.

THE CHARACTERISTIC SEQUENCE OF INTEGER-VALUED POLYNOMIALS ON A SUBSET

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Abstract Let V be a discrete valuation domain with quotient field K and let E be an infinite subset of V. The V-module

$$\mathrm{Int}(E,V)=\{f\in K[X]\mid f(E)\subseteq V\}$$

of integer-valued polynomials on E is isomorphic to $\oplus_{k=0}^{\infty} I_k g_k$ where the g_k are monic polynomials in $V[X]$ and the I_k are the characteristic ideals of $\mathrm{Int}(E,V)$. We compute here the valuation of these ideals I_k in the case where E is a homogeneous subset of V and we give explicit formulas in several particular cases.

1. Introduction

Two papers about integer-valued polynomials on an arbitrary subset E of a Dedekind domain D appeared recently. The first one by Bárbácioru [1] essentialy extends the results of Cahen [2] which concern the case where $E=D$. The second paper by Bhargava [3] is more general, but in some sense gives less precise results than [1]. The aim of the present paper is to use results of [3] to improve results of [1]. For notation, definitions and well known results we refer to [4].

Let D be a Dedekind domain and let E be an infinite subset of D. Let K denote the quotient field of D and $\mathrm{Int}(E,D)$ denote the ring of *integer-valued polynomials* on E, that is:

$$\mathrm{Int}(E,D)=\{f\in K[X]\mid f(E)\subseteq D\}.$$

We are interested in the D-module structure of $\mathrm{Int}(E,D)$. Bárbácioru's main result is the following (in the case where the residue fields are finite): for each integer $n\geq 0$,

$$\mathrm{Int}_n(E,D)=\{f\in\mathrm{Int}(E,D)\mid \deg(f)\leq n\}=\bigoplus_{k=0}^{n} J_k^{(n)} f_k$$

where $J_0^{(n)},\ldots,J_n^{(n)}$ are fractional ideals of D and $f_0,\ldots,f_n$ are monic polynomials in $D[X]$ [1, Theorem 1]. It follows from the proofs of [1] that the ideals $J_k^{(n)}$ depend a priori on the integer n. A natural question raised by this result is then: does the ideal $J_k^{(n)}$ actually depend on n? The answer is no because, if such fractional ideals $J_k^{(n)}$ exist, then they are necessarily the characteristic ideals I_k of $\mathrm{Int}(E,D)$.

Recall that, for each integer $k \geq 0$, the *characteristic ideal* I_k of $\mathrm{Int}(E,D)$ is the fractional ideal formed by 0 and the set of leading coefficients of polynomials in $\mathrm{Int}(E,D)$ of degree $\leq k$ (see [4, §II.1]).

The fact that $J_k^{(n)} = I_k$ is easy to check by induction on k.

In fact, among several results of Bhargava [3], there is the following assertion (without any assumption on the residue fields) which clearly contains Bárbácioru's assertion:

$$\mathrm{Int}(E,D) = \bigoplus_{k=0}^{\infty} I_k g_k$$

where the I_k are the characteristic ideals of $\mathrm{Int}(E,D)$ and the g_k are monic polynomials in $D[X]$ [3, Theorems 12 and 13].

But, on the other hand, there is one interesting point in [1] that we do not find in [3]: an attempt to characterize, and actually compute, the fractional ideals I_k. In order to do that, Bárbácioru first notices that to study the D-module $\mathrm{Int}_n(E,D)$, we may replace E by the set

$$E_n = \{x \in K \mid \forall f \in \mathrm{Int}_n(E,D),\ f(x) \in D\}$$

since

$$\mathrm{Int}_n(E,D) = \mathrm{Int}_n(E_n,D).$$

Of course,

$$E_0 = K,$$

and, for $n \geq 1$,

$$E \subseteq E_n \subseteq D$$

because X belongs to $\mathrm{Int}_n(E,D)$.
Since E is infinite, the D-module $\mathrm{Int}_n(E,D)$ is finitely generated (see for instance [4, Proposition II.1.1]), and hence, 'by continuity' of the integer-valued polynomials which generate $\mathrm{Int}_n(E,D)$ (see for instance [4, Proposition III.2.1]), there is a nonzero ideal A of D such that

$$\text{for each } x \in E_n,\quad x + A = \{x + a \mid a \in A\} \subseteq E_n.$$

