Extensions of simple cohomological Mackey functors

Serge Bouc

Abstract: This is a report on some recent joint work with Radu Stancu, to appear in [4]. It is an expanded version of a talk given at the RIMS workshop *Cohomology of finite groups and related topics*, February 18-20, 2015.

1. Cohomological Mackey functors

1.1. Let G be a finite group, and k be a commutative ring. There are many equivalent definitions of *Mackey functors* for G over k. For the "naive" one, this is an assignment $H \mapsto M(H)$ of a k-module $M(H)$ to any subgroup H of G, together with k-linear maps

$$M(H) \xrightarrow{t_H^K} M(K) \xrightarrow{r_H^K} M(H), \quad M(H) \xrightarrow{c_{x,H}} M({}^xH)$$

whenever $H \leq K \leq G$ and $x \in G$, subject to a list of compatibility conditions, e.g. *transitivity* of transfers and restrictions, or the *Mackey formula* (see [6] for details).

A Mackey functor M is called *cohomological* if

$$\forall H \leq K \leq G, \quad t_H^K \circ r_H^K = |K : H| Id_{M(K)} \quad .$$

The cohomological Mackey functors for G over k form a category $\mathbf{M}_k^c(G)$.

1.2. Examples :

- Let V be a kG-module. The *fixed points functor* FP_V is defined by $M(H) = V^H$, for any $H \leq G$, and by

  $$\forall H \leq K \leq G, \quad r_H^K : V^K \hookrightarrow V^H, \quad t_H^K = \mathrm{Tr}_H^K : V^H \to V^K \quad ,$$

  and by $c_{x,H}(v) = x \cdot v$, for $x \in G$.

  More generally, for $n \in \mathbb{N}$, the cohomology functor $H^n(-, V)$ is a cohomological Mackey functor.

- Let k be a field of characteristic p, let G be a finite p-group. The *simple* cohomological Mackey functors for G over k are the functors $S_X = S_X^G$, where $X \leq G$ (up to G-conjugation), defined by

  $$\forall H \leq G, \quad S_X(H) = \begin{cases} k & \text{if } H =_G X, \\ \{0\} & \text{otherwise.} \end{cases}$$

1.3. Yoshida's Theorem

- Let $\mathbf{perm}_k(G)$ denote the full subcategory of kG-**Mod** consisting of finitely generated *permutation* kG-modules.
- Let $\mathbf{Fun}_k(G)$ denote the category of (contravariant) k-linear functors from $\mathbf{perm}_k(G)$ to k-**Mod**.
- If $M \in \mathbf{M}_k^c(G)$, the functor $\tilde{M} : V \mapsto \mathrm{Hom}_{\mathbf{M}_k^c(G)}(FP_V, M)$ is an object of $\mathbf{Fun}_k(G)$.

1.4. Theorem [Yoshida [7]] **:** *The functor $M \mapsto \tilde{M}$ is an equivalence of categories from $\mathbf{M}_k^c(G)$ to $\mathbf{Fun}_k(G)$.*

1.5. The (cohomological) Mackey algebra

- [Thévenaz-Webb [6]] The (cohomological) Mackey functors for G over k are exactly the modules over the (cohomological) *Mackey algebra.*
- Consider the Hecke algebra $Y_k(G) = \mathrm{End}_{kG}(\underset{H \le G}{\oplus} kG/H)$. This k-algebra is called *the Yoshida algebra* of G over k. It is isomorphic to the cohomological Mackey algebra. In other words, *the category $\mathbf{M}_k^c(G)$ is equivalent to $Y_k(G)$-*$\mathbf{Mod}$.
- The algebra $Y_k(G)$ is a free k-module of rank $\sum\limits_{H,K \le G} |H\backslash G/K|$. In particular, when k is a field, the algebra $Y_k(G)$ is a finite dimensional k-algebra.

2. Complexity

Let k be a field, and A be a finite dimensional k-algebra. Then every finitely generated A-module M admits a resolution

$$\cdots \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$$

by finitely generated projective A-modules.

2.1. Definition : *The module M has* polynomial growth *if there exists such a resolution and numbers c, d, e such that* $\forall n \in \mathbb{N},\ \dim_k P_n \le cn^d + e$. *The lower bound of such d's is called the* complexity *of M.*

The module M has exponential growth *if for any such resolution, there exist numbers $c > 0$, $d > 1$, and e such that* $\forall n \in \mathbb{N},\ \dim_k P_n \ge cd^n + e$.

