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Jean-Luc Chabert

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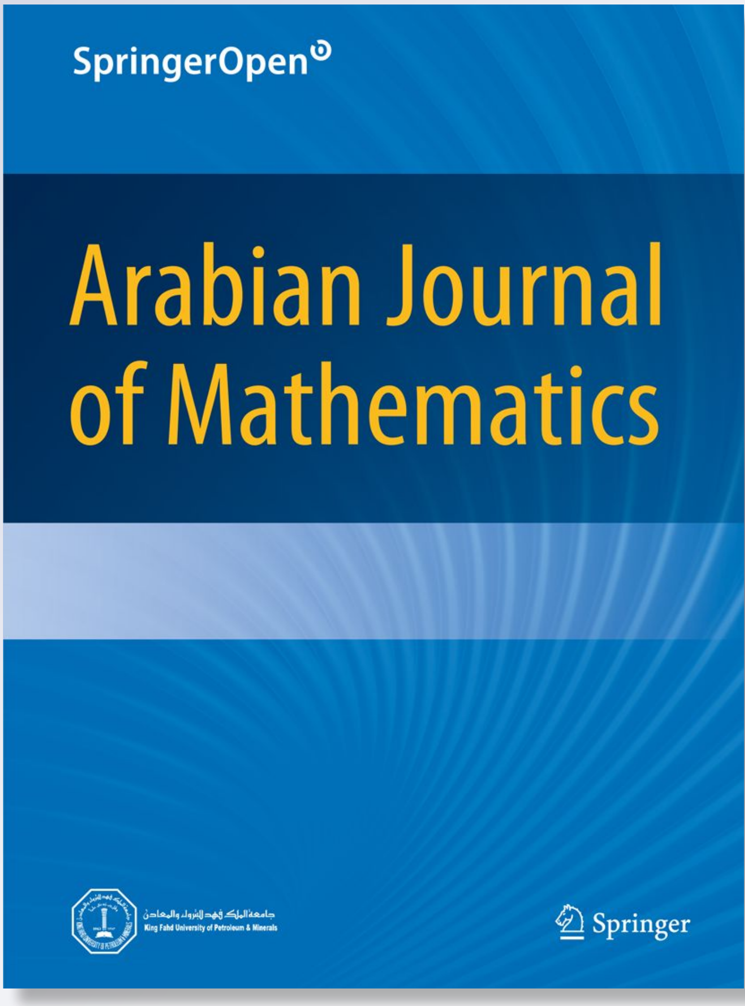

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RESEARCH ARTICLE

Jean-Luc Chabert

Does Int($\mathbb{Z}$) have the stacked bases property?

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Abstract It is known that a Prüfer domain either with dimension 1 or with finite character has the stacked bases property. Following Brewer and Klinger, some rings of integer-valued polynomials provide, for every $n \geq 2$, examples of n-dimensional Prüfer domains without finite character which have the stacked bases property. But, the following question is still open: does the two-dimensional Prüfer domain $\text{Int}(\mathbb{Z}) = \{f \in \mathbb{Q}[X] \mid f(\mathbb{Z}) \subseteq \mathbb{Z}\}$ have the stacked bases property? By means of the UCS-property, we reduce the question to the search for some 2×2 matrices with coefficients in Int($\mathbb{Z}$).

Mathematics Subject Classification 13F20 · 13C10 · 13F05

الملخص

من المعلوم أن أي مجال بروفر ذا بعد 1 أو ذا سمة منتهية يمتلك خاصية القواعد المكدَّسة. باتباع بريور وكلنقر، تقدم بعض حلقات كثيرات الحدود ذات القيم الصحيحة، لكل $n \geq 2$، أمثلة على مجالات بروفر ذات بعد n بدون سمة منتهية ولكنها تمتلك خاصية القواعد المكدَّسة. لكن السؤال التالي ما زال مفتوحاً: هل يمتلك مجال بروفر $\text{Int}(\mathbb{Z}) = \{f \in \mathbb{Q}[X] | f(\mathbb{Z}) \subseteq \mathbb{Z}\}$ ذو البعد 2 خاصية القواعد المكدَّسة؟ باستخدام خاصية UCS، نختزل هذا السؤال إلى البحث عن مصفوفات 2×2 توجد معاملاتها في $\text{Int}(\mathbb{Z})$.

1 The stacked bases property

If you cannot solve the proposed problem, try to solve first some related problem. Could you imagine a more accessible related problem? ... G. Pólya [8]

The following result is well known. It is itself a generalization of the structure theorem for the finitely generated abelian groups.