Such a subset E_n is called by McQuillan [5, §2] a *homogeneous subset* of D with ideal A. Since the subsets E_n are homogeneous subsets, in almost all the proofs of [1], the subset E is supposed itself to be homogeneous.

We may notice that in the case where the Dedekind domain D has finite residue fields, a homogeneous subset is necessarily of the form

$$E = \bigcup_{i=1}^{r} b_i + A$$

where $b_1, \ldots, b_r$ are elements of D pairwise non-congruent modulo A (this is no more true with infinite residue fields).

Moreover, to study the characteristic ideals I_n we may localize because, for each ideal M of D, $(\mathrm{Int}(E,D))_M = \mathrm{Int}(E,D_M)$ [4, Proposition I.2.7], and hence, the fractional ideals $(I_n)_M$ are the characteristic ideals of $\mathrm{Int}(E, D_M)$. Then, D_M is a discrete valuation domain and, if E is the union of r cosets modulo M^l, that is

$$E = \bigcup_{i=1}^{r} b_i + M^l,$$

letting $s = |D/M^l|$, we find in [1, Theorem 4] that

$$(I_n)_M = M^{-S(n)} D_M \quad \text{with} \quad S(n) = l \sum_{\alpha \geq 0} \left[\frac{n}{r s^\alpha} \right].$$

This formula is correct for the classical case where $E = D$ provided one considers that it corresponds to the union of q cosets modulo M (where $q = |D/M|$), so that

$$S(n) = \sum_{\alpha \geq 0} \left[\frac{n}{q^{\alpha+1}} \right]$$

which is the formula given by Pólya [6] (see, for instance, [4, Corollary II.2.9]). In fact, the formula fails for $l > 1$. For example, if $D = \mathbb{Z}$, $M = 2\mathbb{Z}$, and $E = 4\mathbb{Z}$, then $l = 2$, $r = 1$, $s = 4$, and $S(2) = 2 \sum_{\alpha \geq 0} \left[\frac{2}{4^\alpha} \right] = 4$, while $\frac{X(X-4)}{2^5} \in \mathrm{Int}(E, V)$. The correct value is indeed 5 (as follows from Proposition 3.3).

The aim of this paper is to give a correct formula. In order to do this, we use a general result of Bhargava [3] that we first recall.

2. Bhargava's revisited result

HYPOTHESIS. From now on, V denotes a discrete valuation domain and E an infinite subset of V. We denote by K the quotient field of V, v the valuation of K associated to V, M the maximal ideal of V, and t a generator of M.

DEFINITION [4, §IX.3]. The *characteristic sequence* of $\mathrm{Int}(E, V)$ is the sequence of positive integers $\{-v(I_n)\}_{\{n \in \mathbb{N}\}}$ where I_n denotes the characteristic ideal of $\mathrm{Int}(E, V)$ (that is, the fractional ideal formed by 0 and the leading coefficients of the elements of $\mathrm{Int}(E, V)$ with degree $\leq n$).

Similarly to [3], we set the following.

DEFINITION. A *v-ordered sequence of elements of* E is a (finite or infinite) sequence $\{a_n\}_{n \geq 0}$ of elements of E such that, for $n > 0$,

$$v\left(\prod_{k=0}^{n-1} (a_n - a_k) \right) = \inf_{a \in E} v\left(\prod_{k=0}^{n-1} (a - a_k) \right).$$

There always exist infinite v-ordered sequences of elements of E. Such sequences may be constructed inductively on n: choose any element a_0 in E, choose a_1 in E such that $v(a_1 - a_0) = \inf_{a \in E} v(a - a_0)$, and so on. We may notice that a V.W.D.W.O. sequence in V [4, Definition II.2.1] is a v-ordered sequence of E in the case where $E = V$ and V/M is finite.

