A functorial presentation of units of Burnside rings

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Abstract: Let $B^\times$ be the biset functor over $\mathbb{F}_2$ sending a finite group G to the group $B^\times(G)$ of units of its Burnside ring $B(G)$, and let $\widehat{B^\times}$ be its dual functor. The main theorem of this paper gives a characterization of the cokernel of the natural injection from $B^\times$ in the dual Burnside functor $\widehat{\mathbb{F}_2B}$, or equivalently, an explicit set of generators $\mathcal{G}_\mathcal{S}$ of the kernel L of the natural surjection $\mathbb{F}_2B \to \widehat{B^\times}$. This yields a two terms projective resolution of $\widehat{B^\times}$, leading to some information on the extension functors $\mathrm{Ext}^1(-, B^\times)$. For a finite group G, this also allows for a description of $B^\times(G)$ as a limit of groups $B^\times(T/S)$ over sections (T, S) of G such that T/S is cyclic of odd prime order, Klein four, dihedral of order 8, or a Roquette 2-group. Another consequence is that the biset functor $B^\times$ is not finitely generated, and that its dual $\widehat{B^\times}$ is finitely generated, but not finitely presented. The last result of the paper shows in addition that $\mathcal{G}_\mathcal{S}$ is a minimal set of generators of L, and it follows that the lattice of subfunctors of L is uncountable.

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1. Introduction

The Burnside ring $B(G)$ is a fundamental invariant attached to a finite group G. It is a commutative (unital) ring, and most of its structural properties have been described several decades ago, e.g. the prime spectrum by Dress ([10]), or the primitive idempotents of the algebra $\mathbb{Q}B(G)$ by Gluck ([12]) and Yoshida ([19]).

An important missing item in this list is the group of multiplicative units $B(G)^\times$. More precisely, it follows from Burnside's theorem that $B(G)^\times$ is a finite elementary abelian 2-group, but the rank of this group is known only under additional assumptions on G, by the work of many different people ([13], [14], [20], [18], [6], [1]). A very efficient algorithm for computing this rank has also been obtained in [2]. However, no general formula for the rank of $B(G)^\times$ is known so far. Let us recall that, according to an observation of tom Dieck ([17], Proposition 1.5.1) based on a theorem of Dress ([10]), Feit-Thompson's theorem ([11]) is equivalent to the assertion that $B^\times(G)$ has order 2 if $|G|$ is odd.

It was observed in [6] that the assignment $G \mapsto B(G)^\times$ is a biset functor $B^\times$ with values in $\mathbb{F}_2$-vector spaces, and that $B^\times$ embeds in the $\mathbb{F}_2$-

dual functor $\mathrm{Hom}_{\mathbb{Z}}(B, \mathbb{F}_2)$ of the Burnside functor. In the present paper, we give (Theorem 3.4) a characterization of the image of this embedding $\imath : B^\times \to \mathrm{Hom}_{\mathbb{Z}}(B, \mathbb{F}_2)$, or equivalently, we describe generators for the kernel L of the natural transposed surjection $\jmath : \mathbb{F}_2 B \to \mathrm{Hom}_{\mathbb{Z}}(B^\times, \mathbb{F}_2)$. This yields a functorial presentation

$$0 \longrightarrow L \longrightarrow \mathbb{F}_2 B \xrightarrow{\jmath} \mathrm{Hom}_{\mathbb{Z}}(B^\times, \mathbb{F}_2) \longrightarrow 0$$

of the (dual) functor $\mathrm{Hom}_{\mathbb{Z}}(B^\times, \mathbb{F}_2)$. Unfortunately, the generators we obtain for L are not linearly independent, so at least at this stage, the previous exact sequence doesn't provide any obvious formula for the $\mathbb{F}_2$-dimension of the evaluations of the functor $B^\times$.

It still admits some interesting consequences, that we develop in the three last sections. Section 4 is concerned with the extension groups $\mathrm{Ext}^1(M, B^\times)$ for some biset functors M over $\mathbb{F}_2$. In Section 5, we give a sectional characterization of $B^\times(G)$ as a limit of groups $B^\times(T/S)$, where (T, S) runs through sections (T, S) of G such that T/S is cyclic of odd prime order, Klein four, dihedral of order 8, or a Roquette 2-group. Finally, in Section 6, it is shown that the biset functor $B^\times$ is not finitely generated, and that its dual is finitely generated (by a single element), but not finitely presented. We also show that the generating set we obtain for the above functor L is minimal, and that the lattice of subfunctors of L is uncountable. These last results may be seen as another sign of the difficulty of determining $B^\times(G)$ for an arbitrary finite group G, at least with our present methods.

2. Review of Burnside rings and biset functors

2.1. Burnside rings. We quickly recall first some basic definitions on Burnside rings. Missing details can be found in [3].

Let G be a finite group. The Burnside ring $B(G)$ of G is the Grothendieck ring of the category of finite (left) G-sets, for relations given by disjoint union decompositions. In other words $B(G)$ is the quotient of the free abelian group on the set of isomorphism classes of finite G-sets by the subgroup generated by all the elements of the form $[X \sqcup Y] - [X] - [Y]$, where $[X]$ denote the isomorphism class of the G-set X.

The multiplication in $B(G)$ is induced by the cartesian product of G-sets. In other words $[X][Y] = [X \times Y]$ for any two finite G-set X and Y. The identity element is the (isomorphism class of a) G-set of cardinality 1.

The additive group $B(G)$ is free with basis the set of isomorphism classes of transitive G-sets. Each transitive G-set is isomorphic to a G-set of the form

G/H, where H is a subgroup of G. The transitive G sets G/H and G/K are isomorphic if and only if the subgroups H and K of G are conjugate. For sake of simplicity, we denote by G/H instead of $[G/H]$ the image in $B(G)$ of the transitive G-set G/H. With this abuse of notation, the abelian group $B(G)$ has a basis consisting of the elements G/H, where H runs through a set of representatives of conjugacy classes of subgroups of G.

The commutative $\mathbb{Q}$-algebra $\mathbb{Q}B(G) = \mathbb{Q}\otimes_{\mathbb{Z}} B(G)$ is split semisimple. Its primitive idempotents (see [12] or [19]) are indexed by the subgroup of G, up to conjugation. The idempotent e_H^G indexed by $H \leq G$ is equal to

$$e_H^G = \frac{1}{|N_G(H)|}\sum_{L\leq H} |L|\mu(L,H)\, G/L \quad ,$$

where μ is the Möbius function of the poset of subgroups of G. Its defining property is that

$$\forall K \leq G,\ |(e_H^G)^K| = \begin{cases} 1 & \text{if } K =_G H,\\ 0 & \text{otherwise.}\end{cases}$$

Here the map $\alpha \in \mathbb{Q}B(G) \mapsto |\alpha^K| \in \mathbb{Q}$ is the linear form sending (the class of a finite) G-set X to the cardinality $|X^K|$ of the set $X^K|$ of K-fixed points in X.

2.2. Biset functors. (see [7] for details) For finite groups G and H, an (H,G)-biset is a $H\times G^{op}$-set, i.e. a set U endowed with a left H-action and a right G-action which commute. The Burnside group $B(H\times G^{op})$ of (finite) (H,G)-bisets is denoted by $B(H,G)$.

Let $\mathcal{C}$ denote the following category:

- The objects of $\mathcal{C}$ are all the finite groups.
- For finite groups G and H, the set of morphisms $\mathrm{Hom}_{\mathcal{C}}(G,H)$ is equal to $B(H,G)$.
- The composition of morphisms in $\mathcal{C}$ is defined by bilinearity from the standard "tensor product" (also called *composition*) of bisets: for finite groups G, H and K, for a (K,H)-biset V and an (H,G)-biset U, set

  $$V\times_H U = (V\times U)/H \quad ,$$

  where H acts on the right on $(V\times U)$ by $(v,u)h = (vh, h^{-1}u)$. Then $V\times_H U$ is a (K,G)-biset in the obvious way.
- The identity element of G is (the class of) the set G, viewed as a (G,G)-biset by left and right multiplication.

A *biset functor* is an additive functor from $\mathcal{C}$ to the category of all $\mathbb{Z}$-modules. Biset functors, together with natural transformations between them, form an abelian category $\mathcal{F}$. More generally, for a commutative (unital) ring k, one can consider the k-linearization $k\mathcal{C}$ of $\mathcal{C}$, i.e. the category with the same objects, and such that

$$\mathrm{Hom}_{k\mathcal{C}}(G,H) = k \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathcal{C}}(G,H) = kB(H,G) \ \ ,$$

the composition in $k\mathcal{C}$ being defined as the k-bilinear extension of the composition in $\mathcal{C}$. A *biset functor over* k is a k-linear functor from $k\mathcal{C}$ to the category k-Mod of all k-modules. These functors, together with natural transformations between them, form an abelian category $\mathcal{F}_k$.

There is a one to one parametrization $(H,V) \mapsto S_{H,V}$ of simple bisets functors over k (up to isomorphism of functors) by pairs (H,V) of a finite group H and a simple $k\mathrm{Out}(H)$-module V (up to isomorphism of such pairs). If G is a finite group, and if $S_{H,V}(G) \neq \{0\}$, then H is isomorphic to a subquotient of G.

Let F be a biset functor, and let G, H be finite groups. We define *the opposite biset* U^{op} as the (G,H)-biset U, where the action of $(g,h) \in G \times H$ on $u \in U$ is defined by

$$g \cdot u \cdot h = h^{-1} u g^{-1} \ \ .$$

This definition extends uniquely to a k-linear map $\alpha \mapsto \alpha^{op}$ from $kB(H,G)$ to $kB(G,H)$.

For $\alpha \in kB(H,G)$ and $m \in F(G)$, we denote by $\alpha(m)$ or simply $\alpha\, m$ the image of m by the map $F(\alpha)$. In particular, when U is a finite (H,G)-biset, we set $U\, m = F\big([U]\big)(m)$.

When F is a biset functor over k, and M is a k-module, the M-dual of F is the functor $\mathrm{Hom}_k(F,M)$ defined by

$$\left\{ \begin{array}{l} \forall G,\ \mathrm{Hom}_k(F,M)(G) = \mathrm{Hom}_k\big(F(G),M\big) \\ \forall G,H,\ \forall \alpha \in kB(H,G),\ \mathrm{Hom}_k(F,M)(\alpha) = {}^tF(\alpha^{op}) \ \ . \end{array} \right.$$

When p is a prime number, let $k\mathcal{C}_p$ denote the full subcategory of $k\mathcal{C}$ consisting of finite p-groups. A k-linear functor from $k\mathcal{C}_p$ to k-Mod is called a *p-biset functor* (over k). The abelian category of p-biset functors over k is denoted by $\mathcal{F}_{p,k}$.