The module M has intermediate growth *in all other cases.*

2.2. Lemma [Link with extensions] **:** *Let A be a finite dimensional algebra over a field k, and M be a finitely generated A-module.*

1. *If*
$$\cdots \to P_n \to P_{n-1} \cdots \to P_0 \to M \to 0$$
is a minimal projective resolution of M, then
$$P_n \cong \bigoplus_{S \in \mathrm{Irr}(A)} P_S^{\dim_k \mathrm{Ext}^n_A(M,S)/\dim_k \mathrm{End}_A(S)} \quad ,$$
where $\mathrm{Irr}(A)$ is a set of representatives of isomorphism classes of simple A-modules, and P_S denotes a projective cover of S.
2. *In particular M has polynomial growth $\iff$ $\forall S \in \mathrm{Irr}(A)$, $\exists (c, d, e)$ such that $\forall n \in \mathbb{N}$, $\dim_k \mathrm{Ext}^n_A(M, S) \leq c\, n^d + e$.*
3. *Let $0 \to L \to M \to N \to 0$ be a short exact sequence of finitely generated A-modules. If any two of L, M, N have polynomial growth, so does the third.*

2.3. Definition [Poco groups] **:** *Let k be a field of positive characteristic p. A finite group G is called a* poco group *over k if any finitely generated cohomological Mackey functor for G over k has polynomial growth.*

2.4. Theorem [B. [3]] **:** *Let G be a finite group, and k be be a field of characteristic $p > 0$. The following conditions are equivalent:*

1. *The group G is a poco group over k.*
2. *Let S be a Sylow p-subgroup of G. Then :*
   - *If $p > 2$, the group S is cyclic.*
   - *If $p = 2$, the group S has sectional rank at most 2.*

2.5. Remark : A 2-group has sectional rank at most 2 if and only if it is cyclic or metacyclic (Blackburn [2], Andersen-Oliver-Ventura [1]).

3. Construction of functors

3.1. Let k be a field of characteristic $p > 0$, and G be a finite group. By

Yoshida's equivalence $\mathbf{M}_k^c(G) \cong \mathbf{Fun}_k(G)$, cohomological Mackey functors for G over k can be viewed as functors

$$\mathbf{perm}_k(G) \longrightarrow k\text{-}\mathbf{Mod} \quad .$$

When H is another finite group, any k-linear functor

$$F : \mathbf{perm}_k(H) \longrightarrow \mathbf{perm}_k(G)$$

induces a functor

$$\mathbf{M}_k^c(G) \cong \mathbf{Fun}_k(G) \longrightarrow \mathbf{Fun}_k(H) \cong \mathbf{M}_k^c(H) \quad .$$

3.2. In particular, when U is a (finite) (G,H)-biset, the functor

$$t_U : W \in \mathbf{perm}_k(H) \mapsto kU \otimes_{kH} W \in \mathbf{perm}_k(G)$$

induces a functor $L_U : \mathbf{M}_k^c(G) \to \mathbf{M}_k^c(H)$. Similarly, the functor

$$h_U : W' \in \mathbf{perm}_k(G) \mapsto \mathrm{Hom}_{kG}(kU, W') \in \mathbf{perm}_k(H)$$

induces a functor $R_U : \mathbf{M}_k^c(H) \to \mathbf{M}_k^c(G)$.

3.3. Properties

- The functors L_U and R_U are *exact.*
- As t_U is left adjoint to h_U, the functor L_U *is left adjoint to* R_U.
- Let U' be another finite (G,H)-biset. Then

$$L_{U \sqcup U'} \cong L_U \oplus L_{U'}, \quad R_{U \sqcup U'} \cong R_U \oplus R_{U'} \quad .$$

- Let Id_G denote the identity (G,G)-biset. Then L_{Id_G} and R_{Id_G} are isomorphic to *the identity functor.*
- If K is another finite group, and V is an (H,K)-biset, then

$$L_V \circ L_U \cong L_{U \times_H V}, \quad R_U \circ R_V \cong R_{U \times_H V} \quad .$$

3.4. Examples

- Let H be a subgroup of G, and U denote the set G, as a (G,H)-biset. Then $L_U \cong \mathrm{Res}_H^G$, and $R_U \cong \mathrm{Ind}_H^G$.
- Let H be a subgroup of G, and U denote the set G, as an (H,G)-biset. Then $L_U \cong \mathrm{Ind}_H^G$, and $R_U \cong \mathrm{Res}_H^G$.
- Let $N \trianglelefteq G$, let $H = G/N$, and let U denote the set H, as a (G,H)-biset. Then $L_U = \rho_{G/N}^G$, and $R_U = \jmath_{G/N}^G$.
- Let $N \trianglelefteq G$, let $H = G/N$, and let U denote the set H, as an (H,G)-biset. Then $L_U = \imath_{G/N}^G$, and $R_U = \rho_{G/N}^G$.
- Let $f : G \to H$ be a group isomorphism, and U denote the set H, as a (G,H)-biset. Then $L_U \cong \mathrm{Iso}(f)$ and $R_U \cong \mathrm{Iso}(f^{-1})$.