From our point of view, the main result of Bhargava is the following [3, Theorem 1].

PROPOSITION 2.1 (Bhargava) *The sequence $\{w_E(n)\}_{\{n\in\mathbb{N}\}}$ defined by $w_E(n) = \sum_{k=0}^{n-1} v(a_n - a_k)$ where $\{a_n\}_{\{n\in\mathbb{N}\}}$ is a v-ordered sequence of E does not depend on the choice of the sequence $\{a_n\}$.*

This is an easy consequence of the fact that the sequence $w_E(n)$ is the characteristic sequence of $\mathrm{Int}(E,V)$, since the characteristic sequence of $\mathrm{Int}(E,V)$ only depends on $\mathrm{Int}(E,V)$. For the sake of completness we give a straightforward proof of these assertions (shorter than Barghava's proof).

PROPOSITION 2.2 *Assume $\{a_n\}_{\{n\in\mathbb{N}\}}$ is a v-ordered sequence. Then the polynomials*

$$f_n(X) = \prod_{k=0}^{n-1} \frac{X - a_k}{a_n - a_k}$$

form a basis of the V-module $\mathrm{Int}(E,V)$.

Proof. By construction, for each $a \in E$, $v(f_n(a)) \geq v(f_n(a_n))$, and hence, $f_n(E) \subseteq V$. Moreover, $f_n(a_0) = f_n(a_1) = \ldots = f_n(a_{n-1}) = 0$ and $f_n(a_n) = 1$. It then follows that the f_n, $n \in \mathbb{N}$, form a basis of $\mathrm{Int}(E,V)$ (as in the classical case of $\mathrm{Int}(\mathbb{Z})$ [4, Proposition I.1.1]).

COROLLARY 2.3 *Assume $\{a_n\}_{\{n\in\mathbb{N}\}}$ is a v-ordered sequence of elements of E. Then,*

1. $v(I_n) = -\sum_{k=0}^{n-1} v(a_n - a_k)$,
2. $w_E(n) = -v(I_n)$,
3. *if $\{a_n\}_{\{n\in\mathbb{N}\}}$ is a v-ordered sequence of elements of E, then the polynomials $t^{-w_E(n)} \prod_{k=0}^{n-1}(X - a_k)$ form a basis of $\mathrm{Int}(E,V)$.*

Now, let us return to the particular case where E is supposed to be homogeneous.

3. Bárbácioru's corrected formula

We begin with some easy remarks concerning the function w_E. For a fixed subset E of V, $w_E(n)$ is an increasing function of n. More precisely, for all m and $n \in \mathbb{N}$,

$$w_E(m+n) \geq w_E(m) + w_E(n).$$

Moreover, if $E \subseteq F \subseteq V$, then, for each $n \in \mathbb{N}$,

$$0 \leq w_V(n) \leq w_F(n) \leq w_E(n).$$

We now consider translations and homotheties: for each $a \in V$, let $a + E = \{a + x \mid x \in E\}$ and, for each $l \in \mathbb{N}$, let $t^l E = \{t^l x \mid x \in E\}$.

PROPOSITION 3.1 *Let E be a subset of V.*

1. *For each $a \in V$, $w_{a+E}(n) = w_E(n)$.*
2. *For each $l \in \mathbb{N}$, $w_{t^l E}(n) = w_E(n) + ln$.*

