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ASYMPTOTIC BEHAVIOR OF CHARACTERISTIC SEQUENCES OF INTEGER-VALUED POLYNOMIALS

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Abstract.
Let D be the ring of integers of a number field K and let E be an infinite subset of D. The D-module $\mathrm{Int}(E, D)$ of integer-valued polynomials on E is isomorphic to $\oplus_{n=0}^{\infty} \mathfrak{I}_n g_n$ where g_n is a monic polynomial in $D[X]$ of degree n and $\mathfrak{I}_n$ is a fractional ideal of D. For each maximal ideal $\mathfrak{m}$ of D, let $v_{\mathfrak{m}}$ be the corresponding valuation of K; we determine here the asymptotic behavior of the characteristic sequences $\{v_{\mathfrak{m}}(\mathfrak{I}_n)\}_{n \in \mathbb{N}}$ in the case where E is a homogeneous subset of D. In order to do this, we first study some properties of ultrametric matrices; then we prove explicit formulas in the case where D is a Dedekind domain with infinite residue fields; finally, we extend these results to the case of number fields.

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1 INTRODUCTION

Let D be the ring of integers of a number field K and let E be an infinite subset of D. We denote by $\mathrm{Int}(E, D)$ the ring of *integer-valued polynomials on* E, that is,

$$\mathrm{Int}(E, D) = \{f \in K[X] \mid f(E) \subseteq D\}.$$

Recall that the *characteristic ideal of index* n of $\mathrm{Int}(E, D)$ is the fractional ideal $\mathfrak{I}_n$ of D formed by 0 and the leading coefficients of polynomials in $\mathrm{Int}(E, D)$ of degree $\leq n$.

We know [2, Theorems 12 and 13] that there exist monic polynomials $g_n \in D[X]$ of degree n such that

$$\mathrm{Int}(E, D) \simeq \oplus_{n=0}^{\infty} \mathfrak{I}_n g_n.$$

The aim of this paper is to determine these characteristic ideals $\mathfrak{I}_n$, that is, if we denote by $v_{\mathfrak{m}}$ the valuation of K corresponding to a maximal ideal $\mathfrak{m}$ of D, to determine $v_{\mathfrak{m}}(\mathfrak{I}_n) = \inf\{v_{\mathfrak{m}}(x) \mid x \in \mathfrak{I}_n\}$.

By the way, note that, for each polynomial $f = a_0 + a_1 X + \cdots + a_n X^n \in K[X]$ of degree n, if

$$v_{\mathfrak{m}}(f) = \inf\{v_{\mathfrak{m}}(a_i) \mid 0 \leq i \leq n\} \text{ and } v_{\mathfrak{m}}(f(E)) = \inf\{v_{\mathfrak{m}}(f(x)) \mid x \in E\},$$

then one has:

$$v_{\mathfrak{m}}(f) \leq v_{\mathfrak{m}}(f(E)) \leq v_{\mathfrak{m}}(f) - v_{\mathfrak{m}}(\mathfrak{I}_n).$$

Since, for each maximal ideal $\mathfrak{m}$ of D, one has $\mathrm{Int}(E, D)_{\mathfrak{m}} = \mathrm{Int}(E, D_{\mathfrak{m}})$ [5, Proposition I.2.7], $(\mathfrak{I}_n)_{\mathfrak{m}}$ is the characteristic ideal of index n of $\mathrm{Int}(E, D_{\mathfrak{m}})$. Thus, to determine the *characteristic sequences* $\{v_{\mathfrak{m}}(\mathfrak{I}_n) \mid n \in \mathbb{N}\}$, we may restrict our study to the local case.

For the classical case where $E = D$, Pólya [8] gave the following formula:

$$v_{\mathfrak{m}}(\mathfrak{I}_n) = \sum_{k>0} \left[\frac{n}{q_{\mathfrak{m}}^k}\right],$$

where $q_{\mathfrak{m}}$ denotes the cardinal of the residue field $D/\mathfrak{m}$ and $[x]$ denotes the entire part of x.

In fact, to obtain some substantial results, we have to add an hypothesis on E. We assume that E is a *homogeneous* subset of D in the sense given by McQuillan [7, §3], i.e., there exists a nonzero ideal $\mathfrak{a}$ of D such that, for each $x \in E$, $x + \mathfrak{a} = \{x + a \mid a \in \mathfrak{a}\} \subseteq E$. We then say that E is homogeneous with respect to the ideal $\mathfrak{a}$; equivalently, E is the union of cosets of $\mathfrak{a}$. With such an hypothesis, we may still restrict our study to the local case:

Proposition 1.1 *Let D be the ring of integers of a number field K and let E be a homogeneous subset of D with respect to an ideal $\mathfrak{a}$. Let $\mathfrak{m}$ be a maximal ideal of D and let α be the exponent of $\mathfrak{m}$ in the decomposition of $\mathfrak{a}$. Then*

$$(\mathrm{Int}(E,D))_{\mathfrak{m}} = \mathrm{Int}\left(\overline{E}, D_{\mathfrak{m}}\right)$$

where $\overline{E}$ is the following homogeneous subset of $D_{\mathfrak{m}}$:

$$E + \mathfrak{m}^{\alpha} D_{\mathfrak{m}} = \{x + y \mid x \in E,\ y \in \mathfrak{m}^{\alpha} D_{\mathfrak{m}}\}.$$

In particular,
— if $\mathfrak{a}$ is not contained in $\mathfrak{m}$, then $\mathrm{Int}(E,D)_{\mathfrak{m}} = \mathrm{Int}(D_{\mathfrak{m}})$;
— if $\mathfrak{I}_n$ denotes the characteristic ideal of $\mathrm{Int}(E,D)$ of index n, then $(\mathfrak{I}_n)_{\mathfrak{m}}$ is the characteristic ideal of index n of $\mathrm{Int}\left(E + \mathfrak{m}^{\alpha} D_{\mathfrak{m}}, D_{\mathfrak{m}}\right)$.

Proof. Let $\mathfrak{m}$ be a maximal ideal of D. We have $\mathrm{Int}(E, D_{\mathfrak{m}}) = \mathrm{Int}(\overline{E}, D_{\mathfrak{m}})$ where $\overline{E}$ denotes the closure of E in $D_{\mathfrak{m}}$ with respect to the $\mathfrak{m}$-adic topology [5, Theorem IV.1.15]. Obviously, $\overline{E} \subseteq E + \mathfrak{m}^{\alpha} D_{\mathfrak{m}}$. We are going to show that $E + \mathfrak{m}^{\alpha} D_{\mathfrak{m}} \subseteq \overline{E}$. Write $\mathfrak{a} = \mathfrak{m}^{\alpha}\mathfrak{b}$ where $\mathfrak{b}$ is not contained in $\mathfrak{m}$. Let $x \in E$ and $y \in \mathfrak{m}^{\alpha} D_{\mathfrak{m}}$. To prove that $x + y \in \overline{E}$, we construct a sequence $\{y_n\}_{n \geq 1}$ of elements of $\mathfrak{a}$ such that $x + y_n$ tends to $x + y$ in the $\mathfrak{m}$-adic topology, equivalently, such that $y - y_n$ tends to 0. The ideals $\mathfrak{m}^n$ and $\mathfrak{b}$ are relatively prime and there are $u_n \in \mathfrak{m}^n$ and $v_n \in \mathfrak{b}$ such that $u_n + v_n = 1$. Take $y_n = v_n y$. Then, $y_n \in \mathfrak{m}^{\alpha} \cap \mathfrak{b} = \mathfrak{a}$, and $y - y_n = u_n y \in \mathfrak{m}^n D_{\mathfrak{m}}$. $\square$

