The diagonal p-permutation functor kR_k

Serge Bouc

Abstract

Let k be an algebraically closed field of positive characteristic p. We describe the full lattice of subfunctors of the diagonal p-permutation functor kR_k obtained by k-linear extension from the functor R_k of linear representations over k. This leads to the description of the "composition factors" S_P of kR_k, which are parametrized by finite p-groups (up to isomorphism), and of the evaluations of these particular simple diagonal p-permutation functors over k.

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1 Introduction

Let k be an algebraically closed field of positive characteristic p, and R be a commutative and unital ring. We consider[1] the assignment $G \mapsto \mathsf{R}R_k(G)$ sending a finite group G to $\mathsf{R}R_k(G) = \mathsf{R} \otimes_{\mathbb{Z}} R_k(G)$, where $R_k(G)$ is the Grothendieck group of finite dimensional kG-modules. When G and H are finite groups, if M is a (kH, kG)-bimodule which is projective as a right kG-module, tensoring with M over kG preserves exact sequences, hence induces a well defined group homomorphism $R_k(M) : R_k(G) \to R_k(H)$, and an R-linear map $\mathsf{R}R_k(M) : \mathsf{R}R_k(G) \to \mathsf{R}R_k(H)$, both simply denoted $V \mapsto M \otimes_{kG} V$. In particular, one checks easily that this endows $\mathsf{R}R_k$ with a structure of *diagonal p-permutation functor* over R (see [2], [3], [4]). Our main goal is Theorem 1.3, describing the full lattice of subfunctors of kR_k (ordered by inclusion of subfunctors).

As a byproduct of this description, we get a "composition series" of kR_k, with simple subquotients S_P indexed by finite p-groups P up to isomorphism. From this, the evaluations of these particular simple functors can be computed, giving a new proof of the last corollary of [4].

To be more precise, we first need some notation and some definitions:

Notation 1.1: *Let $\mathcal{P}$ be the set of isomorphism classes of finite p-groups. For a finite p-group P, we denote by $[P]$ its isomorphism class. For a subset $\mathcal{C}$ of $\mathcal{P}$, we say (abusively) that P belongs to $\mathcal{C}$, and we write $P \in \mathcal{C}$, if $[P] \in \mathcal{C}$.*

For a finite group G, we denote by k_G the trivial kG-module k, and by $\langle k_G \rangle$ the subfunctor of kR_k generated by $k_G \in kR_k(G)$. We observe that $\langle k_G \rangle$ only depends on the isomorphism class of G.

[1] mainly in the case $\mathsf{R} = k$, but a few results deal with the case of an arbitrary R.

For a finite group G, we denote by $\mathcal{E}_{\mathsf{R}}(G)$ the essential algebra *of G over R, defined by*

$$\mathcal{E}_{\mathsf{R}}(G) = \mathsf{R}T^{\Delta}(G,G)/\sum_{|H|<|G|} \mathsf{R}T^{\Delta}(G,H) \circ \mathsf{R}T^{\Delta}(H,G),$$

where $T^{\Delta}(H,G)$ denotes the Grothendieck group of diagonal p-permutation (kH,kG)-bimodules (see [2], [3], [4] for details).

Definition 1.2: *We say that a subset $\mathcal{C}$ of $\mathcal{P}$ is* closed *if for any finite p-groups P and Q*

$$Q \leq P \text{ and } [P] \in \mathcal{C} \implies [Q] \in \mathcal{C}.$$

In other words, if a p-group is isomorphic to a subgroup of a p-group in $\mathcal{C}$, then it belongs to $\mathcal{C}$.

Our main theorem is the following:

Theorem 1.3: *Let $\mathcal{L}$ denote the lattice of subfunctors of kR_k, ordered by inclusion of subfunctors. Let $\mathcal{F}$ denote the lattice of closed subsets of $\mathcal{P}$, ordered by inclusion of subsets. Then the maps*

$$\Psi : F \in \mathcal{L} \mapsto \Psi(F) = \big\{[P] \in \mathcal{P} \mid k_P \in F(P)\big\} \in \mathcal{F}, \text{ and}$$
$$\Theta : \mathcal{C} \in \mathcal{F} \mapsto \Theta(\mathcal{C}) = \sum_{[P]\in\mathcal{C}} \langle k_P \rangle \in \mathcal{L},$$

are well defined isomorphisms of lattices, inverse to each other.