Proposition 1.1 *Let D be a principal ideal domain. If M is a free D-module with finite rank m, then every submodule N of M is also a free D-module with rank $n \leq m$. Moreover, there exists a basis $(e_1, \ldots, e_m)$ of M and elements $a_1, \ldots, a_n$ of D such that $(a_1e_1, \ldots, a_ne_n)$ is a basis of N. We still may ask that a_j divides a_{j+1} for $1 \leq j \leq n-1$.*

J.-L. Chabert (✉)
Départment de Mathématiques, LAMFA CNRS-UMR 6140,
Université de Picardie, 80039 Amiens, France
E-mail: jean-luc.chabert@u-picardie.fr

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Such a result has been generalized to Dedekind domains D. Since a submodule of a free D-module is not necessarily a free module (for instance, the non-zero ideals of D are only rank-one projective modules), the previous proposition may be generalized only in the following way:

Proposition 1.2 *Let D be a Dedekind domain. If M is a free D-module with finite rank m, then every submodule N of M is a projective D-module with rank $n \leq m$. Moreover, there exist* stacked bases decompositions *of M and N*:

$$M = P_1 \oplus \cdots \oplus P_m \quad \textit{and} \quad N = \mathfrak{I}_1 P_1 \oplus \cdots \oplus \mathfrak{I}_n P_n$$

where the P_i's are rank-one projective modules and the $\mathfrak{I}_j$'s are non-zero ideals of D. We still may ask that $\mathfrak{I}_j$ divides $\mathfrak{I}_{j+1}$ for $1 \leq j \leq n-1$.

If we want to extend such a result to the non-noetherian case by replacing the Dedekind domain by a Prüfer domain, we notice that a submodule of a free module with finite rank is not necessarily finitely generated, so that we have to assume it. On the other hand, we know that every finitely generated submodule of a free module on a Prüfer domain is a projective module with finite rank. So that the generalization to Prüfer domains would be given by the answer to the following question: do Prüfer domains have the stacked bases property (property that we recall now)?

Definition 1.3 [6, § V.4] A domain D is said to have *the stacked bases property* if, for every free D-module M with finite rank m and every finitely generated submodule N of M with rank $n \leq m$ there exist rank-one projective D-modules $P_1, \ldots, P_m$ and non-zero ideals $\mathfrak{I}_1, \ldots, \mathfrak{I}_n$ of D such that:

$$M = P_1 \oplus \cdots \oplus P_m, \quad N = \mathfrak{I}_1 P_1 \oplus \cdots \oplus \mathfrak{I}_n P_n$$

and

$$\mathfrak{I}_{j+1} \subseteq \mathfrak{I}_j \quad \text{for } 1 \leq j \leq n-1.$$

Clearly, in the previous definition, we may replace the hypothesis that M is a free module with finite rank m either by M is a projective module with finite rank m, or by M is equal to D^m.

As said, we are interested in Prüfer domains which have the stacked bases property. In fact, we will replace the stacked bases property by an equivalent one for Prüfer domains, namely the UCS-property. The following study is based on a paper by Brewer and Klinger [3]. We first recall results concerning the UCS-property in Sect. 2, then results on rings of integer-valued polynomials in Sect. 3. Our own contribution is in Sect. 4.

2 The UCS-property

We first recall the notion of content:

Definition 2.1 Let R be a ring, m and n be positive integers.

(1) The *content* $c_R(A)$, or briefly $c(A)$, of a matrix $A \in \mathcal{M}_{m\times n}(R)$ is the ideal of R generated by the coefficients of A.
(2) The *content* $c_R(N)$, or briefly $c(N)$, of a finitely generated sub-R-module N of a free R-module M of finite rank is the content of the matrix formed by the components of any system of generators of N with respect to any basis of M.
(3) The submodule N of M is said to have *unit content* when $c(N) = R$.

Assume that the domain D has the stacked bases property and consider a finitely generated submodule N of D^m. Then, with notation of Definition 1.3, $c(N) = \mathfrak{I}_1$. If N has unit content, then $\mathfrak{I}_1 = D$, and hence, the rank-one projective module P_1 is a submodule of N which is a summand of D^m. Let us introduce this property as a definition:

Definition 2.2 [6, § V.4] The ring R is said to have the *unit content summand property*, or briefly, the *UCS-property* if, for every m, every finitely generated submodule N of R^m with unit content contains a rank-one projective submodule which is a summand of R^m.

Instead of UCS-property, some authors speak of *BCS-property* (see for instance [3]).We have just said that the stacked bases property implies the UCS-property. The converse holds for Prüfer domains:

Proposition 2.3 [1] (see also [6, Thm 4.8]) *A Prüfer domain has the stacked bases property if and only if it has the UCS-property.*

Thus, we are led to study the UCS-property. The following proposition gives an equivalent way to consider the UCS-property.

