FATOU EXTENSION *of a commutative ring* A - A commutative ring B containing A such that each formal power series $\alpha \in A[[X]]$ which is B-rational is in fact A-rational. Recall that a formal power series α is A-*rational* if there exist two polynomials $P, Q \in A[X]$ such that $Q(0) = 1$ and $\alpha = \frac{P}{Q}$, that is, α is equal to the formal expansion of $P \sum_{n=0}^{\infty} (Q-1)^n$. For instance, if $K \subseteq L$ is a field extension, then L is a Fatou extension of K.

Fatou extensions are well characterized in the integral case. Thus, from now on, A is supposed to be an integral domain with quotient field K. The example above shows that an integral domain B containing A is a Fatou extension of A if and only if the ring $B \cap K$ is a Fatou extension of A. If the integral domain A is *Noetherian*, then its quotient field K is a Fatou extension of A, and hence, every integral domain containing A is a Fatou extension of A. Many rings are Noetherian: for instance, every finitely generated $\mathbb{Z}$-algebra is Noetherian.

For a rational function $R \in K(X)$, there are several representations of the form $R = \frac{P}{Q}$ where $P, Q \in K[X]$. Such a representation is said to be a) unitary if the nonzero coefficient of Q corresponding to the lowest degree is 1, b) *irreducible* if P and Q are relatively prime in $K[X]$, c) with coefficients in A if $P, Q \in A[X]$. Let $A(X)$ denote the set of rational functions with a unitary representation with coefficients in A, and let $A((X))$ denote the Laurent power series, that is,

$$A((X)) = \{ \sum_{n \geq n_0}^{\infty} a_n X^n \mid n_0 \in \mathbb{Z},\, a_n \in A \}$$

[these notations extend the classical notations $K(X)$ and $K((X))$].

To say that the integral domain B is a Fatou extension of A is nothing else than to write:

$$B(X) \cap A((X)) = A(X)\,;$$

in other words, each rational function $R \in L(X)$, where L denotes the quotient field of B, which has a unitary representation with coefficients in B and a Laurent expansion at 0 with coefficients in A, has a unitary representation with coefficients in A.

A rational function $R \in K(X)$ has a unique unitary and irreducible representation. With respect to this representation, there are two main results. 1) The ring $A(X)$ is the set of elements of $K(X) \cap A((X))$ which admit a unitary and irreducible representation whose coefficients are *integral* over A. 2) For every element of $K(X) \cap A((X))$, the coefficients of the unitary and irreducible representation are *almost integral* over A. Recall that an element x of K is almost integral over A if there exists a nonzero element d of A such that dx^n belongs to A for each positive integer n. Each element of K which is integral over A is almost integral over A.

An integral domain B containing A is a Fatou extension of A if and only if each element of K which is both integral over B and almost integral over A is integral over A [1]. The Noetherian case considered above follows from the fact that if A is Noetherian, then each element of K which is almost integral over A is integral over A.

The definition of Fatou extensions may be easily extended to semi-ring extensions. Then, $\mathbb{Q}_+$ is a Fatou extension of $\mathbb{N}$, while $\mathbb{Z}$ is not a Fatou extension of $\mathbb{N}$, nor $\mathbb{R}_+$ of $\mathbb{Q}_+$ [2].

Moreover, the notion may be considered for formal power series in non-commuting variables, which have applications in system and control theory [3]. It turns out that the previous characterization in the integral case still holds.

[1] CAHEN, P.-J. and CHABERT, J.-L., 'Eléments quasi-entiers et extensions de Fatou', *J. Algebra* **36** (1975), 185-192.

[2] BERSTEL, J. and REUTENAUER, C., *Rational series and their languages*, Springer, 1988.

[3] SALOMAA, A. and SOITTOLA, M., *Automata-theoretic aspects of formal power series*, Springer, 1978.

FATOU RING - An integral domain A with quotient field K such that if each rational function $R \in K(X)$, which has a Taylor expansion at 0 with coefficients in A, has a unitary and irreducible representation with coefficients in A; that is, for each $R \in K(X) \cap A[[X]]$, there are $P, Q \in A[X]$ such that $R = \frac{P}{Q}$, $Q(0) = 1$ and P and Q are relatively prime in $K[X]$. Fatou's lemma [1]: $\mathbb{Z}$ is a Fatou ring.

Equivalently, A is a Fatou ring means: if a sequence $\{a_n\}_{n\in\mathbb{N}}$ of elements of A satifies a linear recursion formula

$$a_{n+s} + q_1 a_{n+s-1} + \ldots + q_s a_n = 0 \quad \text{for } n \geq 0$$

where $q_1, \ldots, q_s \in K$ and s is the least possible, then $q_1, \ldots, q_s \in A$.

If A is a Fatou ring, then its quotient field K is a *Fatou extension* of A, but the converse does not hold. It is the reason why sometimes Fatou rings are called *strong Fatou rings*, while the domains A such that K is a Fatou extension of A are called *weak Fatou rings* [in this latter case, every $R \in K(X) \cap A[[X]]$ has a unitary (not necessarily irreducible) representation with coefficients in A.]

The coefficients of the unitary and irreducible representation of every element of $K(X) \cap A((X))$ are almost integral over A (see **Fatou extension**). An integral domain A is a Fatou ring if and only if every element of K which is almost integral over A belongs to A [2], such domains are said to be *completely integrally closed*. For instance, a *Noetherian* domain is completely integrally closed if and only if it is integrally closed. The rings of integers of number fields are completely integrally closed, and hence, Fatou rings.

The notion may be extended by considering formal power series in non-commuting variables. The characterization of this generalized property is still an open question.

[1] FATOU, P., 'Sur les séries entières à coefficients entiers', *C. R. Acad. Sci. Paris*, Série A **138** (1904), 342-344.

[2] EILENBERG, S., *Automata, Languages and Machines*, vol. A, Academic Press, New York, 1974.