2 Some evaluations of diagonal p-permutation functors.

We start with a result of independent interest on diagonal p-permutation functors. It gives a way to compute the evaluation of a diagonal p-permutation functor at a direct product of a p'-group and a p-group. This requires the following notation:

Notation 2.1: *Let L be a p'-group and Q be a p-group, and set $H = L\times Q$. For a (finite dimensional) kL-module V, let $\overrightarrow{V} = V\otimes_k kQ$, viewed as a (kH, kQ)-bimodule for the action*

$$\forall (l,x)\in H = L\times Q,\ \forall y,z\in Q,\ \forall v\in V,\ (l,x)\cdot(v\otimes z)\cdot y = lv\otimes xzy.$$

Similarly, let V^* denote the k-dual of V, viewed as a right kL-module, and $\overleftarrow{V} = V^*\otimes_k kQ$, viewed as a (kQ, kH)-bimodule for the action

$$\forall (l,x)\in H = L\times Q,\ \forall y,z\in Q,\ \forall \alpha\in V^*,\ y\cdot(\alpha\otimes z)\cdot(l,x) = \alpha l\otimes yzx.$$

Let also $\mathrm{Irr}_k(L)$ denote a set of representatives of isomorphism classes of irreducible kL-modules. With this notation:

Proposition 2.2:

1. *The bimodules $\overrightarrow{V}$ and $\overleftarrow{V}$ are diagonal p-permutation bimodules.*
2. *If V and W are irreducible kL-modules, then $\overleftarrow{W}\otimes_{kH}\overrightarrow{V}$ is isomorphic to the identity (kQ, kQ)-bimodule kQ, if W and V are isomorphic, and it is zero otherwise.*
3. *The direct sum $\bigoplus_V \big(\overrightarrow{V}\otimes_{kQ}\overleftarrow{V}\big)$, for $V\in\mathrm{Irr}_k(L)$, is isomorphic to the identity (kH, kH)-bimodule kH.*

Proof: 1. Indeed, the group $H\times Q$ has a normal Sylow subgroup $S = Q\times Q$, and the restrictions to S of $\overrightarrow{V}$ and $\overleftarrow{V}$ are both isomorphic to a direct sum of $\dim_k V$ copies of the identity (kQ, kQ)-bimodule kQ.

2. One checks easily that

$$\overleftarrow{W}\otimes_{kH}\overrightarrow{V} = (W^*\otimes_k kQ)\otimes_{k(L\times Q)}(V\otimes_k kQ)\cong (W^*\otimes_{kL} V)\otimes_k (kQ\otimes_{kQ} kQ)$$

as (kQ, kQ)-bimodules. Assertion 2 follows, since $W^*\otimes_{kL} V = 0$ unless V and W are isomorphic, and since $V^*\otimes_{kL} V\cong k$ as k is algebraically closed. Moreover $kQ\otimes_{kQ} kQ\cong kQ$.

3. Similarly

$$\begin{aligned}\overrightarrow{V}\otimes_{kQ}\overleftarrow{V} &= (V\otimes_k kQ)\otimes_{kQ}(V^*\otimes_k kQ)\cong (V\otimes_k V^*)\otimes_k (kQ\otimes_{kQ} kQ)\\ &\cong \mathrm{End}_k(V)\otimes_k kQ\end{aligned}$$

as (kH, kH)-bimodules. Now the direct sum $\bigoplus_{V\in\mathrm{Irr}_k(L)}\mathrm{End}_k(V)$ is the decomposition of the semisimple group algebra kL as a direct product of its Wedderburn components. It follows that $\bigoplus_V\big(\overrightarrow{V}\otimes_{kQ}\overleftarrow{V}\big)\cong kL\otimes kQ\cong k(L\times Q)$ is isomorphic to the identity (kH, kH)-bimodule. □

In the following corollary, we view $\overrightarrow{V}$ and $\overleftarrow{V}$ as elements of $T^\Delta(H, Q)$ and $T^\Delta(Q, H)$, respectively.