More generally, if $\mathcal{T}$ is a class of finite groups closed under taking subquotients, one can consider the full subcategory $k\mathcal{T}$ of $k\mathcal{C}$ consisting of groups in $\mathcal{T}$. The abelian category of k-linear functors from $k\mathcal{T}$ to k-Mod is denoted by $\mathcal{F}_{\mathcal{T},k}$.

Let F be an object of $\mathcal{F}_{\mathcal{T},k}$. An *element* of F is a pair (R,v), where $R \in \mathcal{T}$ and $v \in F(R)$. For a set $\mathcal{G}$ of elements of F, the subfunctor $\langle \mathcal{G} \rangle$ of F

generated by $\mathcal{G}$ is defined as the intersection of all subfunctors L of F such that $v \in L(R)$ for all $(R, v) \in \mathcal{G}$. Its evaluation at a group $H \in \mathcal{T}$ is given by

$$\langle \mathcal{G} \rangle(H) = \sum_{\substack{(R,v)\in\mathcal{G} \\ \alpha\in kB(H,R)}} F(\alpha)(v) \ .$$

The functor F is *finitely generated* if there exists a finite set $\mathcal{G}$ of elements of F such that $\langle \mathcal{G} \rangle = F$. The functor F is *finitely presented* if there exists an exact sequence $Q \to P \to F \to 0$ in $\mathcal{F}_{\mathcal{T},k}$, where P and Q are finitely generated projective functors.

Let F be an object of $\mathcal{F}_{T,k}$, and $H \in \mathcal{T}$. The *residue* (or *Brauer quotient*) $\overline{F}(H)$ of F at H is the k-module defined by

$$\overline{F}(H) = F(H) \Big/ \sum_{\substack{T\in\mathcal{T} \\ |T|<|H| \\ \alpha\in kB(H,T)}} F(\alpha)\big(F(T)\big) \ .$$

2.3. Elementary bisets. Bisets of one of the following forms are called *elementary bisets*:

- Let H be a subgroup of a finite group G. The set G, viewed as a (G, H)-biset (by multiplication), is called *induction* (from H to G), and denoted by Ind_H^G. The set G, viewed as an (H, G)-biset (by multiplication), is called *restriction* (from G to H), and denoted by Res_H^G.
- Let N be a normal subgroup of a finite group G. The set G/N, viewed as a $(G, G/N)$-biset (by multiplication on the right, and projection followed by multiplication on the left), is called *inflation* (from G/N to G) and denoted by $\mathrm{Inf}_{G/N}^G$. The set G/N, viewed as a $(G/N, G)$-biset (by multiplication on the left, and projection followed by multiplication on the right), is called *deflation* (from G to G/N), and denoted by $\mathrm{Def}_{G/N}^G$.
- Let $\varphi : G \to G'$ be an isomorphism of groups. The set G', viewed as a (G', G)-biset (by multiplication on the left, and multiplication by the image under φ on the right), is called *transport by isomorphism* (by φ), and denoted by $\mathrm{Iso}(\varphi)$ or $\mathrm{Iso}_G^{G'}$ if φ is clear from the context.

In addition to these elementary bisets, it is convenient to distinguish the following ones, when (T, S) is a section of a finite group G, i.e. a pair of subgroups of G with $S \trianglelefteq T$:

- The set G/S, viewed as a $(G, T/S)$-biset, is called *induction-inflation* (from T/S to G), and denoted by $\mathrm{Indinf}_{T/S}^G$. It is isomorphic to the composition $\mathrm{Ind}_T^G \times_T \mathrm{Inf}_{T/S}^T$.

- The set $S\backslash G$, viewed as a $(T/S, G)$-biset, is called *deflation-restriction* (from G to T/S), and denoted by $\mathrm{Defres}^G_{T/S}$. It is isomorphic to the composition $\mathrm{Def}^T_{T/S} \times_T \mathrm{Res}^G_T$.

For finite groups G and H, any transitive (H, G)-biset is isomorphic to a biset of the form $(H \times G)/X$, where X is a subgroup of $H \times G$, and the biset structure is given by $h \cdot (a, b)X \cdot g = (ha, g^{-1}b)X$ for h, a in H and g, b in G. We set

$$\begin{aligned} p_1(X) &= \{h \in H \mid \exists g \in G,\ (h, g) \in X\} \\ k_1(X) &= \{h \in H \mid (h, 1) \in X\} \\ p_2(X) &= \{g \in G \mid \exists h \in H,\ (h, g) \in X\} \\ k_2(X) &= \{g \in G \mid (1, g) \in X\} \ . \end{aligned}$$

With this notation, we have $k_1(X) \trianglelefteq p_1(X)$ and $k_2(X) \trianglelefteq p_2(X)$, and there is a canonical group isomorphism $f : p_2(X)/k_2(X) \to p_1(X)/k_1(X)$ sending $gk_2(X)$ to $hk_1(X)$ if $(h, g) \in X$. Moreover (see [7], Lemma 2.3.26), there is an isomorphism of (H, G)-bisets

$$(\mathbf{2}.4) \qquad (H \times G)/X \cong \mathrm{Indinf}^H_{p_1(X)/k_1(X)} \,\mathrm{Iso}(f)\, \mathrm{Defres}^G_{p_2(X)/k_2(X)}$$

where the concatenation on the right hand side denotes the composition of bisets. In other words, any transitive biset is isomorphic to a composition of elementary bisets.

It follows that if $\mathcal{T}$ is a class of finite groups closed under taking subquotients, if $F \in \mathcal{F}_{\mathcal{T},k}$ and $H \in \mathcal{T}$, then

$$\begin{aligned} \overline{F}(H) &= F(H)/ \sum_{\substack{A \trianglelefteq B \leq H \\ (B,A) \neq (H,\mathbf{1})}} \mathrm{Indinf}^H_{B/A} F(B/A) \\ &= F(H)/\Big(\sum_{\substack{B < H \\ B\,\mathrm{maximal}}} \mathrm{Ind}^H_B F(B) + \sum_{\substack{\mathbf{1} < A \trianglelefteq H \\ A\,\mathrm{minimal}}} \mathrm{Inf}^H_{H/A} F(H/A) \Big) \ . \end{aligned}$$

2.5. Faithful elements. (see [7], Section 6.3) Let F be a biset functor over a commutative ring k. For a finite group G, let

$$\partial F(G) = \{u \in F(G) \mid \forall \mathbf{1} \neq N \trianglelefteq G,\ \mathrm{Def}^G_{G/N} u = 0\} \ .$$

The k-submodule $\partial F(G)$ is called the submodule of *faithful elements* of $F(G)$. If is always a direct summand of $F(G)$. More precisely, the element

$$f^G_{\mathbf{1}} = \sum_{N \trianglelefteq G} \mu_{\trianglelefteq G}(\mathbf{1}, N)\, [(G \times G)/L]$$

of $kB(G,G)$ is an idempotent endomorphism of G in the category $k\mathcal{C}$, and one can show that $\partial F(G) = F(f_{\mathbf{1}}^G)\big(F(G)\big)$.

2.6. Genetic bases of p-groups. ([7] Definition 6.4.3, Lemma 9.5.2 and Theorem 9.6.1) Let p be a prime number, and P be a finite p-group. For a subgroup Q of P, let $Z_P(Q) \geq Q$ denote the subgroup of $N_P(Q)$ defined by

$$Z_P(Q)/Q = Z\big(N_P(Q)/Q\big) \ \ .$$

The subgroup Q is a *genetic* subgroup of P if the two following properties hold:

- The group $N_P(Q)/Q$ is a Roquette p-group, i.e. it has *normal p-rank 1*. Recall that the Roquette p-groups of order p^n are the cyclic p-groups C_{p^n} $(n \geq 0)$, and in addition when $p = 2$, the generalized quaternion groups Q_{2^n} $(n \geq 3)$, the dihedral groups D_{2^n} $(n \geq 4)$ and the semidihedral groups SD_{2^n} $(n \geq 4)$.
- For any $x \in G$, the intersection ${}^xQ \cap Z_P(Q)$ is contained in Q if and only if ${}^xQ = Q$.

For two genetic subgroups Q and R of P, write

$$Q \,\widehat{=}_P\, R \Leftrightarrow \exists x \in P \text{ such that } {}^xQ \cap Z_P(R) \leq R \text{ and } R^x \cap Z_P(Q) \leq Q \ \ .$$

One can show that this defines an equivalence relation on the set of genetic subgroups of P. *A genetic basis* of P is a set of representatives of equivalence classes of genetic subgroups of P for the relation $\widehat{=}_P$.

2.7. Rational p-biset functors. ([7] Theorem 10.1.1, Definition 10.1.3 and Theorem 10.1.5) Let p be a prime number, and F be a p-biset functor over a commutative ring k. If P is a finite p-group and $\mathcal{B}$ is a genetic basis of P, the map

$$\mathcal{I}_\mathcal{B} : \bigoplus_{Q \in \mathcal{B}} \mathrm{Indinf}_{N_P(Q)/Q}^P : \bigoplus_{Q \in \mathcal{B}} \partial F\big(N_P(Q)/Q\big) \to F(P)$$

is split injective, with left inverse

$$\mathcal{D}_\mathcal{B} : \bigoplus_{Q \in \mathcal{B}} f_{\mathbf{1}}^{N_P(Q)/Q} \circ \mathrm{Defres}_{N_P(Q)/Q}^P : F(P) \to \bigoplus_{Q \in \mathcal{B}} \partial F\big(N_P(Q)/Q\big) \ \ .$$

The functor F is called *rational* if for any finite p-group P, the map $\mathcal{I}_\mathcal{B}$ is an isomorphism for *some* - equivalently for *any* - genetic basis $\mathcal{B}$ of P. Rational p-biset functors form a Serre subcategory of $\mathcal{F}_{p,k}$. Moreover, for any k-module M, the M-dual $\mathrm{Hom}_k(F, M)$ of a rational p-biset functor F is rational.

