a_k}{a_n-a_k}\quad \textit{with } c_n\in\widehat{K} \textit{ and } \lim_{n\to+\infty}v(c_n)=+\infty \tag{59}$$

Moreover, $\inf_{x\in S}\ v(\varphi(x))=\inf_{n\in\mathbb{N}}\ v(c_n)$.

The generalized binomial polynomials $\prod_{k=0}^{n-1}\frac{X-a_k}{a_n-a_k}$ $(n\geq 0)$ form an orthonormal basis of the Banach space $\mathscr{C}(\widehat{S},\widehat{K})$. The coefficients c_n are unique and may be obtained inductively by:

$$c_n=\varphi(a_n)-\sum_{k=0}^{n-1}c_k\prod_{h=0}^{k-1}\frac{a_n-a_h}{a_k-a_h}\ . \tag{60}$$

Once one knows Formula (59), it is easy to prove that:

Proposition 39. [18, Th. 2.7] *Every basis of the V-module* $\mathrm{Int}(S,V)=\mathscr{C}(\widehat{S},\widehat{V})\cap K[X]$ *is an orthonormal basis of the Banach space $\mathscr{C}(\widehat{S},\widehat{K})$.*

8.2 Polynomial Approximation in $\mathscr{C}^r(\widehat{S},\widehat{K})$ and $LA_\alpha(\widehat{S},\widehat{K})$

Recall that, in ultrametric analysis, the Banach space $\mathscr{C}^r(\widehat{S},\widehat{K})$ of *functions of class $\mathscr{C}^r$* is formed by the function $f:\widehat{S}\to\widehat{K}$ such that $\Phi^k(f)$ may be extended continuously to $\widehat{S}^{k+1}$. Proposition 38 may be generalized in the following way:

Proposition 40. [12, Th. 21] *Assume that the precompact subset S has no isolated points. Then, every basis of the V-module* $\mathrm{Int}^{\{r\}}(S,V) = \mathscr{C}^r(\widehat{S},\widehat{V}) \cap K[X]$ *is an orthonormal basis of the Banach space* $\mathscr{C}^r(\widehat{S},\widehat{K})$.

Example 7.

1. The following polynomials form a basis of the $\mathbb{Z}$-module $\mathrm{Int}^{\{1\}}(\mathbb{Z})$ and thus an orthonormal basis of $\mathscr{C}^1(\mathbb{Z}_p,\mathbb{Q}_p)$ for all p : $\prod_{p\in\mathbb{P}} p^{\left[\frac{\ln n}{\ln p}\right]} \binom{x}{n}$ (Johnson [32]).
2. The following polynomials are the first terms of a basis of $\mathrm{Int}^{\{1\}}(\mathbb{P},\mathbb{Z})$:

$$1\ ,\ X-1,\ (X-1)(X-2)\ ,\ \tfrac{1}{2}(X-1)(X-2)(X-3)\ ,$$
$$\tfrac{1}{4}(X-1)(X-2)(X-3)(X-5)\ ,\ \tfrac{1}{48}(X-1)(X-2)(X-3)(X-5)(X-7),\ldots \text{ [24]}$$

Let $\alpha \in \mathbb{R}^+$. The Banach space $LA_\alpha(\widehat{S},\widehat{K})$ of *locally analytic functions of order* α from $\widehat{S}$ to $\widehat{K}$ is formed by the function $f : \widehat{S} \to \widehat{K}$ such that, for each $s \in S$, the restriction of f to $S \cap B(s,\alpha)$ is extendable to an analytic function on the whole ball $B(s,\alpha)$. Note that, if $a \in V$ is such that $v(a) = \alpha$, the polynomials of $K[X]$ which are in $LA_\alpha(\widehat{S},\widehat{V})$ are the elements of $\mathrm{Int}_a(S,V)$.

Proposition 41. [12, Th. 28] *Assume that the precompact subset S has no isolated points and let $a \in V$ and $\alpha = v(a)$. Then, every basis of the V-module* $\mathrm{Int}_a(S,V)$ *is an orthonormal basis of the Banach space* $LA_\alpha(\widehat{S},\widehat{K})$.

9 Generalizations

Let again D be a Dedekind domain with quotient field K.