Corollary 2.3: *Let* R *be a commutative (unital) ring, and* F *be a diagonal* p*-permutation functor over* R*. Let* L *be a* p'*-group, and* Q *be a* p*-group. Then:*

1. $F(L \times Q) \cong \bigoplus_{V \in \mathrm{Irr}_k(L)} F(Q)$*. More precisely, the maps*

$$\varphi \in F(L \times Q) \mapsto \bigoplus_{V \in \mathrm{Irr}_k(L)} F(\overleftarrow{V})(\varphi) \in \bigoplus_{V \in \mathrm{Irr}_k(L)} F(Q)$$

and

$$\psi = \bigoplus_{V \in \mathrm{Irr}_k(L)} \psi_V \in \bigoplus_{V \in \mathrm{Irr}_k(L)} F(Q) \mapsto \sum_{V \in \mathrm{Irr}_k(L)} F(\overrightarrow{V})(\psi_V) \in F(L \times Q)$$

are isomorphisms of R*-modules, inverse to each other.*

2. *If* L *is non-trivial, then the essential algebra* $\mathcal{E}_{\mathsf{R}}(L \times Q)$ *is zero.*

Proof: 1. Indeed, setting $H = L \times Q$ as above, for irreducible kL-modules V and W, we have that

$$F(\overleftarrow{W}) \circ F(\overrightarrow{V}) = F(\overleftarrow{W} \otimes_{kH} \overrightarrow{V}), \text{ and } F(\overrightarrow{V}) \circ F(\overleftarrow{V}) = F(\overrightarrow{V} \otimes_{kQ} \overleftarrow{V}),$$

so the result follows from Assertions 2 and 3 of Proposition 2.2.

2. Assertion 3 of Proposition 2.2 tells us that the identity bimodule kH is equal to a sum of morphisms which factor through Q, and $|Q| < |H|$ if L is non-trivial. Assertion 2 of Corollary 2.3 follows. □

Remark 2.4:

1. In short, Corollary 2.3 says that

$$F(L \times Q) \cong R_k(L) \otimes_{\mathbb{Z}} F(Q).$$

2. The (non-)vanishing of $\mathcal{E}_{\mathsf{R}}(H)$ for an arbitrary finite group H is studied in detail in Section 3 of [4], under additional conditions on R. The proof given here is much simpler and explicit for the case of a direct product $H = L \times Q$ of a p'-group L and a p-group Q, and doesn't require any additional assumption on R.

We now consider a version of Brauer's induction theorem relative to the prime p:

Theorem 2.5: *Let G be a finite group. Then the cokernel of the induction map*

$$\mathop{\oplus}_{H} \mathrm{Ind}_H^G : \bigoplus_H R_k(H) \to R_k(G)$$

where H runs through Brauer p-elementary subgroups of G, i.e. subgroups of the form $H = P \times C$, where P is a p-group and C is a cyclic p'-group, is finite and of order prime to p.

Proof: Let $(K, \mathcal{O}, k)$ be a p-modular system, such that K is big enough for G. We have a commutative diagram

$$\begin{array}{ccc} \bigoplus_H R_K(H) & \xrightarrow{\oplus_H \mathrm{Ind}_H^G} & R_K(G) \\ {\scriptstyle \oplus_H d_H}\downarrow & & \downarrow {\scriptstyle d_G} \\ \bigoplus_H R_k(H) & \xrightarrow[\oplus_H \mathrm{Ind}_H^G]{} & R_k(G) \end{array}$$

where the vertical arrows d_H and d_G are the respective decomposition maps. Let C_K (resp. C_k) denote the cokernel of the top (resp. bottom) horizontal map. By Theorem 12.28 of [9], the group C_K is finite, and of order prime to p. By Theorem 16.33 of [9], the vertical arrows are surjective. So d_G induces a surjective group homomorphism $C_K \twoheadrightarrow C_k$, hence C_k is finite and of order prime to p. □

Corollary 2.6: *Let G be a finite group. Then*

$$kR_k(G) = \sum_H \mathrm{Ind}_H^G \, kR_k(H),$$

where H runs through the Brauer p-elementary subgroups of G.