Proof. Let τ (resp., σ) be the K-automorphism of $K(X)$ such that $\tau(X) = X - a$ (resp., $\sigma(X) = X/t^l$). Then $\tau(\mathrm{Int}(E,V)) = \mathrm{Int}(a+E,V)$ (resp., $\sigma(\mathrm{Int}(E,V)) = \mathrm{Int}(t^l E, V)$). Indeed, if $f(X)$ belongs to $\mathrm{Int}(E,V)$, then $\tau(f(X)) = f(X-a)$ belongs to $\mathrm{Int}(a+E,V)$ (resp., $\sigma(f(X)) = f(X/t^l)$ belongs to $\mathrm{Int}(t^l E, V)$). If $\{f_n\}_{n\in\mathbb{N}}$ is a basis of $\mathrm{Int}(E,V)$, then $\{\tau(f_n)\}$ (resp., $\{\sigma(f_n)\}$) is a basis of $\mathrm{Int}(a+$

$E, V)$ (resp., $\mathrm{Int}(t^l E, V)$). If the leading coefficient of f_n is α, then the leading coefficient of $\tau(f_n)$ (resp., $\sigma(f_n)$) is α (resp., α/t^{ln}).

The case where $E = V$ is well known. We first recall a notation.

NOTATION [4, §II.2]. For each $q \in \mathbb{N}^*$ and each $n \in \mathbb{N}$, let

$$w_q(n) = \sum_{\alpha>0} \left[\frac{n}{q^a}\right]$$

where $[x]$ denotes the entire part of x. We extend this notation to the case where q is infinite with $w_\infty(n) = 0$ for each $n \in \mathbb{N}$.

PROPOSITION 3.2 [4, Corollaries I.3.7 and II.2.9]. *Let q be the cardinal of the residue field V/M (q is finite or infinite). Then, for each $n \in \mathbb{N}$, one has $w_V(n) = w_q(n)$.*

This leads us to a first formula.

PROPOSITION 3.3 *If $E = a + M^l$ with $a \in V$ and $l \in \mathbb{N}$, then*

$$w_E(n) = w_q(n) + ln.$$

Indeed, $E = a + t^l V$.

Here are two basic technical lemmas. The first one is quite general.

LEMMA 3.4 *Let $\{a_k\}$ be a v-ordered sequence of E and $E_1 = (b+M^l) \cap E$ be the intersection of E with some coset modulo M^l. Then the subsequence formed by the elements a_k in E_1 is a v-ordered sequence of E_1.*

Proof. Note that, even if the sequence $\{a_k\}$ is infinite, the subsequence formed by the elements a_k in E_1 may be finite. If this subsequence is empty, or contains only one element, there is nothing to prove. Suppose, by induction, that the first n elements $a_{k_0}, a_{k_1}, \ldots, a_{k_{n-1}}$ of this subsequence form a v-ordered sequence of E_1. If there is a next one, a_{k_n}, we prove that $a_{k_0}, a_{k_1}, \ldots, a_{k_n}$ is a v-ordered sequence of E_1. We set $N = k_n$. For each α in E_1, we have

$$\sum_{k=0}^{N-1} v(\alpha - a_k) = \sum_{k<N,\, a_k \in E_1} v(\alpha - a_k) + \sum_{k<N,\, a_k \notin E_1} v(\alpha - a_k).$$

If $a_k \notin E_1$, we have $v(b - a_k) < l$, while $v(\alpha - b) \geq l$, thus

$$\sum_{k<N,\, a_k \notin E_1} v(\alpha - a_k) = \sum_{k<N,\, a_k \notin E_1} v(b - a_k)$$

and this sum is independent of the choice of α in E_1.
By hypothesis, $\sum_{k=0}^{N-1} v(\alpha - a_k)$ is minimal for $\alpha = a_{k_n}$. Hence

$$\sum_{k<N,\, a_k \in E_1} v(\alpha - a_k) = \sum_{i=0}^{n-1} v(\alpha - a_{k_i})$$

is also minimal for this choice of α.

HYPOTHESIS. From now on, we assume that E is of the following form:

$$E = \cup_{i=1}^{r} b_i + M^l$$

where $r \in \mathbb{N}^*$, $l \in \mathbb{N}$ and $b_1, \ldots, b_r$ are pairwise non-congruent modulo M^l.