3. The main theorem

3.1. Notation: *Let $\mathcal{R}$ denote the class of finite groups which are isomorphic to one of the following groups:*

- *A cyclic group C_p of odd prime order p.*
- *A cyclic group C_4 of order 4.*
- *An elementary abelian group $(C_2)^2$ of order 4.*
- *A dihedral 2-group D_{2^n} of order $2^n \geq 8$.*
- *A semidihedral 2-group SD_{2^n} of order $2^n \geq 16$.*

For each group R in $\mathcal{R}$, let ε_R denote the element of $B(R)$ defined by:

- *$\varepsilon_R = R/\mathbf{1} - R/R$ if $R \cong C_p$.*
- *$\varepsilon_R = R/\mathbf{1} - R/S$ if $R \cong C_4$, where S is the unique subgroup of order 2 of R.*
- *$\varepsilon_R = R/\mathbf{1} - (R/A + R/B + R/C) + 2R/R$ if $R \cong (C_2)^2$, where A, B, C are the the three subgroups of order 2 of R.*
- *$\varepsilon_R = (R/I - R/IZ) - (R/J - R/JZ)$, if $R \cong D_{2^n}$, where I and J are non conjugate non central subgroups of order 2 of R, and Z is the center of R.*
- *$\varepsilon_R = R/I - R/IZ$ if $R \cong SD_{2^n}$, where I is a non central subgroup of order 2 of R, and Z is the center of R.*

Let moreover $\bar{\varepsilon}_R$ denote the image of ε_R in $\mathbb{F}_2B(G)$.

3.2. Remark: When R is cyclic, elementary abelian of order 4, or dihedral, the elements $\varepsilon_R \in B(R)$ already appear in [9] (Notation 3.6) and [5] (Corollary 6.5 and Notation 6.9, where $\varepsilon_{D_{2^n}}$ is denoted δ_n). Also observe that $\varepsilon = f_{\mathbf{1}}^R R/\mathbf{1} \in B(R)$ if R is neither dihedral nor semidihedral. Moreover $\varepsilon_R = f_{\mathbf{1}}^R (R/I - R/J)$ if $R \cong D_{2^n}$, and $\varepsilon_R = f_{\mathbf{1}}^R R/I$ when $R \cong SD_{2^n}$. Hence $f_{\mathbf{1}}^R \varepsilon_R = \varepsilon_R$ for any $R \in \mathcal{R}$.

3.3. Notation: *Let $\mathbb{F}_{2,+}$ denote the additive group of $\mathbb{F}_2$, and*

$$s \in \{\pm 1\} \mapsto s_+ \in \mathbb{F}_{2,+}$$

be the group isomorphism is defined by $(-1)_+ = 1_{\mathbb{F}_2}$ and $(+1)_+ = 0_{\mathbb{F}_2}$. Let $t \in \mathbb{F}_{2,+} \mapsto t_\times \in \{\pm 1\}$ denote the inverse group isomorphism.

Recall that for a finite group G, any unit element $u \in B^\times(G)$ in the Burnside ring of G defines a linear form $\imath_G(u)$ on $B(G)$, with values in $\mathbb{F}_2$, by

$$\imath_G(u)(G/H) = |u^H|_+ \ \ .$$

When G runs through finite groups, these maps $\imath_G$ form an injective morphism of biset functors $\imath : B^\times \to \mathrm{Hom}_\mathbb{Z}(B, \mathbb{F}_2)$ ([6], Proposition 7.2).

Moreover, by a theorem of Yoshida ([20], Proposition 6.5), a linear form $\varphi : B(G) \to \mathbb{F}_2$ lies in the image of $\imath_G$ if and only if for any subgroup H of G, the map

$$\widetilde{\varphi}_H : x \in N_G(H)/H \mapsto \varphi\big(G/H\langle x\rangle\big) - \varphi(G/H)$$

is a group homomorphism from $N_G(H)/H$ to $\mathbb{F}_{2,+}$.

3.4. Theorem: *Let G be a finite group, and $\varphi \in \mathrm{Hom}_\mathbb{Z}\big(B(G), \mathbb{F}_2\big)$. The following assertions are equivalent:*

1. *φ lies in the image of the natural map $\imath_G : B^\times(G) \to \mathrm{Hom}_\mathbb{Z}\big(B(G), \mathbb{F}_2\big)$.*
2. *$(\mathrm{Defres}^G_{T/S}\varphi)(\varepsilon_{T/S}) = 0_{\mathbb{F}_2}$ for every section (T, S) of G such that $T/S \in \mathcal{R}$.*

Proof: $\boxed{1 \Rightarrow 2}$ We check that if $\varphi \in \mathrm{Im}\,\imath_G$, then $(\mathrm{Defres}^G_{T/S}\varphi)(\varepsilon_{T/S}) = 0_{\mathbb{F}_2}$ for each section (T, S) of G such that $T/S \in \mathcal{R}$. Since $\mathrm{Defres}^G_{T/S}\varphi \in \mathrm{Im}\,\imath_{T/S}$ as $\imath$ is a morphism of biset functors, we can assume that $G \in \mathcal{R}$, and check that $\varphi(\varepsilon_G) = 0_{\mathbb{F}_2}$ when $\varphi = \imath_G(u)$ for some $u \in B^\times(G)$. In this case $\varphi(G/H) = |u^H|_+$ for any subgroup H of G.

But since $f_\mathbf{1}^G\,\varepsilon_G = \varepsilon_G$ by Remark 3.2, and since $f_\mathbf{1}^G = (f_\mathbf{1}^G)^{op}$, we have

$$\begin{aligned}\varphi(\varepsilon_G) &= i_G(u)(f_\mathbf{1}^G\,\varepsilon_G)\\ &= \big(f_\mathbf{1}^G i_G(u)\big)(\varepsilon_G)\\ &= i_G(f_\mathbf{1}^G\,u)(\varepsilon_G) \ \ .\end{aligned}$$

Moreover $f_\mathbf{1}^G\,u \in \partial B^\times(G)$ by definition. But $\partial B^\times(G) = \{0\}$ if $|Z(G)| > 2$ by Lemma 6.8 of [6], and $\partial B^\times(G)$ also vanishes if G is generalized quaternion or semidihedral, by Corollary 6.10 there. It follows that $\partial B^\times(G) = \{0\}$ if $G \in \mathcal{R}$, unless perhaps if $G \cong D_{2^n}$ for some $n \geq 3$. In particular $\varphi(\varepsilon_G) = 0_{\mathbb{F}_2}$ if $G \in \mathcal{R}$, unless G is a dihedral 2-group of order at least 8.

Now by Corollary 6.12 of [6] (and its proof), if $G \cong D_{2^n}$ for $n \geq 3$, the group $\partial B^\times(G)$ has order 2, generated by the element

$$\upsilon_G = 1 - 2(e_I^G + e_J^G) = G/G + G/\mathbf{1} - (G/I + G/J)$$

of $B(G)$. Moreover $\varepsilon_G = f_{\mathbf{1}}^G\,(G/I - G/J)$ by Remark 3.2. Thus

$$\begin{aligned}\varphi(\varepsilon_G) &= i_G(u)\big(f_{\mathbf{1}}^G\,(G/I - G/J)\big)\\ &= \big(f_{\mathbf{1}}^G i_G(u)\big)(G/I - G/J)\\ &= i_G(f_{\mathbf{1}}^G\,u)(G/I - G/J)\\ &= |(f_{\mathbf{1}}^G\,u)^I|_+ - |(f_{\mathbf{1}}^G\,u)^J|_+ \quad .\end{aligned}$$

We can moreover assume that $f_{\mathbf{1}}^G\,u$ is the generator υ_G of $\partial B^\times(G)$, i.e. that $f_{\mathbf{1}}^G\,u = 1 - 2(e_I^G + e_J^G)$. Then clearly $|(f_{\mathbf{1}}^G\,u)^I| = |(f_{\mathbf{1}}^G\,u)^J|$, so $\varphi(\varepsilon_G) = 0_{\mathbb{F}_2}$ also in this case, as was to be shown.

$\boxed{2 \Rightarrow 1}$ We prove the converse by induction on the order of G: we consider $\varphi \in \mathrm{Hom}_{\mathbb{Z}}\big(B(G), \mathbb{F}_2\big)$ such that $(\mathrm{Defres}_{T/S}^G\varphi)(\varepsilon_{T/S}) = 0_{\mathbb{F}_2}$ for every section (T,S) of G such that $T/S \in \mathcal{R}$. Assuming the result holds for groups of order smaller than $|G|$, by transitivity of deflation-restrictions, we can assume that $\mathrm{Defres}_{T/S}^G\varphi$ lies in the image of $B^\times(T/S)$ for any section (T,S) such that $|T/S| < |G|$, in other words any section (T,S) different from $(G,1)$. By Yoshida' theorem above, it follows that the map $\widetilde{\varphi}_H$ is a group homomorphism for any $H \neq \mathbf{1}$. So proving that φ lies in the image of $\imath_G$ amounts to proving that the map $\widetilde{\varphi}_{\mathbf{1}} : G \to \mathbb{F}_2$ is a group homomorphism.

Equivalently, all we have to show is that the map

$$\psi : x \in G \mapsto \widetilde{\varphi}_{\mathbf{1}}(x)_\times \in \{\pm 1\}$$

is a group homomorphism. Now ψ is a central function on G, with values in $\{\pm 1\}$, and moreover $\psi(1) = 1$ since $\widetilde{\varphi}_1(1) = 0_{\mathbb{F}_2}$. Hence all we have to show is that ψ is a generalized character of G. Indeed in this case, there exist integers $n_\chi \in \mathbb{Z}$, indexed by the irreducible complex characters of G such that $\psi = \sum\limits_{\chi \in \mathrm{Irr}(G)} n_\chi \chi$. Moreover

$$\sum_{\chi \in \mathrm{Irr}(G)} n_\chi^2 = \frac{1}{|G|} \sum_{x \in G} |\psi(x)|^2 = 1 \quad ,$$

so $n_\chi = 0$, except for a single irreducible character χ for which $n_\chi = \pm 1$. Since $\psi(1) = 1$, it follows that $\psi = \chi$, so ψ is a character of G, of degree 1, hence a group homomorphism from G to $\{\pm 1\}$.

Now by Brauer's induction theorem, the map ψ is a generalized character of G if and only if its restriction to any Brauer elementary subgroup H of G

is a generalized character of H. But for $x \in H$

$$\begin{aligned}(\mathrm{Res}_H^G \psi)(x) = \widetilde{\varphi}_1(x)_\times &= \varphi(G/\langle x\rangle)_\times/\varphi(G/\mathbf{1})_\times\\ &= \varphi\Big(\mathrm{Ind}_H^G\big(H/\langle x\rangle - H/\mathbf{1}\big)\Big)_\times\\ &= \Big((\mathrm{Res}_H^G\varphi)\big(H/\langle x\rangle - H/\mathbf{1}\big)\Big)_\times \quad .\end{aligned}$$

It follows that if H is a proper subgroup of G, since $\mathrm{Res}_H^G\varphi \in \mathrm{Im}\,\imath_H$, the map $\mathrm{Res}_H^G\psi$ is a generalized character of H. In other words we can assume that G itself is Brauer elementary.