Corollary 2.7: *Let F be a subfunctor of kR_k, and G be a finite group. Then $F(G)$ is an ideal of the algebra $kR_k(G)$.*

Proof: By Corollary 2.6, there is a set $\mathcal{S}$ of Brauer elementary subgroups of G and elements $w_H \in kR_k(H)$, for $H \in \mathcal{S}$, such that

$$k_G = \sum_{H \in \mathcal{S}} \mathrm{Ind}_H^G w_H.$$

Now let $u \in F(G)$. Then in $kR_k(G)$, we have

$$u = k_G \cdot u = \sum_{H \in \mathcal{S}} \mathrm{Ind}_H^G (w_H \cdot \mathrm{Res}_H^G u).$$

Since F is a subfunctor of kR_k, we know that $\mathrm{Res}^G_H F(G) \subseteq F(H)$ and $\mathrm{Ind}^G_H F(H) \subseteq F(G)$. So it is enough to prove that $F(H)$ is an ideal of $kR_k(H)$, for each H in $\mathcal{S}$.

Now each $H \in \mathcal{S}$ is in particular of the form $H = L \times Q$, where L is a (cyclic) p'-group, and Q is a p-group. In particular $R_k(H) = \mathrm{Inf}_L^{L\times Q} R_k(L)$. Let V and W be kL-modules, and let $\widetilde{V}$ denote the (kL, kL)-bimodule $\mathrm{Ind}_{\Delta(L)}^{L\times L} V$. Then $\widetilde{V} \otimes_k kQ$ is a (kH, kH)-diagonal p-permutation bimodule. Moreover, one can check easily that $\widetilde{V} \otimes_{kL} W \cong V \otimes_k W$ as kL-modules, and it follows that there is an isomorphism of kH-modules

$$(\widetilde{V} \otimes_k kQ) \otimes_{k(L\times Q)} \mathrm{Inf}_L^{L\times Q} W \cong \mathrm{Inf}_L^{L\times Q}(V \otimes_k W).$$

Written differently, this reads

$$kR_k\big(\widetilde{V} \otimes_k kQ\big)\big(\mathrm{Inf}_L^{L\times Q} W\big) = \mathrm{Inf}_L^{L\times Q}(V \otimes_k W).$$

In the algebra $kR_k(H) = \mathrm{Inf}_L^{L\times Q} kR_k(L)$, the right hand side is nothing but the product $V\cdot W$, and the left hand side is given by the action of the diagonal p-permutation bimodule $\widetilde{V} \otimes_k kQ$ on $\mathrm{Inf}_L^{L\times Q} W$. It follows more generally that if $\varphi \in F(H)$, then the product $V \cdot \varphi$ is obtained from φ by applying $\widetilde{V} \otimes_k kQ \in T^\Delta(H, H)$, so $V \cdot \varphi \in F(H)$. Thus $kR_k(H) \cdot F(H) \subseteq F(H)$, as was to be shown. □

Notation 2.8: *For finite groups G and H, we write $H \hookrightarrow G$ if H is isomorphic to a subgroup of G.*

Lemma 2.9: *Let P and Q be finite p-groups. Then*

$$\langle k_P\rangle(Q) = \begin{cases} k & \text{if } Q \hookrightarrow P \\ 0 & \text{otherwise.} \end{cases}$$

In particular $\langle k_Q\rangle \leq \langle k_P\rangle$ if and only if $Q \hookrightarrow P$.

Proof: By definition of $\langle k_P\rangle$, we have $\langle k_P\rangle(Q) = kT^\Delta(Q, P)(k_P)$. Moreover since P and Q are p-groups, the group $T^\Delta(Q, P)$ is equal to the Burnside group $B^\Delta(Q, P)$ of left-right free (Q, P)-bisets, so $kT^\Delta(Q, P) = kB^\Delta(Q, P)$. Now if X is a diagonal subgroup of $Q \times P$, the set of (right-)orbits of P on the (Q, P)-biset $(Q \times P)/X$ is a Q-set isomorphic to $Q/p_1(X)$, where $p_1(X)$ is the first projection of X, so we have

$$k\big((Q \times P)/X\big) \otimes_{kP} k \cong k\big(Q/p_1(X)\big).$$

As an element of $R_k(Q)$, this is equal to $|Q : p_1(X)| k_Q$, so it is equal to 0 in $kR_k(Q)$ unless $p_1(X) = Q$, and then it is equal to k_Q. But saying that there exists a diagonal subgroup $X \leq Q \times P$ such that $p_1(X) = Q$ amounts to saying that Q is isomorphic to a subgroup of P.