3.5. Sketch of proof of Theorem 2.4

Recall that k is a field of characteristic $p > 0$, that G is a finite group, and S is a Sylow p-subgroup of G.

- Use the functors Ind_S^G and Res_S^G to reduce to the case where $G = S$ is a p-group.
- Let (B,A) be a *section* of G (i.e. $A \trianglelefteq B \leq G$). The set G/A is a $(G, B/A)$-biset, and the set $A\backslash G$ is a $(B/A, G)$-biset. The corresponding functors $L_{G/A}$, $R_{G/A}$, $L_{A\backslash G}$ and $R_{A\backslash G}$ allow for a reduction to the case where G is *elementary abelian*.
- The case of cyclic groups and Klein four group was settled by M. Samy Modeliar ([5]). In particular, these groups are poco groups.
- Describe the subfunctor structure of $\mathrm{Ind}_H^G S_1^H$, leading to long exact sequences of Ext groups. These sequences show that the functor S_1^G has exponential growth if $G \cong (C_p)^m$, when $p > 2$ and $m \geq 2$, or $p = 2$ and $m \geq 3$.
- Use induction on the order of a 2-group G, to complete the case $p = 2$.

4. Presentation of some Ext algebras

Let p be a prime number, and $G \cong (C_p)^n$, $n \geq 1$.

- Let $X \leq G$ with $|X| = p$. Then there exists a unique non split extension $\alpha_X^G : 0 \to S_1^G \to \binom{S_X^G}{S_1^G} \to S_X^G \to 0$ in $\mathbf{M}_{\mathbb{F}_p}^c(G)$.

Let $\gamma_X^G \in \mathrm{Ext}^2_{\mathbf{M}^c_{\mathbb{F}_p}(G)}(S_1^G, S_1^G)$ denote the class of the splice

$$\alpha_X^G(\alpha_X^G)^* : 0 \to S_1^G \to \begin{pmatrix} S_X^G \\ S_1^G \end{pmatrix} \to \begin{pmatrix} S_1^G \\ S_X^G \end{pmatrix} \to S_1^G \to 0 \quad .$$

- When $p > 2$ and $\varphi : G \to \mathbb{F}_p$ is a group homomorphism, let U_φ^G be the vector space $\mathbb{F}_p \oplus \mathbb{F}_p$, on which $g \in G$ acts by $g(x, y) = \big(x + \varphi(g)y, y\big)$. There is a unique (cohomological) Mackey functor T_φ^G for G over $\mathbb{F}_p$ such that $T_\varphi(H) = \{0\}$ if $1 < H \leq G$, and $T_\varphi^G(1) \cong U_\varphi^G$. It fits in an extension
$$0 \to S_1^G \to U_\varphi^G \to S_1^G \to 0$$
in $\mathbf{M}^c_{\mathbb{F}_p}(G)$. Let $\tau_\varphi^G \in \mathrm{Ext}^1_{\mathbf{M}^c_{\mathbb{F}_p}(G)}(S_1^G, S_1^G)$ denote the class of this extension.

4.1. The algebra $\mathcal{E}_k = \mathrm{Ext}^*_{\mathbf{M}^c_k(G)}(S_1^G, S_1^G)$

4.2. Theorem [B. Stancu [4]] **:** *Let k be a field of characteristic $p > 0$, and $G \cong (C_p)^n$. Let $\mathcal{E}_k$ denote the algebra $\mathrm{Ext}^*_{\mathbf{M}^c_k(G)}(S_1^G, S_1^G)$. Then:*

1. *The extension of scalars from $\mathbb{F}_p$ to k induces an isomorphism of k-algebras $\mathcal{E}_k \cong k \otimes_{\mathbb{F}_p} \mathcal{E}_{\mathbb{F}_p}$.*
2. *The algebra $\mathcal{E}_{\mathbb{F}_p}$ is generated by the elements γ_X^G, where $X \leq G$ with $|X| = p$, together, when $p > 2$, with the elements τ_φ^G, where $\varphi : G \to \mathbb{F}_p^+$.*

4.3. Presentation of $\mathcal{E}_k$ for $p = 2$

4.4. Theorem [B. [3]] **:** *Let k be a field of characteristic 2, and $G \cong (C_2)^m$. Then the graded algebra $\mathcal{E}_k = \mathrm{Ext}^*_{\mathbf{M}^c_k(G)}(S_1^G, S_1^G)$ admits the following presentation:*