Then there exists a prime number p such that $G = C \times P$, where P is a p-group, and C is a p'-cyclic group. In particular G is nilpotent, so $G = Q \times R$, where Q is a 2-group and R is nilpotent of odd order. Then any subgroup T of G is equal to $A \times B$, for some subgroup A of Q and some subgroup B of R. If B is non trivial, there exists a normal subgroup C of B of (odd) prime index l. Set $S = A \times C$. Then (T, S) is a section of G, and $T/S \cong C_l$. Thus

$$(\mathrm{Defres}_{T/S}^G\varphi)(\varepsilon_{T/S}) = 0_{\mathbb{F}_2} = \varphi\big(\mathrm{Indinf}_{T/S}^G(T/S - T/T)\big) = \varphi(G/S) - \varphi(G/T) \quad .$$

It follows by induction that $\varphi\big(G/(A\times B)\big) = \varphi\big(G/(A\times\mathbf{1})\big)$ for any subgroup A of Q and any subgroup B of R. In particular, for $x \in Q$ and $y \in R$, the subgroup $\langle (x,y)\rangle$ of $G = Q \times R$ is equal to $\langle x\rangle \times \langle y\rangle$, and

$$\begin{aligned}\psi(x,y) &= \varphi\big(G/(\langle x\rangle \times \langle y\rangle)\big)_\times/\varphi(G/\mathbf{1})_\times\\ &= \varphi\big(G/(\langle x\rangle \times \mathbf{1})\big)_\times/\varphi(G/\mathbf{1})_\times\\ &= \psi(x,1) \quad .\end{aligned}$$

In other words it is enough to prove that the restriction of ψ to $Q \times \mathbf{1}$ is a generalized character. Equivalently, all we have to do is consider the case where $G = Q \times \mathbf{1}$, i.e. we can assume that G is a 2-group.

We consider the linearization morphism $B(G) \to R_{\mathbb{Q}}(G)$, which is surjective by the Ritter-Segal theorem ([15], [16], [4]), and hence fits in a short exact sequence

$$0 \longrightarrow K(G) \longrightarrow B(G) \longrightarrow R_{\mathbb{Q}}(G) \longrightarrow 0 \quad .$$

By Corollary 6.16 of [5], the kernel $K(G)$ of the linearization morphism is linearly generated by the elements of the form $\mathrm{Indinf}_{T/S}^G\varepsilon_{T/S}$, where (T, S) is a section of G such that T/S is elementary abelian of order 4 or dihedral of order at least 8. Since

$$\varphi(\mathrm{Indinf}_{T/S}^G\varepsilon_{T/S}) = (\mathrm{Defres}_{T/S}^G\varphi)(\varepsilon_{T/S}) = 0_{\mathbb{F}_2} \quad ,$$

the linear form φ vanishes on $K(G)$, so we can consider φ as an element of $\mathrm{Hom}_{\mathbb{Z}}\big(R_{\mathbb{Q}}(G),\mathbb{F}_2\big)$.

Now the functor $R_{\mathbb{Q}}$ is a *rational 2-biset functor* (see Definition 10.1.3 and Proposition 9.6.12 of [7]), so its $\mathbb{F}_2$-dual $\mathrm{Hom}_{\mathbb{Z}}(R_{\mathbb{Q}},\mathbb{F}_2)$ is also rational. It follows that if $\mathcal{G}$ is a genetic basis of G, we have

$$\varphi=\sum_{S\in\mathcal{G}}\mathrm{Indinf}_{N_G(S)/S}^{G}f_{\mathbf{1}}^{N_G(S)/S}\mathrm{Defres}_{N_G(S)/S}^{G}\,\varphi\ \ . \tag{3.5}$$

For each $S\in\mathcal{G}$, the group $N_G(S)/S$ is a Roquette 2-group, i.e. it is cyclic, generalized quaternion of order at least 8, dihedral or semidihedral of order at least 16. Recall moreover that $\mathrm{Im}\,\imath_R=\mathrm{Hom}_{\mathbb{Z}}\big(R_{\mathbb{Q}}(R),\mathbb{F}_2\big)$ if R is trivial, of order 2, or dihedral of order at least 16. If we show that $f_{\mathbf{1}}^{N_G(S)/S}\mathrm{Defres}_{N_G(S)/S}^{G}\,\varphi=0$ when $N_G(S)/S$ is not one of these Roquette 2-groups, then Equation 3.5 shows that $\varphi\in\mathrm{Im}\,\imath_G$, and we are done.

So all we have to do is to show that $f_{\mathbf{1}}^{N_G(S)/S}\mathrm{Defres}_{N_G(S)/S}^{G}\,\varphi=0$ when $R=N_G(S)/S$ is cyclic of order at least 4, generalized quaternion, or semidihedral. In each of these cases R has a unique central subgroup Z of order 2, and $f_{\mathbf{1}}^{R}=R/\mathbf{1}-R/Z\in B(R,R)$.

Let X be any subgroup of R. Then

$$\begin{aligned}(f_{\mathbf{1}}^{N_G(S)/S}\mathrm{Defres}_{N_G(S)/S}^{G}\,\varphi)(R/X)&=(\mathrm{Defres}_{N_G(S)/S}^{G}\,\varphi)\big(f_{\mathbf{1}}^{R}\times_R(R/X)\big)\\&=(\mathrm{Defres}_{N_G(S)/S}^{G}\,\varphi)(R/X-R/ZX)\ \ .\end{aligned}$$

This is zero if $Z\leq X$, so we can assume $Z\not\leq X$, that is $Z\cap X=\mathbf{1}$.

If R is cyclic or generalized quaternion, this implies $X=\mathbf{1}$. Then

$$\begin{aligned}(f_{\mathbf{1}}^{R}\mathrm{Defres}_{R}^{G}\,\varphi)(R/X)&=(\mathrm{Defres}_{R}^{G}\,\varphi)(R/\mathbf{1}-R/Z)\\&=(\mathrm{Defres}_{R}^{G}\,\varphi)\big(\mathrm{Ind}_{D}^{R}(D/\mathbf{1}-D/Z)\big)\\&=(\mathrm{Res}_{D}^{R}\mathrm{Defres}_{R}^{G}\varphi)(D/\mathbf{1}-D/Z)\\&=(\mathrm{Defres}_{D}^{G}\varphi)(\varepsilon_D)\ \ ,\end{aligned}$$

where D is a cyclic subgroup of order 4 of R. By assumption, this is equal to $0_{\mathbb{F}_2}$, hence $f_{\mathbf{1}}^{N_G(S)/S}\mathrm{Defres}_{N_G(S)/S}^{G}\,\varphi=0$ in these cases, as was to be shown.

Now if $R\cong SD_{2^n}$ with $n\geq 4$, the same argument shows that

$$(f_{\mathbf{1}}^{R}\mathrm{Defres}_{R}^{G}\,\varphi)(R/X)=0_{\mathbb{F}_2}$$

if $Z\leq X$ or $X=\mathbf{1}$. Up to conjugation, the only remaining subgroup X of G is $X=I$, and then

$$\begin{aligned}(f_{\mathbf{1}}^{R}\mathrm{Defres}_{R}^{G}\,\varphi)(R/X)&=(\mathrm{Defres}_{R}^{G}\,\varphi)(R/I-R/IZ)\\&=(\mathrm{Defres}_{R}^{G}\,\varphi)(\varepsilon_R)=0_{\mathbb{F}_2}\ \ .\end{aligned}$$

This completes the proof of Theorem 3.4. $\square$

3.6. Corollary: *Let G be a finite group. Then the kernel of the natural map $\jmath_G : \mathbb{F}_2B(G) \to \mathrm{Hom}_{\mathbb{F}_2}\big(B^\times(G), \mathbb{F}_2\big)$ is the group generated by the elements $\mathrm{Indinf}_{T/S}^G \bar{\varepsilon}_{T/S}$, for all sections (T,S) of G such that $T/S \in \mathcal{R}$.*

Proof: This is just a dual reformulation of Theorem 3.4, since the natural map $\jmath_G : \mathbb{F}_2B(G) \to \mathrm{Hom}_{\mathbb{F}_2}\big(B^\times(G), \mathbb{F}_2\big)$ is the transposed map of $\imath_G$, up to the identification of $\mathbb{F}_2B(G)$ with the $\mathbb{F}_2$-bidual of $B(G)$. $\square$

For a finite group R, let $B_R = \mathrm{Hom}_{\mathcal{C}}(R, -) = B(-, R)$ denote the representable biset functor defined by R, and let $\mathbb{F}_2B = \mathbb{F}_2 \otimes_{\mathbb{Z}} B_R \cong \mathrm{Hom}_{\mathbb{F}_2\mathcal{C}}(R, -)$.

3.7. Corollary: *Let $\mathcal{S}$ be the set of finite groups defined by*

$$\mathcal{S} = \{C_p \mid p \text{ odd prime}\} \cup \{C_4\} \cup \{SD_{2^n} \mid n \geq 4\} \ ,$$

(by which we mean that $\mathcal{S}$ contains exactly one group of order p for each odd prime p, one cyclic group of order 4, and one semidihedral group of each order $2^n \geq 16$), and let $\mathcal{G}_{\mathcal{S}} = \{(R, \bar{\varepsilon}_R) \mid R \in \mathcal{S}\}$ be the associated set of elements of $\mathbb{F}_2B$.

1. *Let L be the biset subfunctor of $\mathbb{F}_2B$ generated by $\mathcal{G}_{\mathcal{S}}$. Then there is an exact sequence of biset functors*

$$0 \longrightarrow L \longrightarrow \mathbb{F}_2B \xrightarrow{\jmath} \mathrm{Hom}_{\mathbb{F}_2}(B^\times, \mathbb{F}_2) \longrightarrow 0 \ .$$

2. *When $R \in \mathcal{R}$, let $d_R : \mathbb{F}_2B_R \to \mathbb{F}_2B$ be the morphism of biset functors induced by adjunction from $\bar{\varepsilon}_R \in \mathbb{F}_2B(R)$. Then the sequence*

$$\bigoplus_{R \in \mathcal{S}} \mathbb{F}_2B_R \xrightarrow{d} \mathbb{F}_2B \xrightarrow{\jmath} \mathrm{Hom}_{\mathbb{F}_2}(B^\times, \mathbb{F}_2) \longrightarrow 0$$

   is exact, where d is the sum of all the maps d_R, for $R \in \mathcal{S}$.