Since $\langle k_P \rangle$ is a subfunctor of kR_k, and since $kR_k(Q)$ is one dimensional, generated by k_Q, it follows that $\langle k_P \rangle(Q)$ is non-zero if and only if Q is isomorphic to a subgroup of P, and this occurs if and only if $k_Q \in \langle k_P \rangle(Q)$, that is $\langle k_Q \rangle \leq \langle k_P \rangle$. This completes the proof. □

3 The subfunctors $\langle k_P \rangle$

The following result is a major step in the proof of Theorem 1.3:

Theorem 3.1: *Let F be a subfunctor of kR_k. Then for any finite group G*

$$F(G) = \sum_{\substack{P \in \mathcal{P},\, P \leq G \\ k_P \in F(P)}} \langle k_P \rangle(G).$$

Proof: Let G be a finite group. Saying that $k_P \in F(P)$ amounts to saying that $\langle k_P \rangle \leq F$, so $\sum\limits_{\substack{P \in \mathcal{P}, P \leq G \\ k_P \in F(P)}} \langle k_P \rangle(G) \leq F(G)$. We have to prove the reverse inclusion.

By Theorem 2.6, there is a set $\mathcal{S}$ of Brauer p-elementary subgroups of G, and elements $w_H \in kR_k(H)$, for $H \in \mathcal{S}$, such that

$$k_G = \sum_{H \in \mathcal{S}} \mathrm{Ind}_H^G w_H$$

in $kR_k(G)$. Let $u \in F(G)$. Then

$$u = k_G \otimes_k u = \sum_{H \in \mathcal{S}} \mathrm{Ind}_H^G \big(w_H \otimes_k \mathrm{Res}_H^G u \big). \tag{3.2}$$

Now $\mathrm{Res}_H^G u \in F(H)$, for each $H \in \mathcal{S}$, since F is a subfunctor of kR_k. Then $w_H \otimes_k \mathrm{Res}_H^G u \in F(H)$ also, by Lemma 2.7, since each $H \in \mathcal{S}$ is a product $L_H \times Q_H$, where L_H is a (cyclic) p'-group and Q_H is a p-group. It follows that $u \in \sum\limits_{H \in \mathcal{S}} \mathrm{Ind}_H^G F(H)$ for any $u \in F(G)$, so

$$F(G) = \sum_{H \in \mathcal{S}} \mathrm{Ind}_H^G F(H).$$

Now by Proposition 2.2, for each $H \in \mathcal{S}$, the identity (kH, kH)-bimodule kH splits as

$$kH \cong \bigoplus_{V \in \mathrm{Irr}_k(L_H)} \big(\overrightarrow{V} \otimes_{kQ_H} \overleftarrow{V} \big).$$

It follows that

$$F(H) = \sum_{V \in \mathrm{Irr}_k(L_H)} F(\overrightarrow{V})\big(F(Q_H)\big).$$

Now $F(Q_H) \subseteq kR_k(Q_H) = k$, since Q_H is a p-group. So $F(Q_H)$ is either zero or k. It is non zero exactly when $k_{Q_H} \in F(Q_H)$. So $F(H) = 0$ or

$$F(H) = \sum_{V \in \mathrm{Irr}_k(L_H)} F(\overrightarrow{V})\big(F(Q_H)\big) = \langle k_{Q_H}\rangle(H),$$

and $F(H)$ is non-zero. It follows that

$$F(G) = \sum_{\substack{H \in \mathcal{S} \\ k_{Q_H} \in F(Q_H)}} \mathrm{Ind}_H^G \langle k_{Q_H}\rangle(H) \subseteq \sum_{\substack{P \in \mathcal{P},\, P \leq G \\ k_P \in F(P)}} \langle k_P\rangle(G),$$

as was to be shown. □

Corollary 3.3: *Let F be a subfunctor of kR_k. Then*

$$F = \sum_{\substack{P \in \mathcal{P} \\ k_P \in F(P)}} \langle k_P\rangle.$$

Corollary 3.4: *Let G be a finite group, and P be a Sylow p-subgroup of G. Then:*

1. $kR_k(G) = \langle k_P\rangle(G)$.
2. $\langle k_G\rangle = \langle k_P\rangle$.