NOTATION. For $j \in \{1, \ldots, r\}$ and $\delta_1, \ldots, \delta_r \in \mathbb{N}$, let

$$w_E^j(\delta_1, \ldots, \delta_r) = w_q(\delta_j) + l\delta_j + \sum_{i \neq j} v(b_j - b_i)\delta_i.$$

LEMMA 3.5 *For each $j \in \{1, \ldots, r\}$, let $\delta_j \in \mathbb{N}$ and let $a_{j,1}, \ldots, a_{j,\delta_j}$ be δ_j elements of $b_j + M^l$ which form a v-ordered sequence of elements of $b_j + M^l$. Consider the polynomial*

$$g(X) = \prod_{j=1}^{r} \left(\prod_{k=1}^{\delta_j} (X - a_{j,k}) \right).$$

Then, for each j, one has:

$$\inf \left\{ v(g(x)) \mid x \in b_j + M^l \right\} = w_E^j(\delta_1, \ldots, \delta_r).$$

Proof. Let $x \in b_j + M^l$. For each $i \neq j$ and each $k \in \{1, \ldots, \delta_i\}$, we have $v(x - a_{i,k}) = v(b_j - b_i)$, and hence

$$v(g(x)) = \sum_{k=1}^{\delta_j} v(x - a_{j,k}) + \sum_{i \neq j} v(b_j - b_i)\delta_i.$$

Clearly $\sum_{i \neq j} v(b_i - b_j)\delta_i$ does not depend on the choice of x in $b_j + M^l$. Since $\{a_{j,1}, \ldots, a_{j,\delta_j}\}$ is a v-ordered sequence of elements of $b_j + M^l$, it follows from Proposition 3.3 that the minimal value of $\sum_{k=1}^{\delta_j} v(x - a_{j,k})$ is $w_q(\delta_j) + l\delta_j$.

THEOREM 3.6 *Let V be a discrete valuation domain. Denote by v the corresponding valuation, by M the maximal ideal of V, and by q the cardinal (finite or infinite) of the residue field V/M. Let E be a subset of V such that $E = \cup_{i=1}^{r} b_i + M^l$ where $r \in \mathbb{N}^*$, $l \in \mathbb{N}$ and $b_1, \ldots, b_r \in V$ are pairwise non-congruent modulo M^l. The characteristic sequence $\{w_E(n)\}_{\{n \in \mathbb{N}\}}$ of $\mathrm{Int}(E, V)$ may be computed by means of the following formulas*:

$$w_E(n) = \max_{\delta_1 + \cdots + \delta_r = n} \left(\min_{1 \leq j \leq r} w_E^j(\delta_1, \ldots, \delta_r) \right) \qquad (\delta_1, \ldots, \delta_r \in \mathbb{N})$$

where

$$w_E^j(\delta_1, \ldots, \delta_r) = w_q(\delta_j) + l\delta_j + \sum_{i \neq j} v(b_i - b_j)\delta_i$$

and

$$w_q(\delta_j) = \sum_{\alpha > 0} \left[\frac{\delta_j}{q^\alpha} \right].$$

Proof. Let n be a fixed integer and let

$$\omega(n) = \max_{\delta_1 + \cdots + \delta_r = n} \left(\min_{1 \leq j \leq r} w_E^j(\delta_1, \cdots, \delta_r) \right).$$

Let $\{a_0, a_1, \ldots, a_n\}$ be a v-ordered sequence of elements of E. For each $j \in \{1, \ldots, r\}$, let δ_j be the number of elements a_k $(k < n)$ which are in $b_j + M^l$.