3. *For $R \in \mathcal{R}$, let $\mathbb{F}_2B_Rf_{\mathbf{1}}^R$ be the direct summand of $\mathbb{F}_2B_R$ equal to the image of the idempotent endomorphism $f_{\mathbf{1}}^R \in B(R,R) = \mathrm{End}_{\mathcal{F}}(B_R)$. Then the morphism $d_R : \mathbb{F}_2B_R \to \mathbb{F}_2B$ factors through a morphism $\hat{d}_R : \mathbb{F}_2B_Rf_{\mathbf{1}}^R \to \mathbb{F}_2B$, and this yields an exact sequence of functors*

$$\bigoplus_{R \in \mathcal{S}} \mathbb{F}_2B_Rf_{\mathbf{1}}^R \xrightarrow{\hat{d}} \mathbb{F}_2B \xrightarrow{\jmath} \mathrm{Hom}_{\mathbb{F}_2}(B^\times, \mathbb{F}_2) \longrightarrow 0 \ .$$

   where $\hat{d}$ is the sum of the morphisms $\hat{d}_R$, for $R \in \mathcal{S}$.

Proof: For Assertion 1, all we need to show is that the element $\bar{\varepsilon}_R$ belongs to $L(R)$, when R is elementary abelian of order 4 or dihedral of order at least 8. But one checks easily that $\mathrm{Res}^{SD_{2^n}}_{D_{2^{n-1}}}\bar{\varepsilon}_{SD_{2^n}} = \bar{\varepsilon}_{D_{2^{n-1}}}$ for $n \geq 4$, and that $\mathrm{Res}^{D_8}_{(C_2)^2}\bar{\varepsilon}_{D_8} = \bar{\varepsilon}_{(C_2)^2}$. Now Assertion 2 follows from the fact that L is equal to the image of d, and Assertion 3 from the fact that $f_{\mathbf{1}}^R\varepsilon_R = \varepsilon_R$ by Remark 3.2. □

4. Extensions

4.1. Notation: *Let $\mathcal{T}$ denote the class of finite groups which are subquotient of some group in $\mathcal{S}$, that is the class of groups isomorphic to a group in the following set*

$$\left\{ \begin{array}{c} \{C_p \mid p \text{ odd prime}\} \cup \{C_{2^n} \mid n \geq 0\} \cup \{Q_{2^n} \mid n \geq 3\} \\ \cup \{(C_2)^2\} \cup \{D_{2^n} \mid n \geq 3\} \cup \{SD_{2^n} \mid n \geq 4\} \end{array} \right.$$

In other words $\mathcal{T}$ is the class of groups which are cyclic of odd prime order, elementary abelian of order 4, dihedral of order 8, or a Roquette 2-group.

4.2. Proposition:

1. *Set $\widehat{B^\times} = \mathrm{Hom}_{\mathbb{F}_2}(B^\times, \mathbb{F}_2)$. If S be a biset functor over $\mathbb{F}_2$, such that $S(\mathbf{1}) = \{0\}$, then $\mathrm{Ext}^1_{\mathcal{F}_{\mathbb{F}_2}}(\widehat{B^\times}, S)$ is isomorphic to a subspace of $\prod\limits_{R\in\mathcal{S}} \partial S(R)$, where $\mathcal{S}$ is the set defined in Corollary 3.7.*
2. *Let H be a finite group. If there exists a simple $\mathbb{F}_2\mathrm{Out}(H)$-module V such that $\mathrm{Ext}^1_{\mathcal{F}_{\mathbb{F}_2}}(S_{H,V}, B^\times) \neq \{0\}$, then $H \in \mathcal{T}$.*

Proof: The functor L of Corollary 3.7 is equal to $\mathrm{Im}\, d = \mathrm{Im}\hat{d} = \mathrm{Ker}\, \jmath$, so we have an exact sequence

$$0 \longrightarrow L \longrightarrow \mathbb{F}_2 B \xrightarrow{\jmath} \widehat{B^\times} \longrightarrow 0 \ ,$$

and a surjective morphism $\bigoplus\limits_{R\in\mathcal{S}} \mathbb{F}_2 B_R f_{\mathbf{1}}^R \xrightarrow{\hat{d}} L$. Let M be any biset functor over $\mathbb{F}_2$. Applying first the functor $\mathrm{Hom}_{\mathcal{F}_{\mathbb{F}_2}}(-, M)$ to the short exact sequence, we get the beginning of a long exact sequence

$$0 \to \mathrm{Hom}(\widehat{B^\times}, M) \to \mathrm{Hom}(\mathbb{F}_2 B, M) \to \mathrm{Hom}(L, M) \to \mathrm{Ext}^1(\widehat{B^\times}, M) \to 0 \ ,$$

where Hom and Ext^1 are taken in the category $\mathcal{F}_{\mathbb{F}_2}$. Indeed $\mathrm{Ext}^1(\mathbb{F}_2 B, M)$ is equal to zero as $\mathbb{F}_2 B = \mathrm{Hom}_{\mathcal{F}_{\mathbb{F}_2}}(\mathbf{1}, -)$ is representable, hence projective in $\mathcal{F}_{\mathbb{F}_2}$. Moreover $\mathrm{Hom}(\mathbb{F}_2 B, M) \cong M(\mathbf{1})$, so we get an exact sequence

$$(\mathbf{4.3}) \quad 0 \to \mathrm{Hom}(\widehat{B^\times}, M) \to M(\mathbf{1}) \to \mathrm{Hom}(L, M) \to \mathrm{Ext}^1(\widehat{B^\times}, M) \to 0 \ .$$

In the case $M = S$, since $S(\mathbf{1}) = \{0\}$, this gives an isomorphism $\mathrm{Hom}(L, S) \cong \mathrm{Ext}^1(\widehat{B^\times}, S)$. On the other hand, since L is a quotient of $\bigoplus_{R \in \mathcal{S}} \mathbb{F}_2 B_R f_{\mathbf{1}}^R$, we get an inclusion

$$\mathrm{Hom}(L, S) \hookrightarrow \mathrm{Hom}(\bigoplus_{R \in \mathcal{S}} \mathbb{F}_2 B_R f_{\mathbf{1}}^R, S) \cong \prod_{R \in \mathcal{S}} \mathrm{Hom}(\mathbb{F}_2 B_R f_{\mathbf{1}}^R, S) \ .$$

Moreover $\mathrm{Hom}(\mathbb{F}_2 B_R f_{\mathbf{1}}^R, S) \cong f_{\mathbf{1}}^R S(R) = \partial S(R)$. This proves Assertion 1.

For Assertion 2, we take $M = S_{H,W}$ in the exact sequence 4.3, where $W = V^*$ is the dual module. If $H \neq \mathbf{1}$, then $S_{H,W}(\mathbf{1}) = \{0\}$. And if $H = \mathbf{1}$, then $W = \mathbb{F}_2$, and by duality

$$\mathrm{Hom}(\widehat{B^\times}, S_{\mathbf{1},\mathbb{F}_2}) \cong \mathrm{Hom}(S_{\mathbf{1},\mathbb{F}_2}, B^\times) \cong \mathbb{F}_2 \cong S_{\mathbf{1},\mathbb{F}_2}(\mathbf{1}) \ .$$

In both cases, we get an isomorphism

$$\mathrm{Hom}(L, S_{H,W}) \cong \mathrm{Ext}^1(\widehat{B^\times}, S_{H,W}) \ .$$

Now by duality, this is isomorphic to $\mathrm{Ext}^1(S_{H,V}, B^\times)$. As before, it embeds into $\prod_{R \in \mathcal{S}} \partial S_{H,W}(R)$. Hence if $\mathrm{Ext}^1(S_{H,V}, B^\times) \neq \{0\}$, then there exists some R in $\mathcal{S}$ such that $\partial S_{H,W}(R) \neq \{0\}$. In particular $S_{H,W}(R) \neq \{0\}$, so H is a subquotient of R. This completes the proof. □

5. Sectional characterization

Let $\mathcal{O}_\mathcal{T}$ denote the forgetful functor $\mathcal{F}_k \to \mathcal{F}_{\mathcal{T},k}$. It was shown in Section 5 of [8] that the functor $\mathcal{O}_\mathcal{T}$ has a right adjoint $\mathcal{R}_\mathcal{T}$ defined as follows[1]. For

[1]The construction in [8] actually dealt only with p-biset functors, for a fixed prime number p, and their restriction to a class of finite p-groups closed under taking subquotients. But it extends *verbatim* to the categories $\mathcal{F}_k$ and $\mathcal{F}_{\mathcal{T},k}$, for any class $\mathcal{T}$ of finite groups closed under taking subquotients.

a finite group G, let $\mathcal{T}(G)$ denote the set of sections (T,S) of G such that $T/S \in \mathcal{T}$. If $F \in \mathcal{F}_{\mathcal{T},k}$, then

$$\mathcal{R}_\mathcal{T}(F)(G) = \lim_{\substack{\longleftarrow \\ (T,S)\in\mathcal{T}(G)}} F(T/S) \;\;,$$

that is the set of sequences of elements $l_{T,S} \in F(T/S)$, for $(T,S) \in \mathcal{T}(G)$, subject to the following conditions:

1. If (T,S) and (T',S') are elements of $\mathcal{T}(G)$ such that $S \leq S' \leq T' \leq T$, then
$$\mathrm{Defres}^{T/S}_{T'/S'} l_{T,S} = l_{T',S'} \;\;,$$
2. If $(T,S) \in \mathcal{T}(G)$ and $x \in G$, then
$${}^x l_{T,S} = l_{{}^xT,{}^xS} \;\;.$$

The biset functor structure on $\mathcal{R}_\mathcal{T}(F)$ is obtained as follows. For a finite group H and a finite (H,G)-biset U, the image $v = U\,l$ of $l \in \mathcal{R}_\mathcal{T}(F)(G)$ by U is the sequence $m_{T,S}$, for $(T,S) \in \mathcal{T}(H)$, defined by

$$m_{T,S} = \sum_{u\in[T\backslash U/G]} (S\backslash Tu)\, l_{T^u,S^u} \;\;,$$

where for a subgroup X of H, we set $X^u = \{g \in G \mid \exists x \in X,\; xu = ug\}$. This makes sense as one can show that (T^u, S^u) is a section of G, and that T^u/S^u is isomorphic to a subquotient of T/S, hence in $\mathcal{T}$ if $T/S \in \mathcal{T}$.