Proof: 1. Indeed $kR_k(G) = \sum_{Q \in \mathcal{P},\, Q \leq G} \langle k_Q\rangle(G)$, by Theorem 3.1 applied to the subfunctor $F = kR_k$. But if $Q \in \mathcal{P}$ and $Q \leq G$, then $Q \hookrightarrow P$, hence $\langle k_Q\rangle \leq \langle k_P\rangle$ by Lemma 2.9. Hence $\langle k_Q\rangle(G) \leq \langle k_P\rangle(G)$, and $kR_k(G) = \langle k_P\rangle(G)$.

2. Since $k_G \in \langle k_G\rangle(G)$, we have $k_G \in \langle k_P\rangle(G)$ by Assertion 1, that is $\langle k_G\rangle \leq \langle k_P\rangle$. Conversely $k_P = \mathrm{Res}_P^G k_G$, so $k_P \in \langle k_G\rangle(P)$, i.e. $\langle k_P\rangle \leq \langle k_G\rangle$. Hence $\langle k_G\rangle = \langle k_P\rangle$. □

4 Proof of Theorem 1.3

We first recall the statement:

Theorem: *Let $\mathcal{L}$ denote the lattice of subfunctors of kR_k, ordered by inclusion of subfunctors. Let $\mathcal{F}$ denote the lattice of closed subsets of $\mathcal{P}$, ordered by inclusion of subsets. Then the maps*

$$\Psi : F \in \mathcal{L} \mapsto \Psi(F) = \big\{[P] \in \mathcal{P} \mid k_P \in F(P)\big\} \in \mathcal{F}, \text{ and}$$
$$\Theta : \mathcal{C} \in \mathcal{F} \mapsto \Theta(\mathcal{C}) = \sum_{[P]\in\mathcal{C}} \langle k_P \rangle \in \mathcal{L},$$

are well defined isomorphisms of lattices, inverse to each other.

Proof: We first check that Ψ is well defined, i.e. that $\Psi(F)$ is a closed subset of $\mathcal{P}$, for any $F \in \mathcal{L}$. This follows from Lemma 2.9: Saying that $P \in \Psi(F)$ amounts to saying that $\langle k_P \rangle \leq F$. Now if $Q \hookrightarrow P \in \Psi(F)$, Lemma 2.9 shows that $\langle k_Q \rangle \leq \langle k_P \rangle \leq F$, so $Q \in \Psi(F)$, as was to be shown.

The maps Ψ and Θ are clearly maps of posets. Moreover Corollary 3.3 shows that $\Theta \circ \Psi(F) = F$, for any $F \in \mathcal{L}$. Conversely, let $\mathcal{C}$ be a closed subset of $\mathcal{P}$. Then

$$\Psi \circ \Theta(\mathcal{C}) = \big\{[P] \in \mathcal{P} \mid k_P \in \sum_{Q\in\mathcal{C}} \langle k_Q \rangle(P)\big\}.$$

So clearly $\mathcal{C} \subseteq \Psi \circ \Theta(\mathcal{C})$, since $k_Q \in \langle k_Q \rangle(Q)$. Conversely, if $[P] \in \Psi \circ \Theta(\mathcal{C})$, that is if k_P belongs to $\sum\limits_{Q\in\mathcal{C}} \langle k_Q \rangle(P)$, there is some $Q \in \mathcal{C}$ such that $\langle k_Q \rangle(P)$ is non zero. Then $P \hookrightarrow Q$, by Lemma 2.9, so $P \in \mathcal{C}$ since $\mathcal{C}$ is closed. So $\Psi \circ \Theta(\mathcal{C}) = \mathcal{C}$, which completes the proof of Theorem 1.3. □

Recall (see [8]) that a lattice $(L, \leq, \vee, \wedge)$ is called *complete* if any subset A of L admits a supremum (denoted as the join $\bigvee\limits_{a\in A} a$) and an infimum (denoted as the meet $\bigwedge\limits_{a\in A} a$). A complete lattice is called *completely distributive* if arbitrary joins distribute over arbitrary meets. An element l of of a lattice L is called *join irreducible* if for any finite subset A of L, the equality $l = \bigvee\limits_{a\in A} a$ implies $l \in A$, and *completely join irreducible* if L is complete and the same condition holds for arbitrary subsets A of L. An element l of a lattice L is called *join prime* if for any finite subset A of L, the inequality $l \leq \bigvee\limits_{a\in A} a$ implies $l \leq a$ for some $a \in A$, and *completely join prime* if L is complete and the same condition holds for arbitrary subsets A of L.