Thus, we may replace D by its localizations $D_{\mathfrak{m}}$, that is, by discrete valuation domains V with finite residue fields. In fact, we first delete the hypothesis on the residue fields and recall in the next section the known results in the local case ([1], [2], [4], [8]). To go further we establish some properties of ultrametric matrices (section 3). These results lead to a complete determination of the characteristic ideals in the case of infinite residue fields (section 4). Finally, we obtain the asymptotic behavior of the characteristic sequences in the case of finite residue fields (section 5).

2 The local case

Hypotheses.
Let K be a field with a discrete valuation v, let V be the corresponding valuation domain, let $\mathfrak{m}$ be the maximal ideal of V, and let q be the cardinal of the residue field $V/\mathfrak{m}$. [We no longer assume that q is finite.]

Let

$$E = \bigcup_{i=1}^{r} b_i + \mathfrak{m}^l \tag{1}$$

be a finite union of cosets of a power of $\mathfrak{m}$, where the $b_i \in V$ are pairewise non-congruent modulo $\mathfrak{m}^l$. [If q is infinite, a homogeneous subset of V may be the union of infinitely many cosets, but we exclude this case.]

Let

$$\mathrm{Int}(E, V) = \{f \in K[X] \mid f(E) \subseteq V\}$$

be the ring of integer-valued polynomials on E. For each $n \in \mathbb{N}$, $\mathfrak{I}_n$ denotes the characteristic ideal of index n of $\mathrm{Int}(E, V)$, that is, the fractional ideal formed by 0 and the leading coefficients of polynomials in $\mathrm{Int}(E, V)$ of degree n. Finally, consider the function $w_E : \mathbb{N} \to \mathbb{N}$ that we have to determine and which is defined by:

$$w_E(n) = -v(\mathfrak{I}_n) = -\inf\{v(x) \mid x \in \mathfrak{I}_n,\ x \neq 0\}. \tag{2}$$

We still have:

$$\mathrm{Int}(E, V) = \oplus_{n\in\mathbb{N}}\ \mathfrak{m}^{-w_E(n)} g_n(X), \tag{3}$$

where $g_n(X) \in V[X]$ is a monic polynomial of degree n [even if $V/\mathfrak{m}$ is infinite, because V is a principal ideal domain [5, Corollary II.1.6]].

We already said that Pólya [8] determined the function w_E in the classical case where V is a localization of the ring of integers of a number field K and where $E = V$:

$$w_V(n) = \sum_{k>0} \left[\frac{n}{q^k}\right].$$

Bárbácioru [1, Theorem 4] proposed an extension of this formula to homogeneous subsets E:

$$w_E(n) = l \sum_{k\geq 0} \left[\frac{n}{rq^{lk}}\right].$$

In fact, the proof is wrong [1, Lemma 3] and the formula is incorrect as soon as $l \neq 1$: if $V = \mathbb{Z}_{(2)}$ and $E = 2^l\mathbb{Z}_{(2)}$, Bárbácioru's formula gives the value $w_E(2) = 2l$, while $\frac{X(X-2^l)}{2^{2l+1}} \in \mathrm{Int}(E, V)$, and hence $w_E(2) \geq 2l + 1$.

Using Barghava's notion of *v-ordered sequence* ([2], [3]), we proved another formula which, for each n, allows an algorithmic computation of $w_E(n)$. We recall this result.

Notations. If q is finite, let

$$w_q(n) = \sum_{k>0} \left[\frac{n}{q^k}\right]. \tag{4}$$

If q is infinite, let

$$w_q(n) \equiv 0. \tag{5}$$

For $i \in \{1, \ldots, r\}$ and for $d_1, \ldots, d_r \in \mathbb{N}$, let:

$$w_E^i(d_1, \ldots, d_r) = w_q(d_i) + l d_i + \sum_{j \neq i} v(b_j - b_i) d_j. \tag{6}$$

Proposition 2.1 *[4, Theorem 3.6] With the previous hypotheses and notation [from (1) to (6)], one has*:

$$w_E(n) = \max_{d_1+\cdots+d_r=n} \left(\min_{1\leq i\leq r} w_E^i(d_1, \ldots, d_r) \right). \tag{7}$$

Hence, for each n, $w_E(n)$ may be computed in finitely many steps since there are only finitely many $(d_1, \ldots, d_r) \in \mathbb{N}^r$ such that $d_1 + \cdots + d_r = n$. Corollary 2.3 below improves the computation if this one is done step by step. To establish this corollary, we need to recall the notion of v-ordered sequence (although Proposition 2.1 shows that we may compute $w_E(n)$ without knowing anything about our initial problem).

Definition. Let F be a subset of V. A *v-ordered sequence* $\{a_k\}_{0\leq k\leq n}$ of elements of F is a sequence such that, for each $m \in \{1, \ldots, n\}$,

$$v\left(\prod_{k=0}^{m-1} (a_m - a_k)\right) = \inf_{x\in F} v\left(\prod_{k=0}^{m-1} (x - a_k)\right).$$

We easily see that for each subset F of V and each $n \in \mathbb{N}$:
— there are v-ordered sequences $\{a_k\}_{0\leq k<n}$ of elements of F,
— any v-ordered sequence $\{a_k\}_{0\leq k<n}$ may be extended to a v-ordered sequence $\{a_k\}_{0\leq k\leq n}$ of $n+1$ elements of F.

Proposition 2.2 *[2, Theorems 1 and 12] and [4, Corollary 2.3] Let F be a subset of V. Whatever the v-ordered sequence $\{a_k\}_{0\leq k\leq n}$ of elements of F is, one has:*

$$w_F(n) = v\left(\prod_{k=0}^{n-1} (a_n - a_k)\right).$$

Moreover, assuming $\{a_k\}_{0\leq k<n}$ is a v-ordered sequence of $E=\bigcup_{1\leq j\leq r} b_j+\mathfrak{m}^l$, we know that:
— for each $j\in\{1,\ldots,r\}$, the subsequence formed by the elements a_k in $b_j+\mathfrak{m}^l$ is a v-ordered sequence of $b_j+\mathfrak{m}^l$ [4, Lemma 3.4];
— if, for each $j\in\{1,\ldots,r\}$, d_j denotes the number of a_k's lying in $b_j+\mathfrak{m}^l$, then (see the first part of the proof of [4, Theorem 3.6]):

$$w_E(n)=\inf_j w_E^j(d_1,\ldots,d_r).$$

Now note that, since there is at least one element $a_n\in E$ which extends this v-ordered sequence, there is at least one element $s\in\{1,\ldots,r\}$ such that

$$w_E(n+1)=\inf_j w_E^j(d_1,\ldots,d_s+1,\ldots,d_r).$$

Since $w_E(n+1)$ is a maximin, we also have:

$$w_E(n+1)=\sup_{1\leq t\leq r}\inf_j w_E^j(d_1,\ldots,d_t+1,\ldots,d_r).$$

Now we may conclude with the following improvement of Proposition 2.1:

Corollary 2.3 *The function $w_E(n)$ may be computed in the following way*

$$w_E(n)=\inf_{1\leq j\leq r} w_E^j(d_1,\ldots,d_r),$$

where the integers $d_1=d_1^{(n)},\ldots,d_r=d_r^{(n)}$ are defined by induction:

$$d_1^{(0)}=\ldots=d_r^{(0)}=0$$

and, for $0\leq k<n$,

$$d_i^{(k+1)}=d_i^{(k)} \textit{ for } i\neq s\ ,\ d_s^{(k+1)}=d_s^{(k)}+1$$

where $s\in\{1,\ldots,r\}$ is such that

$$\inf_{1\leq j\leq r} w_E^j(d_1^{(k)},\ldots,d_s^{(k)}+1,\ldots,d_r^{(k)})=$$

$$\sup_{1\leq t\leq r}\inf_{1\leq j\leq r} w_E^j(d_1^{(k)},\ldots,d_t^{(k)}+1,\ldots,d_r^{(k)}).$$

Remark. The previous step by step computation of $w_E(n)$ has to begin with $n=0$ to be sure that, at each step, $d_1^{(n)},\ldots,d_r^{(n)}$ correspond to a v-ordered sequence: it may happen that for some $d_1,\ldots,d_r$ such that $d_1+\ldots+d_r=n$ and $w_E(n)=\inf_j w_E^j(d_1,\ldots,d_r)$ there do not exist a corresponding v-ordered sequence, and hence, that we cannot compute $w_E(n+1)$ with the previous formula.

For instance, let $V = \mathbb{R}[[T]]$ and $E = \{\alpha \in \mathbb{R}[[T]] \mid \alpha \equiv 0, 1, T^2 \pmod{T^3}\}$. Then we have:
$w_E^1(d_1, d_2, d_3) = 3d_1 + 2d_3,\ w_E^2(d_1, d_2, d_3) = 3d_2,\ w_E^3(d_1, d_2, d_3) = 3d_3 + 2d_1.$
In particular,

$$w_E(5) = w_E^2(2, 2, 1) = 6 = w_E^2(3, 2, 0).$$

But $(3, 2, 0)$ does not come from a v-ordered sequence since

$$w_E(4) = w_E^1(1, 2, 1) = 5 \neq \inf_j w_E^j(2, 2, 0),\ \inf_j w_E^j(3, 1, 0).$$

Moreover, although $(3, 2, 0)$ leads to the value $w_E(5)$, it does not provide the following value $w_E(6)$:

$$w_E(6) = w_E^2(2, 2, 2) = 7 \neq \inf_j w_E^j(4, 2, 0),\ \inf_j w_E^j(3, 3, 0),\ \inf_j w_E^j(3, 2, 1).$$

Nevertheless, there is a partial converse: it may happen that, for some values of n, there is a unique r-uple $(d_1, \ldots, d_r)$ such that $d_1 + \ldots + d_r = n$ and $w_E(n) = \inf_j w_E^j(d_1, \ldots, d_r)$ (see Proposition 4.3). For such an r-uple, we are sure that there is a corresponding v-ordered sequence!

In the case where the $v(b_i - b_j)$ are equal, we deduced an explicit formula from Proposition 2.1:

Proposition 2.4 *[4, Proposition 4.4] If, for $i \neq j$, $v(b_i - b_j) = h$ where h is a fixed integer $(0 \leq h < l)$, then one has the following formula:*

$$w_E(n) = w_q\left(\left[\frac{n}{r}\right]\right) + (l - h)\left[\frac{n}{r}\right] + hn.$$

For example, let p be a prime number, let $V = \mathbb{Z}_{(p)}$, and let E be the set of integers not divisible by p. Then $l = 1$, $r = p - 1$, $h = 0$, $q = p$, and hence, one has [6, Lemme 4]:

$$w_E(n) = w_p\left(\left[\frac{n}{p-1}\right]\right) + \left[\frac{n}{p-1}\right] = \sum_{k \geq 0}\left[\frac{n}{(p-1)p^k}\right].$$

If we do not have such a symmetry for the $v(b_i - b_j)$, then the determination of the previous maximin may be quite difficult. For example, the case where E is the set of integers not divisible by p^2 is not so easy (see Proposition 5.4 below).

To go further we may first consider the case where q is infinite, that is, where $w_q \equiv 0$. We then have a linear programming problem. Let us introduce some notation.

Notation. Let $B = (\beta_{i,j}) \in \mathcal{M}_r(\mathbb{N})$ be the symmetric matrix defined by

$$\beta_{i,j} = v(b_i - b_j) \text{ for } 1 \leq i, j \leq r,\ i \neq j, \text{ and } \beta_{i,i} = l \text{ for } 1 \leq i \leq r. \quad (8)$$

Let

$$W_E(d_1,\dots,d_r) = \begin{pmatrix} w_E^1(d_1,\dots,d_r) \\ \cdots \\ w_E^r(d_1,\dots,d_r) \end{pmatrix} \quad \text{and} \quad \Delta = \begin{pmatrix} d_1 \\ \cdots \\ d_r \end{pmatrix}.$$

Then, if q is infinite, formulas (6) are nothing but:

$$W_E(\Delta) = B\Delta.$$

One way to restore some symmetry is to consider values $d_1,\dots,d_r$, if there exist, such that

$$w_E^1(d_1,\dots,d_r) = \cdots = w_E^r(d_1,\dots,d_r) = w.$$

For such values, one has:

$$B\Delta = w \begin{pmatrix} 1 \\ \dots \\ 1 \end{pmatrix}.$$

If the matrix B is invertible, then necessarily the values d_i are determined with:

$$d_i = \frac{w}{\det(B)} \det \gamma_i(B),$$

where $\gamma_i(B)$ denotes the matrix deduced from B by replacing the ith column by the column $\begin{pmatrix} 1 \\ \cdots \\ 1 \end{pmatrix}$. We will see [Proposition 4.3 below] that, for some n, this common value w gives the maximum. But, two questions immediately appear in the previous considerations:

1) Is the matrix B invertible?

2) Are the numbers $\frac{\det(\gamma_i(B))}{\det B}$ $(1 \leq i \leq r)$ positive?

In the following section, we are going to see that both answers are affirmative. To prove it, we introduce the ultrametric matrices; they form a class of matrices which contains the matrices B. Then, in the fourth section, we use our results on determinants of ultrametric matrices (Propositions 3.5 and 3.7) to determine the sequence $w_E(n)$ in the case where q is infinite (Theorem 4.4). Finally, in the last section, we determine the limit of $\frac{w_E(n)}{n}$ when n tends to infinity in the case where q is finite (Theorem 5.3).