The unit $\eta : \mathrm{Id} \to \mathcal{R}_\mathcal{T} \circ \mathcal{O}_\mathcal{T}$ of the adjunction between $\mathcal{O}_\mathcal{T}$ and $\mathcal{R}_\mathcal{T}$, evaluated for $N \in \mathcal{F}_k$ at a finite group G, is the map $\eta_{N,G}$ from $N(G)$ to $\mathcal{R}_\mathcal{T}\mathcal{O}_\mathcal{T}(N)(G)$ sending $n \in N(G)$ to the sequence of elements $l_{T,S} \in \mathcal{O}_\mathcal{T}(N)(T/S) = N(T/S)$, for $(T,S) \in \mathcal{T}(G)$, defined by

$$l_{T,S} = \mathrm{Defres}^G_{T/S} n \;\;.$$

5.1. Theorem: *The unit of the adjunction between $\mathcal{O}_\mathcal{T} : \mathcal{F}_{\mathbb{F}_2} \to \mathcal{F}_{\mathcal{T},\mathbb{F}_2}$ and $\mathcal{R}_\mathcal{T} : \mathcal{F}_{\mathcal{T},\mathbb{F}_2} \to \mathcal{F}_{\mathbb{F}_2}$ induces an isomorphism of biset functors over $\mathbb{F}_2$*

$$\eta_{B^\times} : B^\times \to \mathcal{R}_\mathcal{T}\mathcal{O}_\mathcal{T}(B^\times) \;\;.$$

In other words, for any finite group G, the map

$$\eta_{B^\times,G} : B^\times(G) \to \lim_{\substack{\longleftarrow \\ (T,S)\in\mathcal{T}(G)}} B^\times(T/S)$$

is an isomorphism.

Proof: Set $\widehat{B} = \mathrm{Hom}_{\mathbb{Z}}(B, \mathbb{F}_2)$, and let C denote the cokernel of the morphism $\imath : B^\times \to \widehat{B}$. The functor $\mathcal{O}_T$ is exact, so its right adjoint $\mathcal{R}_T$ is left exact. Hence we get a commutative diagram

$$\text{(5.2)}\qquad \begin{tikzcd} 0 \arrow[r] & B^\times \arrow[r, "\imath"] \arrow[d, "\eta_{B^\times}"] & \widehat{B} \arrow[r, "\pi"] \arrow[d, "\eta_{\widehat{B}}"] & C \arrow[r] \arrow[d, "\eta_C"] & 0 \\ 0 \arrow[r] & \mathcal{R}_{\mathcal{T}}\mathcal{O}_{\mathcal{T}}(B^\times) \arrow[r, "f"] & \mathcal{R}_{\mathcal{T}}\mathcal{O}_{\mathcal{T}}(\widehat{B}) \arrow[r, "g"] & \mathcal{R}_{\mathcal{T}}\mathcal{O}_{\mathcal{T}}(C) & \end{tikzcd}$$

with exact rows, where π is the projection morphism, $f = \mathcal{R}_{\mathcal{T}}\mathcal{O}_{\mathcal{T}}(\imath)$, and $g = \mathcal{R}_{\mathcal{T}}\mathcal{O}_{\mathcal{T}}(\pi)$.

We first claim that the vertical morphism $\eta_{\widehat{B}}$ is an isomorphism. Indeed, let G be a finite group. The map $\eta_{\widehat{B},G} : \widehat{B}(G) \to \varprojlim_{(T,S)\in\mathcal{T}(G)} \widehat{B}(T/S)$ is the map sending the linear form $\varphi : B(G) \to \mathbb{F}_2$ to the sequence of linear forms $d_{T,S} = \mathrm{Defres}^G_{T/S}\varphi : B(T/S) \to \mathbb{F}_2$, for $(T,S) \in \mathcal{T}(G)$. Since

$$d_{T,S}\big((T/S)/(T/S)\big) = \varphi\big(\mathrm{Indinf}^G_{T/S}(T/S)/(T/S)\big) = \varphi(G/T) \enspace ,$$

and since $(T,T) \in \mathcal{T}$ for any subgroup T of G, we see that $\eta_{\widehat{B},G}$ is injective.

Now let $l = (\psi_{T,S})_{(T,S)\in\mathcal{T}(G)}$ be an element of $\varprojlim_{(T,S)\in\mathcal{T}(G)} \widehat{B}(T/S)$. So for each $(T,S) \in \mathcal{T}(G)$, we get a linear map $\psi_{T,S} : B(T/S) \to \mathbb{F}_2$. For a subgroup U of G, we set $\varphi(G/U) = \psi_{U,U}\big((U/U)/(U/U)\big)$. Then for any $x \in G$, we have $\varphi(G/{}^xU) = \varphi(G/U)$, since

$$\begin{aligned} \varphi(G/{}^xU) &= \psi_{{}^xU,{}^xU}\big(({}^xU/{}^xU)/({}^xU/{}^xU)\big) \\ &= ({}^x\psi_{U,U})\big(({}^xU/{}^xU)/({}^xU/{}^xU)\big) \\ &= \psi_{U,U}\big((U/U)/(U/U)\big) \end{aligned}$$

because ${}^x\psi_{T,S} = \psi_{{}^xT,{}^xS}$ for any $(T,S) \in \mathcal{T}(G)$. It follows that φ extends linearly to an element of $\widehat{B}(G)$. Moreover, for any $(T,S) \in \mathcal{T}(G)$ and any subgroup U/S of T/S, we have $\psi_{U,U} = \mathrm{Defres}^{T/S}_{U/U}\psi_{T,S}$, so

$$\begin{aligned} \mathrm{Defres}^G_{T/S}\varphi\big((T/S)/(U/S)\big) &= \varphi\Big(\mathrm{Indinf}^G_{T/S}\big((T/S)/(U/S)\big)\Big) \\ &= \varphi(G/U) = \psi_{U,U}\big((U/U)/(U/U)\big) \\ &= (\mathrm{Defres}^{T/S}_{U/U}\psi_{T,S})\big((U/U)/(U/U)\big) \\ &= \psi_{T,S}\Big(\mathrm{Indinf}^{T/S}_{U/U}\big((U/U)/(U/U)\big) \\ &= \psi_{T,S}\big((T/S)/(U/S)\big) \enspace . \end{aligned}$$

Hence $\mathrm{Defres}^G_{T/S}\varphi = \psi_{T,S}$, for any $(T,S) \in \mathcal{T}(G)$, so the map $\eta_{\widehat{B},G}$ is surjective. Hence it is an isomorphism, for any finite group G, and $\eta_{\widehat{B}}$ is an isomorphism, as claimed.

Now the Snake's lemma, applied to Diagram 5.2, shows that the morphism $\eta_{B^\times}$ is injective, and that its cokernel is isomorphic to the kernel of η_C. For a finite group G, an element $\bar{\varphi}$ of $\mathrm{Ker}\,\eta_{C,G}$ is represented by a linear form $\varphi : B(G) \to \mathbb{F}_2$ such that $\mathrm{Defres}^G_{T/S}\varphi$ belongs to $\imath\big(B^\times(T/S)\big)$, for any $(T,S) \in \mathcal{T}(G)$, hence in particular for any section (T,S) of G such that $T/S \in \mathcal{R}$. By Theorem 3.4, it follows that $(\mathrm{Defres}^G_{T/S}\varphi)(\epsilon_{T,S}) = 0$. As this holds for any section (T,S) of G with $T/S \in \mathcal{R}$, the form φ lies in the image of $\imath_G$, by Theorem 3.4 again. In other words the element $\bar{\varphi}$ is equal to zero. Hence $\mathrm{Ker}\,\eta_{C,G} = \{0\}$ for any G, so η_C is injective. It follows that $\eta_{B^\times}$ is surjective. Hence it is an isomorphism. This completes the proof of the theorem. □

6. Finite generation and presentation

In this section, we show that the biset functor $B^\times$ is not finitely generated, and that its dual $\widehat{B^\times}$ is finitely generated (by a single element!), but not finitely presented. Recall that k is a commutative ring, and that $kB_R = k \otimes_{\mathbb{Z}} B(-,R)$ is the representable functor $\mathrm{Hom}_{k\mathcal{C}}(R,-)$.

6.1. Lemma: *Let R and H be finite groups. If $\overline{kB_R}(H) \neq \{0\}$, then H is a subquotient of R.*

Proof: Recall that $kB_R(H) = kB(H,R)$ has a k-basis consisting of the transitive bisets $(H \times R)/X$, for X in a set of representatives of conjugacy classes of subgroups of $H \times R$. Let X be one of these groups, and $Y = p_1(X)$ be its first projection. Then $(H \times R)/X = kB_R(\mathrm{Ind}_Y^H)\big((Y \times R)/X\big)$, where we abuse notation in the right hand side by viewing X as a subgroup of $Y \times R$, and $(Y \times R)/X$ as an element of $kB_R(Y)$. Hence if the image of $(H \times R)/X$ in $\overline{kB_R}(H)$ is non zero, then $Y = H$.

In this case let $N = k_1(X)$. Then N is a normal subgroup of $H = p_1(X)$, and $(H \times R)/X = kB_R(\mathrm{Inf}_{H/N}^H)\Big(\big((H/N) \times R\big)/\overline{X}\Big)$, where

$$\overline{X} = \{(hN, g) \mid (h,g) \in X\} \ .$$

Hence if the image of $(H \times R)/X$ in $\overline{kB_R}(H)$ is non zero, we also have $N = \mathbf{1}$. Then $H \cong Y/N \cong T/S$, where $T = p_2(X)$ and $S = k_2(X)$. In particular H is a subquotient of R. This completes the proof. □

6.2. Proposition: *Let k be a commutative ring and $\mathcal{T}$ be a class of finite groups closed under taking subquotients. Let $F \in \mathcal{F}_{\mathcal{T},k}$. The following conditions are equivalent:*

1. *The functor F is finitely generated.*
2. *There exists a finite family $\mathcal{E}$ of groups in $\mathcal{T}$ and an epimorphism $\bigoplus_{R\in\mathcal{E}} kB_R \to F$.*
3. *For any $H \in \mathcal{T}$, the k-module $\overline{F}(H)$ is finitely generated, and there exists an integer $n \in \mathbb{N}$ such that $\overline{F}(H) = \{0\}$ whenever $|H| > n$.*

Proof: The equivalence of 1 and 2 is classical. If 1 holds, then there is a finite set $\mathcal{G}$ of elements of F such that $\langle\mathcal{G}\rangle = F$. For each $(R, v) \in \mathcal{G}$, we get a morphism $\widetilde{v} : kB_R \to F$ associated to $v \in F(R)$ by Yoneda's lemma. The sum of these morphisms

$$\bigoplus_{(R,v)\in\mathcal{G}} \widetilde{v} : \bigoplus_{(R,v)\in\mathcal{G}} kB_R \to F$$

is surjective if $\langle\mathcal{G}\rangle = F$, so 2 holds. Conversely, each representable functor kB_R is generated by the single element (R, Id_R), where $\mathrm{Id}_R \in kB(R, R)$ is the identity endomorphism of R. Hence if 2 holds, then F is a quotient of a finite sum of finitely generated functors, so F is finitely generated, and 1 holds.