Let $g(X) = \prod_{k=0}^{n-1}(X - a_k)$. Lemma 3.4 says that, for each j, the finite subsequence formed by the a_k which lie in $b_j + M^l$ is a v-ordered sequence of elements of $b_j + M^l$. Thus, the hypothesis of Lemma 3.5 is satisfied. It follows that we have

$$\inf_{x \in E}\{v(g(x))\} = \min_{1 \leq j \leq r} w_E^j(\delta_1, \ldots, \delta_r).$$

By definition of a v-ordered sequence, we also have

$$v(g(a_n)) = \inf_{x \in E}\{v(g(x))\}.$$

Consequently,

$$w_E(n) = \sum_{k=0}^{n-1} v(a_n - a_k) = \min_{1 \leq j \leq r} w_E^j(\delta_1, \ldots, \delta_r).$$

Since $\delta_1 + \cdots + \delta_r = n$, we have in particular

$$\omega(n) \geq \min_{1 \leq j \leq r} w_E^j(\delta_1, \ldots, \delta_r)$$

and hence

$$\omega(n) \geq w_E(n).$$

Conversely, let now $d_1, \ldots, d_r \in \mathbb{N}$ be such that

$$d_1 + \ldots + d_r = n$$

and such that

$$\inf_{1 \leq j \leq r} w_E^j(d_1, \ldots, d_r) = \omega(n).$$

For each $j \in \{1, \ldots, r\}$, let $\{a_{j,1}, \ldots, a_{j,d_j}\}$ be a v-ordered sequence of elements of $b_j + M^l$. Then, let

$$g(X) = \prod_{j=1}^{r}\left(\prod_{k=1}^{d_j}(X - a_{j,k})\right).$$

It follows from Lemma 3.5 that, for each $x \in E$, we have

$$v(g(x)) \geq \min_{1 \leq j \leq r} w_E^j(d_1, \ldots, d_r).$$

Thus, $t^{-\omega(n)}g(X)$ belongs to $\mathrm{Int}(E, V)$; and hence, by definition of I_n,

$$\omega(n) \leq w_E(n) = -v(I_n).$$

Finally

$$\omega(n) = w_E(n).$$

REMARKS

a) Theorem 3.6 shows that in order to compute $w_E(n)$, we do not really have to know any v-ordered sequence in E. In fact, we may forget the original question on integer-valued polynomials, we just have to know the integers q, l and $v(b_i - b_j)$. We may also notice that, for each n, $w_E(n)$ may be computed in finitely many steps since there are only finitely many $(\delta_1, \ldots, \delta_r) \in \mathbb{N}^r$ such that $\delta_1 + \cdots + \delta_r = n$.

b) On the other hand, for each n, the computation of $w_E(n)$ may help us to determine a v-ordered sequence of n elements. Among the r-uples $(\delta_1, \ldots, \delta_r) \in \mathbb{N}^r$ such that $\delta_1 + \cdots + \delta_r = n$ and $\inf_{1 \leq j \leq r} w_E^j(\delta_1, \ldots, \delta_r) = w_E(n)$, there is at least one which corresponds to a v-ordered sequence. To construct the corresponding sequence, it suffices to consider, for each $j \in \{1, \ldots, r\}$, a v-ordered sequence of δ_j

elements in $b_j + M^l$. Such sequences are easy to construct : if $\{a_k\}$ is a v-ordered sequence in V, then $\{b_j + a_k t^l\}$ is a v-ordered sequence in $b_j + M^l$. Moreover, v-ordered sequences in V are well known : if q is finite, see for instance [4, Proposition II.2.3], and if q is infinite, any sequence of elements of V which are pairwise non-congruent modulo M is a v-ordered sequence in V.

c) In the case where q is infinite, the problem becomes a classical linear programming problem. Let us consider the symmetric matrix

$$B = (\beta_{i,j}) \in \mathcal{M}_r(\mathbb{N})$$

defined by

$$\begin{cases} \beta_{ii} &= l & \text{for each } i \\ \beta_{ij} &= \beta_{ji} = v(b_j - b_i) & \text{for } i \neq j. \end{cases}$$