9.1 Integer-Valued Polynomials in Several Indeterminates

Let m be a positive integer, let $\underline{S}$ be a subset of D^m, and consider the D-algebra

$$\mathrm{Int}(\underline{S},D) = \{f(X_1,\ldots,X_m) \in K[X_1,\ldots,X_m] \mid f(\underline{S}) \subseteq D\}. \tag{61}$$

Most of the results about $\mathrm{Int}(\underline{S},D)$ which are gathered in [17, Chap. XI] concern subsets $\underline{S}$ of the form $S = \prod_{i=1}^m S_i$. In 2000, Bhargava [11, §12] suggested some ways to define generalized factorials for all subsets $\underline{S}$ and, only recently, several interesting results were published (Evrard [27]).

Following [27], let $\mathfrak{I}_n(\underline{S},D)$ be the D-module generated by *all* the coefficients of the polynomials of total degree n in $\mathrm{Int}(\underline{S},D)$ and let

$$n!_{\underline{S}}^D = \mathfrak{I}_n(\underline{S},D)^{-1} = \{x \in D \mid xf \in D[X_1,\ldots,X_m]\ \forall f \in \mathrm{Int}(\underline{S},D), \deg(f) \leq n\}.$$

One may compute these factorials by means of a generalized notion of v-ordering. For this, we must first assume that no non-zero polynomial $f \in K[X_1, \ldots, X_m]$ is such that $f(\underline{S}) = 0$ (the analog of $\mathrm{Card}(S) = \infty$ for one variable).

Then, write all the monomials in a sequence $(m_l)_{l \geq 0}$ in a way compatible with the total degree, that is such that: $\deg(m_l) < \deg(m_{l'}) \Rightarrow l < l'$.

Finally, for $l \geq 1$ and any sequence $(\underline{x_0}, \ldots, \underline{x}_{l-1})$ of elements of D^m, let

$$\Delta(\underline{x_0}, \ldots, \underline{x}_{l-1}) = \det\left(m_j(\underline{x_i})\right)_{0 \leq i,j < l} \ . \tag{62}$$

Definition 13. A v-ordering of $\underline{S}$ is a sequence $\{\underline{a}_k\}_{k \geq 0}$ of elements of $\underline{S}$ such that, for every $k \geq 1$: $v(\Delta(\underline{a}_0, \ldots, \underline{a}_k)) = \inf_{\underline{x} \in S} v\left(\Delta(\underline{a}_0, \ldots, \underline{a}_{k-1}, \underline{x})\right)$.

Proposition 42. [27] *Let $\{\underline{a}_k\}_{k \geq 0}$ be a sequence of elements of $\underline{S}$ such that, for every $k \geq 0$, $\Delta(\underline{a}_0, \ldots, \underline{a}_k) \neq 0$ and consider the associated sequence of polynomials:*

$$F_k(\underline{X}) = \frac{\Delta(\underline{a}_0, \ldots, \underline{a}_{k-1}, \underline{X})}{\Delta(\underline{a}_0, \ldots, \underline{a}_k)} \ .$$

Then the following assertions are equivalent:

1. *$\{\underline{a}_k\}_{k \geq 0}$ is a v-ordering of $\underline{S}$.*
2. *For every $k \geq 0$, $F_k \in \mathrm{Int}(\underline{S}, D)$.*
3. *$\{F_k(\underline{X})\}_{k \geq 0}$ is a basis of the V-module $\mathrm{Int}(\underline{S}, V)$.*
4. *For every $f(\underline{X}) \in K[\underline{X}]$, if the indices of the monomials of f are $< k$, then*

$$f \in \mathrm{Int}(\underline{S}, V) \Leftrightarrow f(\underline{a}_0), \ldots, f(\underline{a}_{k-1}) \in V \ .$$

Then, we can compute the factorials of $\underline{S}$ by globalization. Properties A, B, C and D still hold for these factorials [27]. We do not know whether Property E is still true and if we have a generalized Property K (Kempner's formula).

9.2 Other Generalizations

9.2.1 Homogeneous Integer-Valued Polynomials

Johnson and Patterson [34] introduced a notion of *projective v-ordering* to construct bases of homogeneous polynomials in (only) two variables. For instance, they considered the $\mathbb{Z}_2$-module:

$$\{f \in \mathbb{Q}_2[X,Y] \mid f \text{ homogeneous}, \ \deg(f) = 3, f(\mathbb{Z}_2 \times \mathbb{Z}_2) \subseteq \mathbb{Z}_2\}$$

and obtained the following basis:

$$Y^3, XY^2, X^2(X-Y), XY(X-Y)/2.$$

9.2.2 Integer-Valued Polynomials on Non-Commutative Algebras

There are several works about algebras of *integer-valued polynomials on quaternions* (Werner [44], Johnson and Pavlovski [35]), but only partial results about the additive structure.