Now if 2 holds, then for each $H \in \mathcal{T}$, the k-module $F(H)$ is a quotient of $\bigoplus_{R\in\mathcal{E}} kB(H, R)$, and each $kB(H, R)$ is a finitely generated k-module. Hence $F(H)$ is a finitely generated k-module. Moreover $\overline{F}(H)$ is a quotient of $\bigoplus_{R\in\mathcal{E}} \overline{kB_R}(H)$, and $\overline{kB_R}(H) = \{0\}$ unless H is a subquotient of R, by Lemma 6.1. In particular $\overline{kB_R}(H) = \{0\}$ if $|H| > |R|$, so $\overline{F}(H) = \{0\}$ if $|H| > n$, where $n = \max\{|R| \mid R \in \mathcal{E}\}$. Hence 3 holds.

Now if 3 holds, there is a finite set of isomorphism classes of finite groups R such that $\overline{F}(R) \neq \{0\}$. Let $\mathcal{U}$ be a set of representatives of this set. For each $R \in \mathcal{U}$, we can lift to $F(R)$ a finite generating set of the k-module $\overline{F}(R)$. We get a finite subset V_R of $F(R)$, and this gives a finite set

$$\mathcal{G} = \{(R, v) \mid R \in \mathcal{U},\ v \in V_R\}$$

of elements of F, which in turns gives a morphism

$$\pi : P = \bigoplus_{(R,v)\in\mathcal{G}} kB_R \to F \quad .$$

Our choice of $\mathcal{U}$ and V_R, for $R \in \mathcal{U}$, shows that the induced morphism $\overline{\pi_H} : \overline{P}(H) \to \overline{F}(H)$ is surjective for any $H \in \mathcal{T}$: if H is not isomorphic to a group in $\mathcal{U}$, then this is trivially true because $\overline{F}(H) = \{0\}$. And otherwise, we can assume $H \in \mathcal{U}$, and then $\overline{\pi_H}$ is surjective because $\overline{\pi_H}(V_H)$ generates $\overline{F}(H)$ by construction.

We deduce by induction on $n = |H|$ that $\pi_H : P(H) \to F(H)$ is surjective for any $H \in \mathcal{T}$. For $n = 1$, this is clear, since

$$\pi_{\mathbf{1}} = \overline{\pi_1} : \overline{P}(\mathbf{1}) = P(\mathbf{1}) \to F(\mathbf{1}) = \overline{F}(\mathbf{1})$$

is surjective. Now assume π_K is surjective for any $K \in \mathcal{T}$ with $|K| < n = |H|$, and let $v \in F(H)$. Since $\overline{\pi_H}$ is surjective, there is an element $w \in P(H)$, a set Σ of proper sections of H (i.e. sections different from $(H, \mathbf{1})$), and elements $v_{T,S} \in F(T/S)$, for $(T,S) \in \Sigma$, such that

$$v = \pi_H(w) + \sum_{(T,S)\in\Sigma} \mathrm{Indinf}_{T/S}^H v_{T,S} \ \ .$$

Since $|T/S| < n$ for any $(T,S) \in \Sigma$, the map $\pi_{T/S} : P(T/S) \to F(T/S)$ is surjective, and there is an element $w_{T,S} \in P(T/S)$ such that $v_{T/S} = \pi_{T/S}(w_{T,S})$. It follows that

$$\begin{aligned} v &= \pi_H(w) + \sum_{(T,S)\in\Sigma} \mathrm{Indinf}_{T/S}^H \pi_{T/S}(w_{T,S}) \\ &= \pi_H(w) + \sum_{(T,S)\in\Sigma} \pi_H\big(\mathrm{Indinf}_{T/S}^H w_{T,S}\big) = \pi_H\big(w + \sum_{(T,S)\in\Sigma} \mathrm{Indinf}_{T/S}^H w_{T,S}\big) \ \ . \end{aligned}$$

Hence π_H is surjective, and this completes the inductive step.

It follows that π is an epimorphism, so 3 implies 2, completing the proof of Proposition 6.2. □

6.3. Corollary: *The biset functor $B^\times$ is not finitely generated.*

Proof: It has been show by Barsotti ([1], Proposition 6.8) that if p is a prime number congruent to 1 mod 4, then $\overline{B^\times}(D_{2p}) \neq \{0\}$, where D_{2p} is a dihedral group of order $2p$ (in Barsotti's terminology, the group D_{2p} is *residual*). It follows that there are arbitrary large finite groups H such that $\overline{B^\times}(H) \neq \{0\}$. By Proposition 6.2, the functor $B^\times$ is not finitely generated. □

Recall from Corollary 3.7 that there is an exact sequence of biset functors

$$(\mathbf{6}.4) \qquad 0 \longrightarrow L \longrightarrow \mathbb{F}_2 B \xrightarrow{\jmath} \widehat{B^\times} \longrightarrow 0 \ \ ,$$

where

$$\mathcal{S} = \{C_p \mid p \text{ odd prime}\} \cup \{C_4\} \cup \{SD_{2^n} \mid n \geq 4\} \ ,$$

and L is the biset subfunctor of $\mathbb{F}_2B$ generated $\mathcal{G}_\mathcal{S} = \{(R, \bar{\varepsilon}_R) \mid R \in \mathcal{S}\}$.

6.5. Proposition:

1. *Let $\mathcal{G} = \{(\mathbf{1}, u)\}$, where u is the non zero element of $\widehat{B^\times}(\mathbf{1}) \cong \mathbb{F}_2$. Then $\langle \mathcal{G} \rangle = \widehat{B^\times}$.*
2. *The functor L is not finitely generated.*
3. *The functor $\widehat{B^\times}$ is not finitely presented.*

Proof: (1) This follows from the fact that $\widehat{B^\times}$ is a quotient of $\mathbb{F}_2B$, and that $\mathbb{F}_2B$ is generated by $(\mathbf{1}, e)$, where $e \in \mathbb{F}_2B(\mathbf{1})$ is the class of a set of cardinality one, endowed with the trivial action of the trivial group. Indeed if $K \leq H$ are finite groups, then $H/K = \mathrm{Indinf}_{K/K}^H \mathrm{Iso}(f_K)(e)$, where $f_K : \mathbf{1} \to K/K$ is the unique group isomorphism.

(2) The exact sequence (6.4) shows that if p is an odd prime number, then $L(C_p) \cong \mathbb{F}_2$ and $L(\mathbf{1}) = \{0\}$. Hence $\overline{L}(C_p) \cong L(C_p) \cong \mathbb{F}_2$, so there exist arbitrary large finite groups H such that $\overline{L}(H) \neq \{0\}$. By Proposition 6.2, the functor L is not finitely generated.

(3) Suppose that there exists an exact sequence in $\mathcal{F}_{\mathbb{F}_2}$

$$N \longrightarrow M \longrightarrow \widehat{B^\times} \longrightarrow 0 \ , \qquad (\mathbf{6.6})$$

where M is projective and N is finitely generated. This gives an exact sequence

$$0 \longrightarrow K \longrightarrow M \longrightarrow \widehat{B^\times} \longrightarrow 0 \ ,$$

where K is the image of N in M. In particular K is finitely generated. Then, since M and $\mathbb{F}_2B$ are projective in $\mathcal{F}_{\mathbb{F}_2}$, Shanuel's lemma gives an isomorphism of functors

$$L \oplus M \cong K \oplus \mathbb{F}_2B \ .$$

Then L is a quotient of $K \oplus \mathbb{F}_2B$, which is finitely generated. Hence L is finitely generated, contradicting 2. So no short exact sequence like (6.6) can exists, hence $\widehat{B^\times}$ is not finitely presented. □

Recall that in Corollary 3.7, the set

$$\mathcal{S} = \{C_p \mid p \text{ odd prime}\} \cup \{C_4\} \cup \{SD_{2^n} \mid n \geq 4\}$$

was introduced. We finally prove that $\mathcal{G}_{\mathcal{S}} = \{(R, \bar{\varepsilon}_R) \mid R \in \mathcal{S}\}$ is a minimal set of generators of L.

6.7. Theorem: *Let $\mathcal{S}'$ be a proper subset of the set $\mathcal{S}$ introduced in Corollary 3.7, and $\mathcal{G}_{\mathcal{S}'} = \{(R, \bar{\varepsilon}_R) \mid r \in \mathcal{S}'\}$. Then $\langle \mathcal{G}_{\mathcal{S}'} \rangle$ is a proper subfunctor of $\langle \mathcal{G}_{\mathcal{S}} \rangle = L$.*

Proof: It suffices to show that for any $R \in \mathcal{S}$, if we set $\mathcal{G}_{\mathcal{S}}^R = \mathcal{G}_{\mathcal{S}} - \{(R, \bar{\varepsilon}_R)\}$ and $L^R = \langle \mathcal{G}_{\mathcal{S}}^R \rangle$, then $\bar{\varepsilon}_R \notin L^R(R)$. So we assume that $\bar{\varepsilon}_R \in L^R(R)$, for contradiction, i.e. that there exists a finite set $\mathcal{E}$ of pairs (H, X), where $H \in \mathcal{S} - \{R\}$ and X is a subgroup of $R \times H$, such that

$$\bar{\varepsilon}_R = \sum_{(H,X) \in \mathcal{E}} \big((R \times H)/X\big) \bar{\varepsilon}_H \quad . \tag{6.8}$$

Since $f_{\mathbf{1}}^R \bar{\varepsilon}_R = \bar{\varepsilon}_R$ for any R, by Remark 3.2, this also reads

$$\bar{\varepsilon}_R = \sum_{(H,X) \in \mathcal{E}} f_{\mathbf{1}}^R \big((R \times H)/X\big) f_{\mathbf{1}}^H \bar{\varepsilon}_H \quad .$$

We now observe that each element H of $\mathcal{S}$ has a unique minimal normal (hence central of prime order) subgroup Z_H. And if $k_1(X) \geq Z_R$, then

$$f_{\mathbf{1}}^R \big((R \times H)/X\big) = f_{\mathbf{1}}^R \mathrm{Inf}_{R/Z_R}^R \mathrm{Def}_{R/Z_R}^R \big((R \times H)/X\big) = 0$$

by Lemma 6.3.2 of [7]. Similarly $\big((R \times H)/X\big) f_{\mathbf{1}}^H = 0$ if $k_2(X) \geq Z_H$. So we can assume that $k_1(X) \cap Z_R = \mathbf{1}$ and $k_2(X) \cap Z_H = \mathbf{1}$ for any $(H, X) \in \mathcal{E}$.