Proposition 2.4 [7] *A ring R has the UCS-property if and only if, for every matrix $B \in \mathcal{M}_{m\times n}(R)$ with unit content, there exists a matrix $C \in \mathcal{M}_{n\times r}(R)$ such that the matrix BC has unit content and all 2×2 minors of BC are zero.*

Recall also:

Definition 2.5 [6, § V.4]

(1) A ring R is said to be *local-global* when whatever $n \geq 1$ and whatever $f \in R[X_1, \ldots, X_n]$ if, for every maximal ideal $\mathfrak{m}$ of R, there exist $x_1, \ldots, x_n \in R$ such that $f(x_1, \ldots, x_n) \notin \mathfrak{m}$, then there exist $y_1, \ldots, y_n \in R$ such that $f(y_1, \ldots, y_n)$ is a unit in R.
(2) A ring R is said to be *almost local-global* if all its proper factor rings are local-global.

Proposition 2.6 [2] (see also [6, Thm 4.7]) *Every almost local-global ring has the UCS property*

Since every one-dimensional integral domain and every domain with finite character (that is, such that every non-zero element is contained in at most finitely many maximal ideals) are almost local-global (see [6, Example 4.3]), we have:

Corollary 2.7 *A Prüfer domain which is either of dimension one or of finite character has the stacked bases property.*

It is then interesting to consider Prüfer domains with Krull dimension ≥ 2 which are not of finite character. Rings of integer-valued polynomials provide such examples.

3 Rings of integer-valued polynomials

Recall that, for every domain D with quotient field K and for every subset E of D, the *ring of integer-valued polynomials* on E with respect to D is the ring:

$$\text{Int}(E, D) = \{f \in K[X] \mid f(x) \in D \; \forall x \in E\}.$$

This ring is well known when D is assumed to be a valuation domain:

Proposition 3.1 [5] *If V is a valuation domain with finite residue field and finite dimension n and if E is a precompact subset of V, then* $\text{Int}(E, V)$ *is a Prüfer domain with dimension $n + 1$.*

Moreover, the prime ideals of $\text{Int}(E, V)$ lying over the maximal ideal $\mathfrak{m}$ of V are the following distinct maximal ideals:

$$\mathfrak{m}_x = \{f \in \text{Int}(E, D) \mid f(x) \in \widehat{\mathfrak{m}}\} \quad (x \in \widehat{E})$$

where $\widehat{\mathfrak{m}}$ and $\widehat{E}$ denote, respectively, the completion of $\mathfrak{m}$ and E. As a consequence, as soon as E is infinite, $\text{Int}(E, V)$ is not of finite character.

Proposition 3.2 [3, Thm 5] *If V is a valuation domain with finite residue field and finite dimension n and if E is a precompact subset of V, then the $n + 1$-dimensional Prüfer domain* $\text{Int}(E, V)$ *is almost local-global, and hence, has the stacked bases property.*

So that, the ring $\text{Int}(E, V)$ is a natural example of Prüfer domain of any dimension, without the finite character property and which has the stacked bases property. But, do all these rings (when they are Prüfer) have the stacked bases property. By globalisation we easily have:

Proposition 3.3 [4, §§ V.2 and VI.1] *Let D be a Dedekind domain with finite residue fields and let E be any subset of D. Then, the ring* $\mathrm{Int}(E, D)$ *is a two-dimensional Prüfer domain. Moreover, the prime ideals of* $\mathrm{Int}(E, D)$ *which contain non-zero elements of D are the following distinct maximal ideals:*

$$\mathfrak{m}_x = \{f \in \mathrm{Int}(E, D) \mid f(x) \in \widehat{\mathfrak{m}}\}$$

where $\mathfrak{m}$ is any maximal ideal of D and x is any element of $\widehat{E}$ ($\widehat{\mathfrak{m}}$ and $\widehat{E}$ denote the completions of $\mathfrak{m}$ and E with respect to the $\mathfrak{m}$-adic topology).

But, already in the easiest and most natural case where $E = D = \mathbb{Z}$, we cannot conclude in the same way because the ring

$$\mathrm{Int}(\mathbb{Z}) = \mathrm{Int}(\mathbb{Z}, \mathbb{Z}) = \{f \in \mathbb{Q}[X] \mid f(\mathbb{Z}) \subseteq \mathbb{Z}\}$$

is a two-dimensional Prüfer domain which is not almost local-global: the factor ring $\mathrm{Int}(\mathbb{Z})/(X^2 + 14)$ is not local-global [3, Example 7].