3 ULTRAMETRIC MATRICES

Definition. A matrix $A = (a_{i,j}) \in \mathcal{M}_r(\mathbb{R})$ is said to be *ultrametric* if both following conditions are satisfied:

1) A is symmetric,
2) for each $i, j, k \in \{1, \ldots, r\}$, one has $a_{i,j} \geq \inf(a_{i,k}, a_{k,j})$.

For example, $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$ is ultrametric if and only if $a \geq b$ and $c \geq b$.

Lemma 3.1 *Let A be an ultrametric matrix. Then, in particular:*
a) *for each $i \in \{1, \ldots, r\}$, $a_{i,i} = \sup_{1 \leq k \leq r} a_{i,k}$.*
b) *if $i, j, k \in \{1, \ldots, r\}$ are distincts and if $a_{i,k} < a_{k,j}$, then $a_{i,j} = a_{i,k}$.*

Proof. We just have to check the second assertion. If $a_{i,k} < a_{k,j}$, then $a_{i,j} \geq a_{i,k}$. On the other hand, $a_{i,k} \geq \inf(a_{i,j}, a_{j,k})$. If moreover $\inf(a_{i,j}, a_{j,k}) = a_{j,k}$, then $a_{i,k} \geq a_{j,k}$, and we have a contradiction. Thus, $\inf(a_{i,j}, a_{j,k}) = a_{i,j}$, $a_{i,k} \geq a_{i,j}$, and we have the equality. □

The class of ultrametric matrices is stable with respect to several operations we are going to consider now.

Notation. a) For each permutation $\sigma \in \Sigma_r$, we consider the operator:

$$\sigma : A = (a_{i,j}) \in \mathcal{M}_r(\mathbb{R}) \mapsto \sigma(A) = \left(a^{\sigma}_{i,j}\right) \in \mathcal{M}_r(\mathbb{R}),$$

where $a^{\sigma}_{i,j} = a_{\sigma(i),\sigma(j)}$.
b) For $i \in \{1, \ldots, r\}$, let

$$\tau_i : A \in \mathcal{M}_r(\mathbb{R}) \mapsto \tau_i(A) \in \mathcal{M}_{r-1}(\mathbb{R})$$

where $\tau_i(A)$ is the matrix deduced from A by deleting the i-th row and the i-th column.
c) For each $(\varepsilon_1, \ldots, \varepsilon_r) \in \mathbb{R}^r$, let

$$t_{\varepsilon_1,\ldots,\varepsilon_r} : A \in \mathcal{M}_r(\mathbb{R}) \mapsto t_{\varepsilon_1,\ldots,\varepsilon_r}(A) = A + \begin{pmatrix} \varepsilon_1 & \cdots & 0 \\ \cdot & \cdots & \cdot \\ 0 & \cdots & \varepsilon_r \end{pmatrix} \in \mathcal{M}_r(\mathbb{R}).$$

If $A \in \mathcal{M}_r(\mathbb{R})$ is ultrametric then, for each $\sigma \in \Sigma_r$, for each $i \in \{1, \ldots, r\}$, and for each $(\varepsilon_1, \cdots, \varepsilon_r) \in (\mathbb{R}_+)^r$, the matrices $\sigma(A)$, $\tau_i(A)$, and $t_{\varepsilon_1,\cdots,\varepsilon_r}(A)$ are ultrametric.

We also have the following immediate result, we give it as a lemma without proof.

Lemma 3.2 *Let* $A \in \mathcal{M}_r(\mathbb{R})$ *and let* $\sigma \in \Sigma_r$. *Then,*

a) $\det(\sigma(A)) = \det(A)$,

b) for each $(\varepsilon_1, \ldots, \varepsilon_r) \in \mathbb{R}^r$, $\sigma\left(t_{\varepsilon_1,\ldots,\varepsilon_r}(A)\right) = t_{\varepsilon_{\sigma(1)},\ldots,\varepsilon_{\sigma(r)}}(\sigma(A))$.

Now let us return to the matrix B defined at the end of section 2. Obviously, B is an ultrametric matrix. More generally, if F is a field with a rank-one valuation ω (that is, such that the value group of ω is a subgroup of $\mathbb{R}$) and if $c_1, \ldots, c_r$ are distinct elements of F, then the elements $a_{i,j} = \omega(c_i - c_j)$ for $1 \leq i \neq j \leq r$, are obviously the non-diagonal coefficients of an ultrametric matrix $A \in \mathcal{M}_r(\mathbb{R})$. In fact, the converse also holds.

Proposition 3.3 *Let* $r \geq 2$ *and let* $\{a_{i,j} \mid 1 \leq i < j \leq r\}$ *be a set of real numbers. Let* F *be a field with a rank-one valuation* ω *such that the value group* $\omega(K^*)$ *contains all the* $a_{i,j}$*'s and the residue field of* ω *contains at least* r *elements. Then the* $a_{i,j}$*'s are the non-diagonal coefficients of an ultrametric matrix* $A \in \mathcal{M}_r(\mathbb{R})$ *if and only if there are* r *elements* $c_1, \ldots, c_r \in F$ *such that* $\omega(c_i - c_j) = a_{i,j}$ *for* $1 \leq i < j \leq r$.

To prove this proposition we first point out a particularity of ultrametric matrices.

Lemma 3.4 *Let* $A = (a_{i,j}) \in \mathcal{M}_r(\mathbb{R})$ *be an ultrametric matrix (with* $r \geq 3$*). If* i_0 *and* $j_0 \in \{1, \ldots, r\}, i_0 \neq j_0$, *are such that* $a_{i_0,j_0} = \sup_{i \neq j} a_{i,j}$, *then, for each* $k \neq i_0, j_0$, *one has:* $a_{i_0,k} = a_{j_0,k}$.

Proof. For each $k \neq i_0, j_0$, one has:

$$a_{i_0,k} \geq \inf\left(a_{i_0,j_0}, a_{j_0,k}\right) = a_{j_0,k}.$$

By symmetry, we have the equality. □

Proof of Proposition 3.3. We have to state the necessary condition. We prove it by induction on r. For $r = 2$ the assertion is obvious. Let $r \geq 3$ and assume the assertion is true for $r - 1$. Let $a = \sup_{i \neq j} a_{i,j}$.

Of course, for each permutation $\sigma \in \Sigma_r$, the assertion is true for the ultrametric matrix $A \in \mathcal{M}_r(\mathbb{R})$ if and only if it is true for the matrix $\sigma(A)$ and, if $c_1, \ldots, c_r$ may be associated to A, then $c_{\sigma(1)}, \ldots, c_{\sigma(r)}$ may be associated to $\sigma(A)$. Thus, we may assume that $a_{1,r} = a$. Since $\tau_r(A) \in \mathcal{M}_{r-1}(\mathbb{R})$ is also ultrametric, it follows from the induction hypothesis that there are $r-1$ elements $c_1, \ldots, c_{r-1} \in F$ such that $a_{i,j} = \omega(c_i - c_j)$ for $1 \leq i < j \leq r - 1$. Moreover, we may assume that, for some $s \in \{1, \cdots, r-1\}$, $a_{1,r} = a_{2,r} = \ldots = a_{s,r} = a$ and $a_{k,r} \neq a$ for $s < k < r$.