Corollary 4.1:

1. *The lattice $\mathcal{L}$ is completely distributive.*

2. *Let F be a subfunctor of kR_k. The following are equivalent in the lattice $\mathcal{L}$:*

   (a) *F is completely join irreducible.*

   (b) *F is completely join prime.*

   (c) *There is a finite p-group P such that $F = \langle k_P \rangle$.*

   Moreover in (c), the p-group P is unique up to isomorphism.

Proof: 1. Indeed, the lattice $\mathcal{L}$ is isomorphic to $\mathcal{F}$, which is clearly completely distributive: its join is union of closed subsets, and its meet is intersection. Now if $\mathcal{C}$ and $(\mathcal{C}_i)_{i\in I}$ are closed subsets of $\mathcal{P}$, then $\mathcal{C} \cap \bigcup_{i\in I} \mathcal{C}_i = \bigcup_{i\in I}(\mathcal{C} \cap \mathcal{C}_i)$.

2. Clearly (b) implies (a). Moreover (c) implies (b): if P is a p-group, then by Lemma 2.9

$$\Psi\big(\langle k_P \rangle\big) = \{Q \in \mathcal{P} \mid Q \hookrightarrow P\}.$$

Hence if $(\mathcal{C}_i)_{i\in I}$ is a set of closed subsets of $\mathcal{P}$, then $\Psi\big(\langle k_P \rangle\big) \subseteq \bigcup_{i\in I} \mathcal{C}_i$ if and only if there exists some $i \in I$ such that $P \in \mathcal{C}_i$, i.e. $\Psi\big(\langle k_P \rangle\big) \subseteq \mathcal{C}_i$. So $\Psi\big(\langle k_P \rangle\big)$ is completely join prime in $\mathcal{F}$, and $\langle k_P \rangle$ is completely join prime in $\mathcal{L}$.

Finally (a) implies (c): if F is completely join irreducible in $\mathcal{L}$, since $F = \sum_{k_P \in F(P)} \langle k_P \rangle$, there is a p-group P such that $F = \langle k_P \rangle$. Moreover P is unique up to isomorphism, by Lemma 2.9. □

Remark 4.2: There are join irreducible elements of $\mathcal{L}$, or equivalently of $\mathcal{F}$, which are *not* completely join irreducible: Let for example $\mathcal{C}$ be the subset of $\mathcal{P}$ consisting of cyclic p-groups. Then $\mathcal{C}$ is closed, and one checks easily that $\mathcal{C}$ is join irreducible in $\mathcal{F}$, but not completely join irreducible.

5 Some simple diagonal p-permutation functors

Proposition 5.1: *Let P be a finite p-group. Then $\langle k_P \rangle$ has a unique (proper) maximal subfunctor J_P, defined for a finite group G by*

$$J_P(G) = \big\{u \in \langle k_P \rangle(G) \mid kT^{\Delta}(P,G)(u) = 0\big\}.$$

Moreover $J_P = \sum_{\substack{Q \hookrightarrow P \\ Q \not\cong P}} \langle k_Q \rangle.$

Proof: First it is easy to check that the assignment $G \mapsto J_P(G)$ defines a subfunctor of kR_k, hence of $\langle k_P\rangle$. Moreover $J_P(P) = 0$. Now Lemma 2.9 implies that $\langle k_P\rangle(P) = k$, so J_P is a proper subfunctor of $\langle k_P\rangle$. If F is a subfunctor of $\langle k_P\rangle$, there are two possibilities: Either $F(P) = k$, and then $k_P \in F(P)$, so $F = \langle k_P\rangle$. Or $F(P) = 0$, and then for any finite group G, we have $kT^\Delta(P,G)\big(F(G)\big) \leq F(P) = 0$, that is $F \leq J_P$.