4 Does Int($\mathbb{Z}$) have the stacked bases property?

We are then interested in the question to know whether $\mathrm{Int}(\mathbb{Z})$ has the stacked bases property. This question raised by Brewer and Klinger [3] at the end of their paper is interesting whatever the answer: either $\mathrm{Int}(\mathbb{Z})$ has the stacked bases property, and this is interesting because $\mathrm{Int}(\mathbb{Z})$ is an interesting natural ring; or $\mathrm{Int}(\mathbb{Z})$ does not have the property and this is still more interesting because it would provide an example of a two-dimensional Prüfer domain which has not the stacked bases property.

Since the almost local-global property is too strong for our example, we consider another property which implies also the UCS-property:

Definition 4.1 A ring R has the *BCU-property* if, for every $m \geq 1$, every finitely generated submodule of R^m with unit content contains a vector x such that $c(Rx) = R$.

Equivalently, a ring R has the BCU-property if, for every matrix $B \in \mathcal{M}_{m\times n}(R)$ with unit content, there exists a column-matrix X such that the column-matrix BX has unit content. The BCU-property implies the UCS-property since, for every vector x of R^m such that $c(Rx) = R$, Rx is a rank-one projective module which is a summand of R^m. Thus, we have the following implications:

$$\begin{array}{lcl} R \text{ is local-global} & \Rightarrow & R \text{ has the BCU property} \\ \Downarrow & & \Downarrow \\ R \text{ is almost local-global} & \Rightarrow & R \text{ has the UCS property} \end{array}$$

Example 4.2 [3]

(1) A principal ideal domain has the BCU-property.
(2) The ring $\mathrm{Int}(\mathbb{Z})/(X^2 + 14)$ does not have the BCU-property.

With respect to this BCU property, we now prove the following technical proposition:

Proposition 4.3 *Let R be a ring and S be a multiplicative subset of R which does not contain zero divisors. We assume that:*

(1) *the ring $S^{-1}R$ has the BCU-property,*
(2) *for every non-zero finitely generated ideal $\mathfrak{I}$ of R such that $\mathfrak{I} \cap S \neq \emptyset$, the ring $R/\mathfrak{I}$ has the BCU-property.*

Then, for every $m \in \mathbb{N}$:

(1) *every submodule of R^m which is finitely generated with unit content contains a submodule with unit content which may be generated by two elements,*
(2) *every submodule of R^m which is a rank-one projective module and a summand of R^m may be generated by two elements.*

Proof Let $m \geq 1$.

(1) Let N be a finitely generated sub-R-module of R^m with unit content. Then, $S^{-1}N$ is a finitely generated sub-$S^{-1}R$-module of $(S^{-1}R)^m$ with unit content. Since $S^{-1}R$ has the BCU-property, there exists a vector $z \in S^{-1}N$ such that $c_{S^{-1}R}(S^{-1}Rz) = S^{-1}R$. Let $z = s^{-1}x$ where $x \in N$ and $s \in S$, and let $\mathfrak{I} = c_R(Rx)$. Then, $S^{-1}\mathfrak{I} = S^{-1}R$, in other words, $\mathfrak{I} \cap S \neq \emptyset$. If $\mathfrak{I} = R$, $c_R(Rx) = R$ and we are done. Else, $\overline{R} = R/\mathfrak{I}$ has the BCU-property by hypothesis. Consequently, since the $\overline{R}$-module $\overline{N} = N/\mathfrak{I}N$ is finitely generated with unit content, there exists an element $\overline{y} \in \overline{N}$ such that $c_{\overline{R}}(\overline{R}\overline{y}) = \overline{R}$. If $y \in N$ is a representative of $\overline{y}$, then one has $c_R(Ry) + \mathfrak{I} = R$. Thus, the R-module $< x, y >$ generated by x and y is a sub-R-module of N such that $c_R(Rx) + c_R(Ry) = R$.
Moreover, note that $c_R(Rx) \cap S \neq \emptyset$.