Now, for $j, j' \in \{1, \cdots, s\}$, if $j \neq j'$, then one has $\omega(c_j - c_{j'}) = a_{j,j'} \geq \inf(a_{j,r}, a_{j',r}) = a$, and hence $\omega(c_j - c_{j'}) = a$. Let $d \in F$ be such that $\omega(d) = a$. For each $j \in \{2, \ldots, s\}$, let $u_j \in F$ be such that $c_j = c_1 + du_j$ $(\omega(u_j) = 0)$. Let $u_1 = 0$. Then the classes of the u_j $(1 \leq j \leq s)$ are distinct elements of the residue field of ω: for $j \neq j'$, $\omega(u_j - u_{j'}) = \omega(c_j - c_{j'}) - \omega(d) = 0$. Now, let $u_r \in F$ be such that its class is distinct from those of $u_1 = 0, u_2, \ldots, u_s$, and let $c_r = c_1 + du_r$.

Let us prove that $\omega(c_r - c_j) = a_{j,r}$ for $1 \leq j \leq r - 1$:
– by construction, $\omega(c_r - c_1) = \omega(d) + \omega(u_r) = a = a_{1,r}$;
– for $2 \leq j \leq s$, $\omega(c_r - c_j) = \omega(d) + \omega(u_r - u_j) = a = a_{j,r}$;
– for $s + 1 \leq j \leq r - 1$, one has: $\omega(c_j - c_1) = a_{1,j},\ a_{1,j} = a_{r,j}$ (Lemma 3.4); $a_{r,j} < a = a_{1,r},\ a_{1,r} = \omega(c_1 - c_r)$; finally, $\omega(c_j - c_1) < \omega(c_1 - c_r)$, and hence, $\omega(c_j - c_r) = \omega(c_j - c_1) = a_{r,j}$. $\square$

In fact, the matrix B considered in the first section has a property stronger than the ultrametric property.

Definition. A matrix $A = (a_{i,j}) \in \mathcal{M}_r(\mathbb{R})$ (with $r \geq 2$) is said to be *strictly ultrametric* if both following conditions are satisfied:
1) A is ultrametric,
2) for each $i \in \{1, \ldots, r\}$, one has:

$$a_{i,i} > \sup_{1 \leq k \leq r,\, k \neq i} a_{i,k}.$$

The class of strictly ultrametric matrices is also stable with respect to the operations σ, τ_i, and $t_{\varepsilon_1,\ldots,\varepsilon_r}$ with $(\varepsilon_1, \ldots, \varepsilon_r) \in (\mathbb{R}_+)^r$. Here is the answer to the first question raised in section 2.

Proposition 3.5 *Let $A \in \mathcal{M}_r(\mathbb{R}_+)$.*
1. If A is a ultrametric, then $\det(A) \geq 0$.
2. If A is strictly ultrametric, then $\det(A) > 0$.

Proof. We prove the assertion by induction on r. If $r = 2$, then $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ with $a \geq b \geq 0$, $c \geq b \geq 0$ ($a > b \geq 0$, $c > b \geq 0$, respectively), and hence $\det(A) = ac - b^2 \geq 0$ (> 0, respectively). Let r be a fixed integer ≥ 3. We assume that the assertion is true for $r - 1$ and we consider an ultrametric matrix $A \in \mathcal{M}_r(\mathbb{R}_+)$. It follows from Lemma 3.2.a that we may assume $a_{1,2} = \sup_{i,j} a_{i,j}$, and from Lemma 3.4 that A is of the following

form:
$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \dots & a_{1,r} \\ a_{1,2} & a_{2,2} & a_{1,3} & \dots & a_{1,r} \\ a_{1,3} & a_{1,3} & & & \\ \cdot & \cdot & & \tau_1\left(\tau_2(A)\right) & \\ a_{1,r} & a_{1,r} & & & \end{pmatrix}.$$

Then

$$\det(A) = \begin{vmatrix} a_{1,1}-a_{1,2} & a_{1,2}-a_{2,2} & 0 & \dots & 0 \\ a_{1,2} & a_{2,2} & a_{1,3} & \dots & a_{1,r} \\ a_{1,3} & a_{1,3} & & & \\ \cdot & \cdot & & \tau_1\left(\tau_2(A)\right) & \\ a_{1,r} & a_{1,r} & & & \end{vmatrix} =$$

$$(a_{1,1}-a_{1,2})\det\left(\tau_1(A)\right) + (a_{2,2}-a_{1,2})\det\left(t_{a_{1,2}-a_{2,2},0,\dots,0}\left(\tau_1(A)\right)\right).$$

It follows from the choice of $a_{1,2}$ that the matrix $t_{a_{1,2}-a_{2,2},0,\dots,0}\left(\tau_1(A)\right)$ is still ultrametric (although $a_{1,2}-a_{2,2}$ may be < 0), and from the induction hypothesis that its determinant is ≥ 0. Thus we have:

$$\det(A) \geq (a_{1,1}-a_{1,2})\det\left(\tau_1(A)\right) \geq 0 \quad (> 0, \text{ respectively})$$

since A and then $\tau_1(A)$ are (strictly) ultrametric. □

Recall now the operators γ_i introduced in the first section.

Notation. For each $i \in \{1,\dots,r\}$, let

$$\gamma_i : A \in \mathcal{M}_r(\mathbb{R}) \mapsto \gamma_i(A) \in \mathcal{M}_r(\mathbb{R}),$$

where $\gamma_i(A)$ is the matrix deduced from A by replacing the ith column by the column $\begin{pmatrix} 1 \\ \cdots \\ 1 \end{pmatrix}$.

Once more, we have a straightforward result:

Lemma 3.6 *For each $i \in \{1,\dots,r\}$ and each permutation $\sigma \in \Sigma_r$, one has:*

$$\sigma\left(\gamma_i(A)\right) = \gamma_{\sigma^{-1}(i)}\left(\sigma(A)\right).$$

Now we give the answer to the second question.

Proposition 3.7 *Let $A \in \mathcal{M}_r(\mathbb{R})$.*
1. If A is ultrametric then, for each $i \in \{1, \dots, r\}$, $\det(\gamma_i(A)) \geq 0$.
2. If A is strictly ultrametric then, for each $i \in \{1, \dots, r\}$, $\det(\gamma_i(A)) > 0$.

Proof. Let us note that we no more assume that the coefficients of A are positive. The proof is analogous to that of Proposition 3.5. If $r = 1$, then $\det(\gamma_1(A)) = 1$. If $r = 2$, then $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ with $a \geq b$ and $c \geq b$ ($a > b$ and $c > b$, respectively), and hence $\det(\gamma_1(A)) = c - b \geq 0$ (> 0, respectively) and $\det(\gamma_2(A)) = a - b \geq 0$ (> 0, respectively).