There are also several works about algebras of *integer-valued polynomials on matrices* (Frisch [30], Werner [45]). Let us recall the only case where we know a basis, that is, the case of integer-valued polynomials on triangular matrices.

Let $\mathscr{M}_n(D)$ denote the ring of $n \times n$ matrices with coefficients in D and, for every subset S of $\mathscr{M}_n(D)$, let

$$\mathrm{Int}(S, \mathscr{M}_n(D)) = \{f(X) \in K[X] \mid f(S) \subseteq \mathscr{M}_n(D)\}. \tag{63}$$

Denoting by $T_n(D)$ the subring of $\mathscr{M}_n(D)$ formed by triangular matrices, Evrard, Fares and Johnson [28] obtained the equality:

$$\mathrm{Int}(T_n(D), \mathscr{M}_n(D)) = \{f \in K[X] \mid f(T_n(D) \subseteq \mathscr{M}_n(D)\} = \mathrm{Int}^{\{n-1\}}(D). \tag{64}$$

It seems that today there is no other published construction of bases of such integer-valued polynomials on non-commutative algebras...

Acknowledgements The author thanks the anonymous referee for many valuable suggestions.

References

1. D. Adam, Simultaneous orderings in function fields, J. Number Theory **112** (2005), 287–297.
2. D. Adam, Finite differences in finite characteristic, J. Algebra **296** (2006), 285–300.
3. D. Adam, Pólya and Newtonian function fields, Manuscripta Math. **126** (2008), 231–246.
4. D. Adam, Polynômes à valeurs entières ainsi que leurs dérivées en caractéristique *p*, Acta Arith. **148** (2011), 351–365.
5. D. Adam and P.-J. Cahen, Newtonian and Schinzel quadratic fields, J. Pure and Appl. Algebra **215** (2011), 1902–1918.
6. D. Adam, J.-L. Chabert, and Y. Fares, Subsets of $\mathbb{Z}$ with simultaneous orderings, Integers **10** (2010), 437–451.
7. D. Adam and Y. Fares, Integer-valued Euler-Jackson's finite differences, Monatsh. Math. (2010), 15-32.
8. Y. Amice, Interpolation *p*-adique, Bull. Soc. math. France **92** (1964), 117–180.
9. M. Bhargava, *P*-orderings and polynomial functions on arbitrary subsets of Dedekind rings, J. reine angew. Math. **490** (1997), 101-127.
10. M. Bhargava, Generalized Factorials and Fixed Divisors over Subsets of a Dedekind Domain, J. Number Theory **72** (1998), 67–75.
11. M. Bhargava, The Factorial Function and Generalizations, Amer. Math. Monthly, **107** (2000), 783-799.
12. M. Bhargava, On *p*-orderings, integer-valued polynomials, and ultrametric analysis, J. Amer. Math. Soc. **22** (2009), 963–993.
13. M. Bhargava, P.-J. Cahen, and J. Yeramian, Finite generation properties for various rings of integer-valued polynomials, J. Algebra **322** (2009), 1129–1150.