We have to determine the function w_E such that

$$w_E(n) = \max_{\delta_1+\cdots+\delta_r=n} \left(\min_{1\leq j\leq r} w_E^j(\delta_1, \ldots, \delta_r) \right)$$

with

$$W_E(\Delta) = \Delta B$$

where

$$\Delta = (\delta_1, \ldots, \delta_r) \in \mathbb{N}^r \quad \text{and} \quad W_E(\Delta) = \big(w_E^1(\Delta), \ldots, w_E^r(\Delta)\big) \in \mathbb{N}^r.$$

REFEREE'S REMARK The previous results may be slightly improved in the case where the residue field is infinite: Lemma 3.5 and Theorem 3.6, in particular, remain valid if E is of the form

$$E = \bigcup_{i=1}^{r} b_i + M^{l_i},$$

where the l_i are positive integers and the b_i are elements of V which are pairwise non-congruent modulo M^l with $l = \inf_{1\leq i\leq r} l_i$. We just have to consider the new following functions

$$w_E^j(\delta_1, \ldots, \delta_r) = w_q(\delta_j) + l_j\delta_j + \sum_{i\neq j} v(b_i - b_j)\delta_i.$$

[Note that if the residue field is finite, we may always assume that all the l_i are equal.]

4. Some explicit formulas

There are some cases where the *maximin* which gives the value for $w_E(n)$ (see Theorem 3.6) may be described by an explicit formula. The first one is the case where $l = 1$, that is, the only case where Bárbácioru's formula is correct (see Section 1).

PROPOSITION 4.1 *If $E = \cup_{j=1}^r b_j + M$ where $b_1, \ldots, b_r$ are pairwise non-congruent modulo M, then*

$$w_E(n) = w_q\left(\left[\frac{n}{r}\right]\right) + \left[\frac{n}{r}\right] = \sum_{\alpha\geq 0} \left[\frac{n}{rq^\alpha}\right].$$

This is a particular case of Proposition 4.2 below where, for each $i \neq j$, $\beta_{ij} = v(b_j - b_i) = 0$. We already encountered an example of such a case in the literature: $\mathrm{Int}(E, V)$ where $V = \mathbb{Z}_{(p)}$, $E = \mathbb{Z} \setminus p\mathbb{Z}$ and p is a prime number. Then $\mathrm{Int}(E, V) = \mathrm{Int}(\overline{E}, V)$ where $\overline{E} = \mathbb{Z}_{(p)} \setminus p\mathbb{Z}_{(p)}$ [4, Theorem IV.1.15], and $\overline{E} = \left\{1 + \mathbb{Z}_{(p)}\right\} + \cdots + \left\{(p-1) + \mathbb{Z}_{(p)}\right\}$ corresponds to $l = 1$, $q = p$, and $r = p - 1$. Hence,

$$w_E(n) = \sum_{\alpha \geq 0} \left[\frac{n}{(p-1)p^\alpha}\right].$$

There are at least two reasons which explain the difficulty in replacing the *maximin* by explicit formulas:
— the gaps of the function w_q are difficult to control unless the residue field is infinite (in this case, $w_q(n) \equiv 0$, see the last remark of the previous section),
— the weights of the cosets modulo M^l may be different unless all the $\beta_{i,j} = v(b_i - b_j)$ are equal.
A first case where the later difficulty is avoided is those where all the $\beta_{i,j}$ are equal to zero.

PROPOSITION 4.2 *If $E = \cup_{i=1}^r b_j + M^l$ where $l \in \mathbb{N}$ and $b_1, \ldots, b_r$ are pairwise non-congruent modulo M, then*

$$w_E(n) = w_q\left(\left[\frac{n}{r}\right]\right) + l\left[\frac{n}{r}\right].$$