Let $r \geq 3$. We assume the assertion is true for $r - 1$. Using the same notation, an analogous computation shows that, for $3 \leq i \leq r$,

$$\det(\gamma_i(A)) = \begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \dots & 1 & \dots & a_{1,r} \\ a_{1,2} & a_{2,2} & a_{1,3} & \dots & 1 & \dots & a_{1,r} \\ a_{1,3} & a_{1,3} & & & & & \\ \cdot & \cdot & & & \gamma_{i-1}(\tau_1(A)) & & \\ a_{1,r} & a_{1,r} & & & & & \end{vmatrix} =$$

$$(a_{1,1}-a_{1,2}) \det(\gamma_{i-1}(\tau_1(A))) + (a_{2,2}-a_{1,2}) \det\left(\gamma_{i-1}\left(t_{a_{1,2}-a_{2,2},0,\dots,0}(\tau_1(A))\right)\right).$$

In the same manner we may conclude with the induction hypothesis.

For $i = 1$, one has:

$$\det(\gamma_1(A)) = \begin{vmatrix} 1 & a_{1,2} & a_{1,3} & \dots & a_{1,r} \\ 1 & a_{2,2} & a_{1,3} & \dots & a_{1,r} \\ 1 & a_{1,3} & & & \\ \cdot & \cdot & & \tau_1(\tau_2(A)) & \\ 1 & a_{1,r} & & & \end{vmatrix} =$$

$$(a_{2,2} - a_{1,2}) \begin{vmatrix} 1 & a_{1,3} & \dots & a_{1,r} \\ 1 & & & \\ \cdot & & \tau_1(\tau_1(A)) & \\ 1 & & & \end{vmatrix} = (a_{2,2} - a_{1,2}) \det(\gamma_1(\tau_2(A))).$$

Once more, we end with the induction hypothesis. The proof is the same for $\gamma_2(A)$. $\square$

4 COMPUTATION OF $W_E(n)$ IN THE CASE WHEN q IS INFINITE

We first introduce another notation.

Notation. For every matrix $A \in \mathcal{M}_r(\mathbb{R})$, we set

$$\delta_i(A) = \det(\gamma_i(A)) \qquad \text{for } i = 1, \ldots, r \tag{9}$$

(where $\gamma_i(A)$ is the matrix deduced from A by replacing every element of the ith column by 1) and

$$\nu(A) = \sum_{i=1}^{r} \delta_i(A) = \sum_{1 \le i \le r} \det\left(\gamma_i(A)\right). \tag{10}$$

Proposition 4.1 *If q is infinite then, for each $n \in \mathbb{N}$, one has:*

$$w_E(n) \le n \frac{\det(B)}{\nu(B)}.$$

This is a particular case of the following lemma which will also be useful for the finite case.

Lemma 4.2 *Assume $B^{\#} \in \mathcal{M}_r(\mathbb{R})$ is a strictly ultrametric matrix such that, for each $\Delta \in \mathbb{N}^r$,*

$$W_E(\Delta) \le B^{\#}\Delta,$$

where $\mathbb{R}^r$ is partially ordered by:

$$(v_1 \ldots, v_r) \le (w_1, \ldots, w_r) \ := \ v_1 \le w_1, \ldots, v_r \le w_r.$$

Then, for each $n \in \mathbb{N}$, one has:

$$w_E(n) \le n \frac{\det(B^{\#})}{\nu(B^{\#})}.$$

Proof. Let $n \in \mathbb{N}$. Fix $d_1, \ldots, d_r \in \mathbb{N}$ such that $d_1 + \cdots + d_r = n$ and consider $w^1, \ldots, w^r$ defined by $\begin{pmatrix} w^1 \\ \cdot \\ \cdot \\ w^r \end{pmatrix} = B^{\#} \begin{pmatrix} d_1 \\ \cdot \\ \cdot \\ d_r \end{pmatrix}$. We then may consider that we have a linear system of $r+1$ equations in the r unknowns $d_1, \ldots, d_r$:

$$\begin{pmatrix} & & \\ & B^{\#} & \\ & & \\ 1 & \cdots & 1 \end{pmatrix} \begin{pmatrix} d_1 \\ \cdot \\ \cdot \\ d_r \end{pmatrix} = \begin{pmatrix} w^1 \\ \cdot \\ \cdot \\ w^r \\ n \end{pmatrix}.$$

There is a compatibility condition

$$\begin{vmatrix} & & & w^1 \\ & B^{\#} & & \cdot \\ & & & w^r \\ 1 & \cdot & 1 & n \end{vmatrix} = 0$$

which is necessarily satisfied. That is,

$$w^1 \det\Big(\gamma_1(B^{\#})\Big) + \ldots + w^r \det\Big(\gamma_r(B^{\#})\Big) - n\det(B^{\#}) = 0,$$

or

$$w^1\delta_1(B^{\#}) + \ldots + w^r\delta_r(B^{\#}) - n\det(B^{\#}) = 0.$$

Since, for each i, $\delta_i(B^{\#}) > 0$ (Proposition 3.6), one has:

$$\left(\inf_{1\le i\le r} w^i\right)\nu(B^{\#}) \le n\det(B^{\#}).$$

On the other hand, by hypothesis $w_E^i(d_1, \ldots, d_r) \le w^i$ for each $i \in \{1, \ldots, r\}$. Thus,

$$\inf_{1\le i\le r} w_E^i(d_1, \ldots, d_r) \le \inf_{1\le i\le r} w^i \le n\frac{\det(B^{\#})}{\nu(B^{\#})}.$$

Since the inequality $\inf_i w_E^i(d_1, \ldots, d_r) \le n\frac{\det(B^{\#})}{\nu(B^{\#})}$ holds for all $d_1, \ldots, d_r$ such that $d_1 + \cdots + d_r = n$, we finally have :

$$w_E(n) = \sup_{d_1+\cdots+d_r=n} \inf_{1\le i\le r} w_E^i(d_1, \ldots, d_r) \le n\frac{\det(B^{\#})}{\nu(B^{\#})}.$$

□

Proposition 4.3 *Assume q is infinite and n is a multiple of $\nu(B)$. Then*

$$w_E(n) = \frac{n}{\nu(B)}\det(B).$$

Moreover, one has $w_E(n) = \inf_j w^j(d_1, \ldots, d_r)$ where $d_1 + \ldots + d_r = n$ if and only if $d_i = \frac{n}{\nu(B)}\delta_i(B)$ for $i = 1, \ldots, r$. In particular, in every v-ordered sequence of n elements there are exactly $d_i = \frac{n}{\nu(B)}\delta_i(B)$ elements in $b_i + \mathfrak{m}^l$.