This forces $k_1(X) = k_2(X) = \mathbf{1}$, unless R is semidihedral and $k_1(X)$ is a non central subgroup of order 2 of R, or H is semidihedral and $k_2(X)$ is a non central subgroup of order 2 of H (both cases may occur simultaneously). In the first case $p_1(X) \leq N_R\big(k_1(X)\big) = k_1(X) Z_R$, and $p_1(X)/k_1(X)$ has order 1 or 2. Similarly, in the second case $p_2(X)/k_2(X)$ has order 1 or 2. In any of these two cases, the morphism $f_{\mathbf{1}}^R \big((R \times H)/X\big) f_{\mathbf{1}}^H$ of $\mathcal{C}_{\mathbb{F}_2}$ factors through a group of order 1 or 2 (see 2.4). Since $L(\mathbf{1}) = L(C_2) = \{0\}$, it follows that $f_{\mathbf{1}}^R \big((R \times H)/X\big) f_{\mathbf{1}}^H \bar{\varepsilon}_H = 0$.

So we can assume that $k_1(X) = k_2(X) = \mathbf{1}$ for any $(H, X) \in \mathcal{E}$. In this case X is a twisted diagonal subgroup of $R \times H$, that is, there is a subgroup K of H, a subgroup S of R, and a group isomorphism $f : K \to S$, such that

$$(R \times H)/X \cong \mathrm{Ind}_S^R \, \mathrm{Iso}(f) \, \mathrm{Res}_K^H \quad . \tag{6.9}$$

We can also assume that $\mathrm{Res}_K^H \bar{\varepsilon}_H \neq 0$, and in particular that $L(K) \neq \{0\}$.

Suppose first that $R = C_p$ for an odd prime number p. Then $S = \mathbf{1}$ or $S = R$. Since $L(\mathbf{1}) = \{0\}$, we have $S = R$. Then $K \cong S = C_p$ is a subgroup of $H \in \mathcal{S}$. The only element H of $\mathcal{S}$ admitting a subgroup of odd prime order p is C_p itself. But $H \in \mathcal{S} - \{C_p\}$ by assumption, so we get a contradiction.

Suppose now that $R = C_4$. Again, since $L(\mathbf{1}) = L(C_2) = \{0\}$, we have $S = R$, and $H \geq K \cong C_4$. Since $H \in \mathcal{S} - \{C_4\}$, it follows that $H = SD_{2^n}$ for some $n \geq 4$. Then K is contained in the unique generalized quaternion subgroup Q of index 2 of H. One checks easily that $\mathrm{Res}^H_Q \bar{\varepsilon}_H = Q/\mathbf{1} - Q/Z_H$, and it follows that $\mathrm{Res}^H_K \bar{\varepsilon}_H = 0$. We get a contradiction also in this case.

We are left with the case $R = SD_{2^n}$, for some $n \geq 4$. Then if $(H, X) \in \mathcal{E}$, the first projection S of the twisted diagonal subgroup X of $R \times H$ is a proper subgroup of R: otherwise indeed, its second projection K is a semidihedral subgroup of $H \in \mathcal{S}$, which can only occur if H itself is semidihedral and isomorphic, hence equal, to R. This is a contradiction, since $H \in \mathcal{S} - \{R\}$.

It follows from (6.8) and (6.9) that $\bar{\varepsilon}_R$ is a sum of elements of the form $\mathrm{Ind}^R_S u_S$, for proper subgroups S of R and elements u_S of $L(S)$. Let D, C, and Q be the subgroups of index 2 of R, where D is dihedral, C is cyclic, and Q is generalized quaternion. We can write

$$\bar{\varepsilon}_R = \mathrm{Ind}^R_D v_D + \mathrm{Ind}^R_C v_C + \mathrm{Ind}^R_Q v_Q \ ,$$

for some $v_D \in L(D)$, $v_C \in L(C)$, and $v_Q \in L(Q)$. We set $Z = Z_R$ for simplicity. If $M \in \{D, C, Q\}$, then $M \geq Z$, and one checks easily that $f^R_{\mathbf{1}} \mathrm{Ind}^R_M = \mathrm{Ind}^R_M f^M_{\mathbf{1}}$. It follows that

$$\bar{\varepsilon}_R = \mathrm{Ind}^R_D f^D_{\mathbf{1}} v_D + \mathrm{Ind}^R_C f^C_{\mathbf{1}} v_C + \mathrm{Ind}^R_Q f^Q_{\mathbf{1}} v_Q \ .$$

Moreover $f^M_{\mathbf{1}} M/N = 0$ if $N \cap Z(M) \neq \mathbf{1}$. It follows that $f^C_{\mathbf{1}} v_C$ is a multiple of $C/\mathbf{1} - C/Z$, and that $f^Q_{\mathbf{1}} v_Q$ is a multiple of $Q/\mathbf{1} - Q/Z$. Hence

$$\bar{\varepsilon}_R = \mathrm{Ind}^R_D w_D + \lambda(R/\mathbf{1} - R/Z) \ ,$$

for some $\lambda \in \mathbb{F}_2$ and some $w_D (= f^D_{\mathbf{1}} v_D) \in \partial L(D)$.

Now cutting the exact sequence

$$0 \longrightarrow L(D) \longrightarrow \mathbb{F}_2 B(D) \longrightarrow \widehat{B^\times}(D) \longrightarrow 0$$

by the idempotent $f^D_{\mathbf{1}}$ gives the exact sequence

$$0 \longrightarrow \partial L(D) \longrightarrow \partial \mathbb{F}_2 B(D) \longrightarrow \partial \widehat{B^\times}(D) \longrightarrow 0 \ .$$

The vector space $\partial \mathbb{F}_2 B(D)$ has a basis consisting of the elements $D/\mathbf{1} - D/Z$, $D/I - D/IZ$, and $D/J - D/JZ$, where I and J are two non conjugate non central subgroups of order 2 of D. The vector space $\partial\widehat{B^\times}(D)$ is isomorphic to the dual of $\partial B^\times(D)$, which is one dimensional (see e.g. Corollary 6.12 of [6], or the proof of Theorem 3.4). It follows that $\partial L(D)$ has dimension 2. Now $\partial L(D)$ contains the two elements $\bar{\varepsilon}_D = (D/I - D/IZ) - (D/J - D/JZ)$ and $D/\mathbf{1} - D/Z = \mathrm{Ind}_{C_4}^D \bar{\varepsilon}_{C_4}$, which are obviously linearly independent. Hence these two elements form a basis of $\partial L(D)$. It follows that

$$\bar{\varepsilon}_R = \alpha \mathrm{Ind}_D^R \bar{\varepsilon}_D + \beta \mathrm{Ind}_{C_4}^R \bar{\varepsilon}_{C_4} + \lambda(R/\mathbf{1} - R/Z) \enspace ,$$

for some α, β in $\mathbb{F}_2$. But one checks easily that $\mathrm{Ind}_D^R \bar{\varepsilon}_D = 0$, for I and J are conjugate in R. Moreover $\mathrm{Ind}_{C_4}^R \bar{\varepsilon}_{C_4} = R/\mathbf{1} - R/Z$. Thus $\bar{\varepsilon}_R = R/I - R/IZ$ is a scalar multiple of $R/\mathbf{1} - R/Z$, which is obviously wrong. This final contradiction completes the proof of Theorem 6.7. □

6.10. Corollary: *Let $2^{\mathcal{S}}$ be the set of subsets of $\mathcal{S}$, ordered by inclusion of subsets, and $[0, L]$ be the poset of subfunctors of L, ordered by inclusion of subfunctors. Let $g : 2^{\mathcal{S}} \to [0, L]$ be the map sending $A \subseteq \mathcal{S}$ to*

$$g(A) = \big\langle \{(H, \bar{\varepsilon}_H) \mid H \in A\} \big\rangle \subseteq L \enspace ,$$

and $f : [0, L] \to 2^{\mathcal{S}}$ be the map sending the subfunctor M of L to

$$f(M) = \{H \in \mathcal{S} \mid \bar{\varepsilon}_H \in M(H)\} \subseteq \mathcal{S} \enspace .$$

Then:

1. *Let A, A' be subsets of $\mathcal{S}$, and M, M' be subfunctors of L. Then*

   $$g(A \cup A') = g(A) + g(A') \text{ and } f(M \cap M') = f(M) \cap f(M') \enspace .$$

   In particular f and g are maps of posets.
2. *$f \circ g = \mathrm{Id}_{2^{\mathcal{S}}}$.*
3. *The poset $[0, L]$ is uncountable.*

Proof: Assertion 1 is straightforward. For Assertion 2, let $A \subseteq \mathcal{S}$ and $A' = f \circ g(A)$. Then clearly $A \subseteq A'$. If this inclusion is strict, let $S \in A' - A$. Then

$$(S, \bar{\varepsilon}_S) \in g(A) = \big\langle \{(H, \bar{\varepsilon}_H) \mid H \in A\} \big\rangle \subseteq g(\mathcal{S}') \enspace ,$$

where $\mathcal{S}' = \mathcal{S} - \{S\}$. Then $g(\mathcal{S}') = g(\mathcal{S}) = L$, and by Theorem 6.7, it follows that $\mathcal{S}' = \mathcal{S}$, a contradiction. Hence $A = A'$, and $f \circ g = \mathrm{Id}_{2^{\mathcal{S}}}$. In particular g is injective, and Assertion 3 follows, since the set of subsets of the (infinite) countable set $\mathcal{S}$ is uncountable. □

6.11. Remark: The map g is not surjective, and not a map of lattices (that is, the image by g of an intersection of subsets need not be the intersection of the images of the subsets). Indeed, if S is any semidihedral group in $\mathcal{S}$, and C its cyclic subgroup of index 2, we have

$$\mathrm{Res}_C^S \bar{\varepsilon}_S = C/\mathbf{1} - C/Z = \mathrm{Ind}_{C'}^C \bar{\varepsilon}_{C'} \quad ,$$

where Z is the center of S and C' the subgroup of order 4 of C. It follows that $u = C/\mathbf{1} - C/Z$ is a non-zero element of $M(C)$, where M is the intersection of the subfunctors of L generated by $\{(S, \bar{\varepsilon}_S)\}$ and $\{(C_4, \bar{\varepsilon}_{C_4})\}$. In other words if $A = \{S\}$ and $A' = \{C_4\}$, we have $0 \neq u \in \big(g(A) \cap g(A')\big)(C)$, so $g(A) \cap g(A') \neq \{0\}$. Now if $g(A) \cap g(A')$ belongs to the image of g, there is a subset A'' of $\mathcal{S}$ such that $g(A) \cap g(A') = g(A'')$, and then

$$A'' = fg(A'') = f\big(g(A) \cap g(A')\big) = fg(A) \cap fg(A') = A \cap A' = \emptyset \quad .$$

This is a contradiction since $g(A'') \neq \{0\}$ but $g(\emptyset) = \{0\}$. It follows that g is not surjective, and not a map of lattices, since $g(A) \cap g(A') \neq g(A \cap A')$.

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