- *The generators γ_x are indexed by the elements x of $G - \{0\}$. They have degree 2.*
- *The relations are the following:*
    1. *If $H < G$ with $|G : H| = 2$, then $\sum\limits_{x \notin H} \gamma_x = 0$.*
    2. *If x and y are distinct elements of $G - \{0\}$, then*
    $$[\gamma_x + \gamma_y, \gamma_{x+y}] = 0 \quad .$$

4.5. Presentation of $\mathcal{E}_k$, for $p > 2$

4.6. Theorem [B. Stancu [4]] **:** *Let k be a field of characteristic $p > 2$, and $G \cong (C_p)^m$. Then the graded algebra $\mathcal{E}_k = \mathrm{Ext}^*_{\mathbf{M}^c_k(G)}(S_1^G, S_1^G)$ admits the following presentation:*

1. *The generators are the elements γ_X in degree 2, for $X \leq G$ such that $|X| = p$, and the elements τ_φ in degree 1, for $\varphi \in \mathrm{Hom}(G, \mathbb{F}_p^+)$.*
2. *The relations are the following:*

   (a) $\tau_{\varphi+\psi} = \tau_\varphi + \tau_\psi$, *for any* φ, ψ *in* $\mathrm{Hom}(G, \mathbb{F}_p^+)$.

   (b) • *If* $p \geq 5$, *then* $\tau_\varphi^2 = 0$ *and* $[\tau_\varphi, \sum\limits_{X \nleq Ker\varphi} \gamma_X] = 0$, *for any* φ *in* $\mathrm{Hom}(G, \mathbb{F}_p^+)$.
   • *If* $p = 3$, *then* $\tau_\varphi^2 = -\sum\limits_{X \nleq Ker\varphi} \gamma_X$, *for any* $\varphi \in \mathrm{Hom}(G, \mathbb{F}_p^+)$.

   (c) $[\gamma_X, \tau_\varphi] = 0$, *for any* $\varphi \in \mathrm{Hom}(G, \mathbb{F}_p^+)$, *for any* $X \leq Ker\varphi$ *with* $|X| = p$.

   (d) $[\gamma_X, \sum\limits_{\substack{Y \leq Q \\ |Y|=p}} \gamma_Y] = 0$, *for any* $Q \leq G$ *with* $|Q| = p^2$ *and any* $X \leq Q$ *with* $|X| = p$.

4.7. Corollary : *The Poincaré series of $\mathcal{E}_k$ is equal to*

$$P(t) = \frac{1}{\big(1-t^2\big)\big(1-3t^2\big)\big(1-7t^2\big)\cdots\big(1-(2^{m-1}-1)t^2\big)}$$

when $p = 2$, and to

$$\frac{1}{\big(1-t\big)\big(1-t-(p-1)t^2\big)\big(1-t-(p^2-1)t^2\big)\cdots\big(1-t-(p^{m-1}-1)t^2\big)}$$

when p is odd.

4.8. Corollary : *Let k be a field of characteristic $p > 0$. When G is an elementary abelian p-group, one can compute explicitly all the extension groups between any two simple cohomological Mackey functors for G over k.*

Acknowledgements: I wish to thank Professor Fumihito Oda for his invitation, and Kinki University, Osaka for support. I also thank Professor Akihiko Hida for the opportunity of giving a talk at the RIMS workshop *Cohomology of finite groups and related topics*, February 18-20, 2015.

References

[1] K. Andersen, B. Oliver, and J. Ventura. Reduced fusion systems over 2-groups. Preprint, 2008.

[2] N. Blackburn. Generalizations of certain elementary theorems on p-groups. *Proc. London Math. Soc.*, 11:1–22, 1961.

[3] S. Bouc. Complexity and cohomology of cohomological Mackey functors. *Adv. Maths*, 221:983–1045, 2009.

[4] S. Bouc and R. Stancu. The extension algebra of some cohomological Mackey functors. Preprint, 2015, Adv. in Maths., to appear, 2015.

[5] M. Samy Modeliar. *Certaines constructions liées aux foncteurs de Mackey cohomologiques*. PhD thesis, Université Paris 7-Denis Diderot, 2005.

[6] J. Thévenaz and P. Webb. The structure of Mackey functors. *Trans. Amer. Math. Soc.*, 347(6):1865–1961, June 1995.

[7] T. Yoshida. On G-functors (II): Hecke operators and G-functors. *J. Math. Soc. Japan*, 35:179–190, 1983.

Serge Bouc, CNRS-LAMFA, 33 rue St Leu, 80039 Amiens Cedex 01, France.
`email: serge.bouc@u-picardie.fr`