Proof. It follows from Proposition 4.1 that we just have to prove the inequality $w_E(n) \geq n\frac{\det(B)}{\nu(B)}$. Write $n = m\nu(B)$ where $m \in \mathbb{N}$. For $i = 1, \ldots, r$, let $d_i = m\delta_i(B)$. Then, $d_i \in \mathbb{N}$ since $\det(\gamma_i(B)) > 0$ (Proposition 3.6), $\sum_{i=1}^r d_i = m\nu(B) = n$, and the d_i's obviously form a solution of the linear system

$$B \begin{pmatrix} d_1 \\ \cdots \\ d_r \end{pmatrix} = m\det(B) \begin{pmatrix} 1 \\ \cdots \\ 1 \end{pmatrix}.$$

Then, for such $d_1, \ldots, d_r$ one has:

$$W_E(\Delta) = m\det(B) \begin{pmatrix} 1 \\ \cdots \\ 1 \end{pmatrix};$$

and hence,

$$\inf_{1\leq i\leq r} w_E^i(d_1, \ldots, d_r) = m\det(B).$$

Finally,

$$w_E(n) = \sup_{d_1+\cdots+d_r=n} \inf_{1\leq i\leq r} w_E^i(d_1, \ldots, d_r) \geq m\det(B) = \frac{n}{\nu(B)}\det(B).$$

Now assume that $d_1, \ldots, d_r$ are such that $d_1 + \ldots + d_r = n$ and $w_E(n) = \inf_j w_E^j(d_1, \ldots, d_r)$. To simplify let us note $w^j = w_E^j(d_1, .., d_r)$. It follows from the previous proof that $(\inf_j w^j) \times \sum_i \delta_i(B) = n\det(B)$ and from the proof of the previous proposition that $\sum_j w^j \times \delta_j(B) = n\det(B)$. Thus, necessarily $w^1 = \ldots = w^r$, and hence, the d_i's are of the form $m\delta_i(B)$ (see the end of § 2), that is, $d_i = \frac{n}{\nu(B)}\delta_i(B)$. The last assertion of the proposition is then an immediate consequence. □

Theorem 4.4 *With the previous hypothesis and notations [from (1) to (10)], assume q is infinite. If*

$$n = m\nu(B) + n_0 \qquad \text{with} \quad m, n_0 \in \mathbb{N},$$

then

$$w_E(n) = w_E(m\nu(B)) + w_E(n_0) = m\det(B) + w_E(n_0).$$

Consequently, to know the sequence $w_E(n)$ we just have to compute $w_E(n_0)$ for $0 < n_0 < \nu(B)$.

Proof. The definition of $w_E(n)$ obviously implies:

$$w_E(n) \geq w_E(m\nu(B)) + w_E(n_0).$$

We have to prove the other inequality. Let $d'_i = m\delta_i(B)$ for $i = 1, \ldots, r$. The previous proof shows that:

$$w_E(m\nu(B)) = w_E^j(d'_1, \ldots, d'_r) \text{ for } j = 1, \ldots, r.$$

Moreover, the previous proposition shows that there is a v-ordered sequence of elements of E with $m\nu(B)$ elements such that d'_i elements are in $b_i + \mathfrak{m}^l$ for $i = 1, \ldots, r$. We extend this sequence to obtain a v-ordered sequence with n elements and denote by d_i the number of elements belonging to $b_i + \mathfrak{m}^l$ $(i = 1, \ldots, r)$. Then, by Proposition 2.2, we have:

$$w_E(n) = \inf_j w_E^j(d_1, \ldots, d_r).$$

Of course $d_i \geq d'_i$: let $d''_i = d_i - d'_i$ for $i = 1, \ldots, r$. Since q is infinite, the functions w_E^j are linear:

$$w_E^j(d_1, \ldots, d_r) = w_E^j(d'_1, \ldots, d'_r) + w_E^j(d''_1, \ldots, d''_r).$$

Since $w_E^j(d'_1, \ldots, d'_r)$ does not depend on j, we also have:

$$\inf_j w_E^j(d_1, \ldots, d_r) = w_E(m\nu(B)) + \inf_j w_E^j(d''_1, \ldots, d''_r).$$

Now, it follows from Proposition 2.1 that

$$w_E(n_0) \geq \inf_j w_E^j(d''_1, \ldots, d''_r).$$

Consequently,

$$w_E(n) \leq w_E(m\nu(B)) + w_E(n_0).$$

Finally we have an equality. □

Example. Let $V = \mathbb{R}[[T]]$ and $E = \{\alpha \in \mathbb{R}[[T]] \mid \alpha \equiv 0, 1, T \pmod{T^2}\}$. If $n = 7m + n_0$, then $w_E(n) = 6m + w_E(n_0)$ with $w_E(n_0) = n_0 - 1$ for $1 \leq n_0 \leq 5$ and $w_E(6) = 4$.

Remark. In the case when q is infinite, we may consider the following more general subsets E of V:

$$E = \bigcup_{i=1}^{r} b_i + \mathfrak{m}^{l_i},$$

where the l_i are not necessarily equal and where the b_i are pairwise non-congruent modulo $\mathfrak{m}^l$ with $l = \inf_i l^i$. All the previous results remain valid when we replace $\beta_{i,i} = l$ by $\beta_{i,i} = l_i$ in Formula (8) because the corresponding matrix B remains a strictly ultrametric matrix.

5 ASYMPTOTIC BEHAVIOR OF $w_E(n)$ IN THE CASE WHEN q IS FINITE

In the case when q is finite, things are slightly more complicated.

Proposition 5.1 *Assume q is finite. Let $B^* = B + \frac{1}{q-1} I_r$. Then, for each $n \in \mathbb{N}$, one has:*

$$w_E(n) \leq n \frac{\det(B^*)}{\nu(B^*)}.$$

This proposition follows from Lemma 4.2 with $B^\# = B^*$ since, for each $d \in \mathbb{N}$, $w_q(d) = \sum_{k>0} \left[\frac{d}{q^k}\right] < \frac{d}{q-1}$.

Lemma 5.2 *Assume q is finite. For each $k \in \mathbb{N}$, let*

$$B^{(k)} = B + \frac{q^k - 1}{q^k(q-1)} I_r.$$

If n is a multiple of $q^{rk}\nu(B^{(k)})$, then $w_E(n) \geq n \frac{\det(B^{(k)})}{\nu(B^{(k)})}$.

Proof. Write $n = mq^{rk}\nu(B^{(k)})$ where $m \in \mathbb{N}$. For $i = 1, \ldots, r$, let $d_i = mq^{rk}\delta_i(B^{(k)})$. Then $d_i \in \mathbb{N}$, $\sum_{i=1}^r d_i = n$, and the d_i's obviously form a solution of the linear system:

$$B^{(k)} \begin{pmatrix} d_1 \\ \cdots \\ d_r \end{pmatrix} = mq^{rk} \det(B^{(k)}) \begin{pmatrix} 1 \\ \cdots \\ 1 \end{pmatrix}.$$

For each $i \in \{1, \ldots, r\}$ and for such $d_1, \ldots, d_r$, one has

$$w_q(d_i) \geq \frac{q^{rk} - 1}{q^{rk}(q-1)} d_i,$$

and hence,

$$w_E^i(d_1, \ldots, d_r) \geq mq^{rk} \det(B^{(k)}).$$

Finally,

$$w_E(n) \geq \inf_{1 \leq i \leq r} w_E^i(d_1, \ldots, d_r) \geq mq^{rk} \det(B^{(k)}) = n \frac{\det(B^{(k)})}{\nu(B^{(k)})}.$$