(2) Let P be a rank-one projective R-module which is a summand of R^m. Then, P may be generated by m elements. Moreover, P has unit content. Indeed, for each $\mathfrak{m} \in \mathrm{Max}(R)$, $P_\mathfrak{m}$ is a free $R_\mathfrak{m}$-module which is a summand of $R_\mathfrak{m}^m$, and hence, $c_R(P)R_\mathfrak{m} = R_\mathfrak{m}$. The first part of the proof shows that P contains a submodule Q with unit content which may be generated by two elements. Then, for every maximal ideal $\mathfrak{m}$ of R, $P_\mathfrak{m}$ is a free $R_\mathfrak{m}$-module with rank one which contains an $R_\mathfrak{m}$-module $Q_\mathfrak{m}$ with unit content. Necessarily, by definition of the content, $P_\mathfrak{m} = Q_\mathfrak{m}$ for every $\mathfrak{m}$, and hence, $P = Q$. □

Corollary 4.4 *Let R be a ring and S be a multiplicative subset of R without zero divisors which satisfy the hypotheses of Proposition 4.3. Then, R has the UCS-property if and only if, for every matrix $B \in \mathcal{M}_{m\times 2}(R)$ with unit content, there exists a matrix $C \in \mathcal{M}_{2\times 2}(R)$ such that BC has unit content and all 2×2 minors of BC are zero.*

Proof Analogously to Proposition 2.4, let us translate the UCS property in terms of matrices. Assume that R has the UCS property and let $B \in \mathfrak{M}_{m\times 2}(R)$ be a matrix with unit content. The submodule N of R^m generated by the columns of B has unit content. By Definition 2.2, N contains a rank-one projective submodule P which is a summand of R^m. By Proposition 4.3, P is generated by two elements. Let $D \in \mathcal{M}_{m\times 2}(R)$ be the matrix formed by the components of these two generators. Then, D has unit content and all 2×2 minors of D are zero: we may verify these properties locally and we know that, for every maximal ideal $\mathfrak{m}$ of R, $P_\mathfrak{m}$ is a rank-one free module which is a summand of $R_\mathfrak{m}^m$. Finally, if $C \in \mathcal{M}_{2\times 2}(R)$ denotes any matrix formed by the (non-unique) coefficients of the two generators of P written as linear combinations of the two generators of N, then $D = BC$.

Conversely, assume that R satisfies the property concerning the matrices and let N be any submodule of R^m with unit content. It follows from Proposition 4.3 that, to prove the UCS property for the module N, we may assume that N is generated by two elements. Then, the matrix $B \in \mathcal{M}_{m\times 2}(R)$ formed by the components of two generators of N has unit content. By hypothesis, there exists $C \in \mathcal{M}_{2\times 2}(R)$ such that $D = BC \in \mathcal{M}_{m\times 2}(R)$ has unit content and the 2×2 minors of D are zero. Let P be the submodule of R^m generated by the two columns of D. The fact that $D = BC$ shows that P is a submodule of N. Let us prove that the properties of the matrix D imply that P is a rank-one projective summand of R^m. Fix any maximal ideal $\mathfrak{m}$ of R. Since D has unit content, there a column of D with a coefficient which is invertible in $R_\mathfrak{m}$. The corresponding vector z of P may be chosen as an element of a basis of the $R_\mathfrak{m}$-module $R_\mathfrak{m}^m$. The condition on the 2×2 minors of D implies that $P_\mathfrak{m} = zR_\mathfrak{m}$, and hence, $(R^m/P)_\mathfrak{m} = R_\mathfrak{m}^m/P_\mathfrak{m}$ is a free $R_\mathfrak{m}$-module with rank $m-1$. Consequently, R^m/P is a projective R-module with rank $m-1$, R^m/P is a summand of R^m and P is a rank-one projective summand of R^m. □

Application to the ring $R = \mathrm{Int}(\mathbb{Z})$.

We may apply the previous proposition and its corollary to the ring $R = \mathrm{Int}(\mathbb{Z})$ by considering the multiplicative subset $S = \mathbb{Z} \setminus \{0\}$. Indeed, on the one hand, $S^{-1}\mathrm{Int}(\mathbb{Z}) = \mathbb{Q}[X]$ is a principal ideal domain, thus it has the BCU-property. On the other hand, following Proposition 3.3, for every ideal $\mathfrak{I}$ of $\mathrm{Int}(\mathbb{Z})$ which contains non-zero elements of $\mathbb{Z}$, the ring $\mathrm{Int}(\mathbb{Z})/\mathfrak{I}$ is zero dimensional, thus, is local-global and, in particular, has the BCU-property. So that:

Theorem 4.5 *The ring $R = \mathrm{Int}(\mathbb{Z}) = \{f \in \mathbb{Q}[X] \mid f(\mathbb{Z}) \subseteq \mathbb{Z}\}$, which is a two-dimensional Prüfer domain, has the stacked bases property if and only if, for every matrix $B \in \mathcal{M}_{m\times 2}(R)$ with unit content, there exists a matrix $C \in \mathcal{M}_{2\times 2}(R)$ such that BC has unit content and all 2×2 minors of BC are zero.*

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