Proof. Since $v(b_i - b_j) = 0$ for $i \neq j$, it follows from the definition of $w_E^j(\delta_1, \ldots, \delta_r)$ that we have

$$w_E^j(\delta_1, \ldots, \delta_r) = w_q(\delta_j) + l\delta_j.$$

Then

$$w_E^j(\delta_1, \ldots, \delta_r) = \varphi(\delta_j) \text{ where } \varphi(\delta) = w_q(\delta) + l\delta$$

is an increasing function of δ. Thus

$$\min_j w_E^j(\delta_1, \ldots, \delta_r) = \varphi(\delta) \text{ where } \delta = \inf_j \delta_j,$$

and

$$\begin{aligned}\max_{\delta_1+\cdots+\delta_r=n}\left(\min_j\left(w_E^j(\delta_1, \ldots, \delta_r)\right)\right) &= \max_{\delta_1+\cdots+\delta_r=n} \varphi\left(\min_j \delta_j\right) \\ &= \varphi\left(\max_{\delta_1+\ldots+\delta_r=n}\left(\min_j \delta_j\right)\right).\end{aligned}$$

Since

$$\max_{\delta_1+\cdots+\delta_r=n}\left(\min_j \delta_j\right) = \left[\frac{n}{r}\right],$$

one has

$$w_E(n) = \varphi\left(\left[\frac{n}{r}\right]\right).$$

Another case where things are relatively easy is those where $r = 2$ since in that case there is only one $\beta_{i,j}$ to consider ($\beta_{1,2} = \beta_{2,1}$).

PROPOSITION 4.3 *If $E = \{b_1 + M^l\} \cup \{b_2 + M^l\}$ with $l \in \mathbb{N}^*$ and $v(b_1 - b_2) < l$, then*

$$w_E(n) = w_q\left(\left[\frac{n}{2}\right]\right) + l\left[\frac{n}{2}\right] + v(b_1 - b_2)\left[\frac{n+1}{2}\right].$$

Proof. For $j = 1, 2$, one has

$$w_E^j(\delta_1, \delta_2) = w_q(\delta_j) + l\delta_j + h\delta_{3-j} \text{ where } h = \beta_{12} = \beta_{21}.$$

Thus, if $\delta_1 + \delta_2 = n$, then

$$w_E^j(\delta_1, \delta_2) = \psi(\delta_j)$$

where

$$\psi(\delta) = w_q(\delta) + l\delta + h(n - \delta) = w_q(\delta) + (l - h)\delta + hn$$

is an increasing function of δ. As in the previous proof:

$$\begin{aligned} \max_{\delta_1+\delta_2=n} \left(\min_j w_E^j(\delta_1, \delta_2) \right) &= \psi \left(\max_{\delta_1+\delta_2=n} \left(\min_j \delta_j \right) \right) = \psi \left(\left[\frac{n}{2} \right] \right) \\ &= w_q \left(\left[\frac{n}{2} \right] \right) + (l - h) \left[\frac{n}{2} \right] + hn \\ &= w_q \left(\left[\frac{n}{2} \right] \right) + l \left[\frac{n}{2} \right] + h \left[\frac{n+1}{2} \right]. \end{aligned}$$

In fact, both previous propositions are particular cases of the following where the $\beta_{i,j}$ are equal to each other.

PROPOSITION 4.4 *If $E = \cup_{j=1}^r b_j + M^l$ where $\beta_{i,j} = v(b_j - b_i) = h$ for each $i \neq j$, $l \in \mathbb{N}$ and $0 \leq h < l$, then*

$$w_E(n) = w_q \left(\left[\frac{n}{r} \right] \right) + (l - h) \left[\frac{n}{r} \right] + hn.$$

Proof. By hypothesis, for each $j \in \{1, \ldots, r\}$, one has $b_j - b_1 = t^h c_j$ where t is a generator of M and the elements $c_1, \ldots, c_r$ of V are pairwise non-congruent modulo M. Let $E_1 = E - b_1$, then $E_1 = t^h E_2$ where $E_2 = \cup_{j=1}^r c_j + M^{l-h}$. Then, Proposition 3.1 shows that

$$w_E(n) = w_{E_1}(n) = w_{E_2}(n) + hn,$$

and Proposition 4.2 shows that

$$w_{E_2}(n) = w_q \left(\left[\frac{n}{r} \right] \right) + (l - h) \left[\frac{n}{r} \right].$$

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