□

Theorem 5.3 *With the previous hypotheses and notations [from (1) to (10)], assume q is finite. Then one has:*

$$\lim_{n\to\infty} \frac{w_E(n)}{n} = \frac{\det(B^*)}{\nu(B^*)} \quad \text{where } B^* = \left(B + \frac{1}{q-1} I_r\right),$$

$$\nu(B^*) = \sum_{i=1}^{r} \delta_i(B^*) \text{ and } \delta_i(B^*) = \det(\gamma_i(B^*)).$$

Proof. Let $\varepsilon > 0$. Since, $\frac{\det(B^{(k)})}{\nu(B^{(k)})}$ tends to $\frac{\det(B^*)}{\nu(B^*)}$ when k tends to infinity, we may fix a k such that

$$\left| \frac{\det(B^{(k)})}{\nu(B^{(k)})} - \frac{\det(B^*)}{\nu(B^*)} \right| \leq \frac{\varepsilon}{2}.$$

Let $n \in \mathbb{N}$ and let

$$m = \left[\frac{n}{q^{rk}\nu(B^{(k)})} \right].$$

Then

$$n_0 = mq^{rk}\nu(B^{(k)}) \leq n < (m+1)q^{rk}\nu(B^{(k)}) = n_1.$$

One has: $w_E(n_0) \leq w_E(n)$ (w_E is an increasing function), $w_E(n) \leq n\frac{\det(B^*)}{\nu(B^*)}$ (Proposition 5.1), and $w_E(n_0) \geq n_0 \frac{\det(B^{(k)})}{\nu(B^{(k)})}$ (Lemma 5.2). Thus,

$$\frac{m}{m+1} \frac{\det(B^{(k)})}{\nu(B^{(k)})} \leq \frac{n_0}{n} \frac{\det(B^{(k)})}{\nu(B^{(k)})} \leq \frac{w_E(n_0)}{n} \leq \frac{w_E(n)}{n} \leq \frac{\det(B^*)}{\nu(B^*)}.$$

In particular,

$$\frac{m}{m+1} \frac{\det(B^{(k)})}{\nu(B^{(k)})} \leq \frac{w_E(n)}{n} \leq \frac{\det(B^*)}{\nu(B^*)}.$$

Finally,

$$\frac{\det(B^*)}{\nu(B^*)} - \frac{m}{m+1} \frac{\det(B^{(k)})}{\nu(B^{(k)})} \leq \frac{1}{m} \frac{\det(B^*)}{\nu(B^*)} + \frac{\varepsilon}{2} \leq \varepsilon$$

as soon as $m \geq \frac{2}{\varepsilon} \frac{\det(B^*)}{\nu(B^*)}$. □

Remark. Let $B \in \mathcal{M}_r(\mathbb{R}_+)$ be a strictly ultrametric matrix and, for each $x \in \mathbb{R}_+$, let $B(x) = B + xI_r$. We are going to see that the function ϕ defined by $\phi(x) = \frac{\det(B(x))}{\nu(B(x))}$ is an increasing function of x. In particular, $\frac{\det(B^{(k)})}{\nu(B^{(k)})}$ is an increasing function of k whose limit at infinity is $\frac{\det(B^*)}{\nu(B^*)}$.

This is a consequence of the previous proofs. Let

$$w_x = \sup_{(d_1,\dots,d_r)\in(\mathbb{R}_+)^r,\ d_1+\dots+d_r=1} \ \inf_{1\le i\le r} \ w_x^i(d_1,\dots,d_r),$$

where

$$\left(w_x^i(d_1,\dots,d_r)\right) = B(x)\begin{pmatrix} d_1 \\ \cdots \\ d_r \end{pmatrix}.$$

The proof of Lemma 4.2 shows that $w_x \le \phi(x)$, and the proof of Proposition 4.3 shows that $w_x \ge \phi(x)$ (the d_i are no more supposed to be integers). Thus, $\phi(x) = w_x$. Since $x \le y$ obviously implies $w_x \le w_y$, we then have:

$$x \le y \Rightarrow \phi(x) \le \phi(y).$$

Proposition 5.4 *Let p be a prime number. If E is the set of integers not divisible by p^2, then*

$$\lim_{n\to+\infty} \frac{w_E(n)}{n} = \frac{\det(B^*)}{\nu(B^*)} = \frac{p(p^2-p+1)}{(p-1)^2(p^2+1)}.$$

Proof.

$$E = \bigcup_{k=1}^{p^2-1} k + p^2\mathbb{Z}$$

and

$$\mathrm{Int}(E,\mathbb{Z})_{(p)} = \mathrm{Int}(\overline{E},\mathbb{Z}_{(p)})$$

where (Proposition 1.1):

$$\overline{E} = \bigcup_{k=1}^{p^2-1} k + p^2\mathbb{Z}_{(p)}.$$

By ordering the elements $b_i \in \{1,\dots,p^2-1\}$ in the following way :

$$1, 1+p, 1+2p, \dots, 1+(p-1)p\,;\ 2, 2+p, 2+2p, \dots, 2+(p-1)p\,;$$

$$3, 3+p, 3+2p, \dots, 3+(p-1)p\,;\ \dots\,;\ p, 2p, \dots, (p-1)p$$

the corresponding matrix $B^* = B + \frac{1}{p-1} I_{p^2-1}$ is of the form:

$$J = \begin{pmatrix} J_p & 0 & \cdots & 0 & 0 \\ 0 & J_p & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & J_p & 0 \\ 0 & 0 & \cdots & 0 & J_{p-1} \end{pmatrix},$$

where, for each $s \in \mathbb{N}^*$, $J_s \in \mathcal{M}_s(\mathbb{R})$ is defined by:

$$J_s = \begin{pmatrix} 1 & \cdots & 1 \\ \cdots & \cdots & \cdots \\ 1 & \cdots & 1 \end{pmatrix} + \frac{p}{p-1} I_s.$$

We easily obtain the following equalities:

$$\det(B^*) = (\det(J_p))^{p-1} \det(J_{p-1}) \ ;$$

for $1 \leq i \leq (p-1)p$,

$$\delta_i(B^*) = (\det(J_p))^{p-2} \det(J_{p-1}) \det(\gamma_1(J_p)) \ ;$$

for $(p-1)p + 1 \leq i \leq p^2 - 1$,

$$\delta_i(B^*) = (\det(J_p))^{p-1} \det(\gamma_1(J_{p-1})) \ ;$$

and for each $s \in \mathbb{N}^*$,

$$\det(J_s) = \left(s + \frac{p}{p-1}\right)\left(\frac{p}{p-1}\right)^{s-1} \text{ and } \det(\gamma_1(J_s)) = \left(\frac{p}{p-1}\right)^{s-1}.$$

Finally,

$$\lim_{n \to +\infty} \frac{w_E(n)}{n} = \frac{\det(B^*)}{\nu(B^*)} = \frac{p(p^2 - p + 1)}{(p-1)^2(p^2+1)}.$$

□

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