14. M. Bhagava and K. Kedlaya, Continuous functions on compact subsets of local fields, Acta Arith. **91** (1999), 191–198.
15. J. Boulanger and J.-L. Chabert, Asymptotic behavior of characteristic sequences of integer-valued polynomials, J. Number Theory **80** (2000), 238-259.
16. J. Boulanger, J.-L. Chabert, S. Evrard and G. Gerboud, The Characteristic Sequence of Integer-Valued Polynomials on a Subset, in *Advances in Commutative Ring Theory*, 161-174, Lecture Notes in Pure and Appl. Math. **205**, Dekker, New York, 1999.
17. P.-J. Cahen and J.-L. Chabert, *Integer-Valued Polynomials*, Amer. Math. Soc. Surveys and Monographs, **48**, Providence, 1997.
18. P.-J. Cahen and J.-L. Chabert, On the ultrametric Stone-Weierstrass theorem and Mahler's expansion, J. Théor. Nombres Bordeaux, **14** (2002), 43–57.
19. P.-J. Cahen, J.-L. Chabert, and K.A. Loper, High dimension Prüfer domains of integer-valued polynomials, J. Korean Math. Soc., **38** (2001), 915–935.
20. L. Carlitz, A class of polynomials, Trans. Amer. math. Soc. **43** (1938), 167–182.
21. J.-L. Chabert, Generalized factorial ideals, Arab. J. Sci. Eng. Sect. C **26** (2001), 51–68.
22. J.-L. Chabert, Factorial groups and Pólya groups in Galoisian extensions of $\mathbb{Q}$, in *Commutative Ring Theory and Applications*, Lecture Notes in Pure and Appl. Math. **231**, 77–86, Marcel Dekker, New York, 2003.
23. J.-L. Chabert, Integer-valued polynomials in valued fields with an application to discrete dynamical systems, in *Commutative Algebra and Applications*, de Gruyter, Berlin, 2009, 103–134.
24. J.-L. Chabert, About polynomials whose divided differences are integer valued on prime numbers, to appear.
25. J.-L. Chabert, S. Evrard, and Y. Fares, Regular Subsets of valued Fields and Bhargava's v-orderings, Math. Zeitschrift, **274** (2013), 263–290.
26. J.-L. Chabert, A.-H. Fan, and Y. Fares, Minimal dynamical systems on a discrete valuation domain, Discrete and Contin. Dyn. Syst. **35** (2009), 777–795.
27. S. Evrard, Bhargava's factorials in several variables, J. Algebra **372** (2012), 134–148.
28. S. Evrard, Y. Fares, and K. Johnson, Integer valued polynomials on lower triangular integer matrices, Monatsh. Math. **170** (2013), 147–160.
29. S. Frisch, Polynomial functions on finite commutative rings, in *Advances in Commutative Ring Theory*, 323–336, Lecture Notes in Pure and Appl. Math. **205**, Dekker, New York, 1999.
30. S. Frisch, Integer-valued polynomials on algebras, J. Algebra **373** (2013), 414–425.
31. K. Johnson, Limits of characteristic sequences of integer-valued polynomials on homogeneous sets, J. Number Theory **129** (2009), 2933-2942.
32. K. Johnson, Computing r-removed P-orderings and P-orderings of order h, Actes des rencontres du C.I.R.M. **2** n°2 (2010), 147–160.
33. K. Johnson, Super-additive sequences and algebras of polynomials, Proc. Amer. math. Soc. **139** (2011), 3431–3443.
34. K. Johnson and D. Patterson, Projective p-orderings and homogeneous integer-valued polynomials, Integers **11** (2011), 597–604.
35. K. Johnson and M. Pavlovski, Integer-valued polynomials on the Hurwitz ring of integral quaternions, Comm. Algebra **40** (2012), 4171–4176.
36. A. J. Kempner, Polynomials and their residue systems, Trans. Amer. Math. Soc. **22** (1921), 240–288.
37. A. Leriche, Cubic, quatric and sextic Pólya fields, J. Number Theory **133** (2013), 59–71.
38. A. Leriche, About the embedding of a number field in a Pólya field, to appear.
39. K. Mahler, An Interpolation Series for Continuous Functions of a p-adic Variable, J. reine angew. Math. **199** (1958), 23–34 and **208** (1961), 70–72.
40. A. Mingarelli, Abstract factorials, arXiv:00705.4299v3 [math.NT] 10 Jul 2012
41. A. Ostrowski, Über ganzwertige Polynome in algebraischen Zahlkörpern, *J. reine angew. Math.* **149** (1919), 117–124.
42. G. Pólya, Über ganzwertige ganze Funktionen, Rend. Circ. Mat. Palermo **40** (1915), 1–16.
43. G. Pólya, Über ganzwertige Polynome in algebraischen Zahlkörpern, J. reine angew. Math. **149** (1919), 97–116.

44. N. Werner, Integer-valued polynomials over quaternions rings, J. Algebra **324** (2010), 1754–1769.
45. N. Werner, Integer-valued polynomials over matrix rings, Comm. Algebra **40** (2012), 4717–4726.
46. M. Wood, *P*-orderings: a metric viewpoint and the non-existence of simultaneous orderings, J. Number Theory **99** (2003), 36–56.
47. H. Zantema, Integer valued polynomials over a number field, Manuscr. Math. **40** (1982), 155–203.