For the last assertion, denote by $]., P]$ the subset of $\mathcal{P}$ consisting of p-groups isomorphic to a subgroup of P, and by $]., P[$ the subset of $]., P]$ consisting of p-groups isomorphic to a *proper* subgroup of P. Then clearly $]., P] \in \mathcal{F}$, and $\Theta(]., P]) = \langle k_P\rangle$, by Lemma 2.9. Also $]., P[\in \mathcal{F}$, and $\Theta\big(]., P[\big) = \sum\limits_{\substack{Q \hookrightarrow P\\ Q \not\cong P}} \langle k_Q\rangle$, by definition of Θ. Now$]., P[$ is clearly the unique maximal proper closed subset of $]., P]$, so $\Theta\big(]., P[\big) = J_P$, by Theorem 1.3. This completes the proof. □

Notation 5.2: *For a finite p-group P, we denote by*

$$S_P = \langle k_P\rangle / J_P$$

the unique simple quotient of $\langle k_P\rangle$.

Lemma 5.3: *Let G be a finite group, and P be a finite p-group.*

1. *If $S_P(G) \neq 0$, then $P \hookrightarrow G$.*
2. *If Q is a finite p-group, then*

$$S_P(Q) = \begin{cases} k & \text{if } Q \cong P \\ 0 & \text{otherwise.} \end{cases}$$

In particular $S_P \cong S_Q$ if and only if $P \cong Q$.

Proof: 1. By Theorem 3.1, we have

$$\langle k_P\rangle(G) = \sum_{\substack{Q\in\mathcal{P},\, Q\leq G\\ k_Q \in \langle k_P\rangle(Q)}} \langle k_Q\rangle(G).$$

Now $k_Q \in \langle k_P\rangle(Q)$ if and only if $\langle k_Q\rangle \leq \langle k_P\rangle$, i.e. $Q \hookrightarrow P$, by Lemma 2.9. So if P is not isomorphic to a subgroup of G, then $Q \leq G$ and $Q \hookrightarrow P$ implies that Q is isomorphic to a proper subgroup of P, and then $\langle k_Q\rangle(G) \leq J_P(G)$, by Proposition 5.1. It follows that $\langle k_P\rangle(G) \leq J_P(G)$, so $S_P(G) = 0$.

2. Indeed, if $S_P(Q) \neq 0$, then in particular $\langle k_P\rangle(Q) \neq 0$, so $Q \hookrightarrow P$ by Lemma 2.9. But also $P \hookrightarrow Q$ by Assertion 1. So $S_P(Q) = 0$ unless $Q \cong P$. Moreover $S_P(P) = \langle k_P\rangle(P)/J_P(P) = k/\{0\} \cong k$, which completes the proof. □

Theorem 5.4:

1. *Let $F_2 < F_1$ be subfunctors of kR_k such that F_1/F_2 is a simple functor. Then there exists a (unique, up to isomorphism) finite p-group P such that $F_1/F_2 \cong S_P$.*

2. *There exists a filtration*

$$0 = F_0 < F_1 < \ldots < F_n < F_{n+1} < \ldots$$

of kR_k by subfunctors F_i, for $i \in \mathbb{N}$, such that:

   (a) $\bigcup_{i=0}^{\infty} F_i = kR_k$.

   (b) *For $i > 0$, the functor F_i/F_{i-1} is simple, isomorphic to S_{P_i}, for a finite p-group P_i.*

   (c) *For every finite p-group P, there is exactly one integer $i > 0$ such that $P_i \cong P$.*

Proof: 1. Set $\mathcal{C}_i = \Psi(F_i)$, for $i \in \{1, 2\}$. Then $\mathcal{C}_1$ and $\mathcal{C}_2$ are closed subsets of $\mathcal{P}$, and $\mathcal{C}_2 \subset \mathcal{C}_1$. Since F_1/F_2 is simple, any subfunctor F of kR_k such that $F_2 \leq F \leq F_1$ is equal either to F_1 or F_2. Then any closed subset $\mathcal{C}$ of $\mathcal{P}$ such that $\mathcal{C}_2 \subseteq \mathcal{C} \subseteq \mathcal{C}_1$ is equal either to $\mathcal{C}_1$ or $\mathcal{C}_2$. If $P \in \mathcal{C}_1 - \mathcal{C}_2$, then $\mathcal{C}_2 \cup]., P]$ is closed, different from $\mathcal{C}_2$, and contained in $\mathcal{C}_1$. So $\mathcal{C}_2 \cup]., P] = \mathcal{C}_1$. Now if $P' \in \mathcal{C}_1 - \mathcal{C}_2$, then $P' \in]., P]$, and $P \in]., P']$, by exchanging the roles of P and P'. It follows that $P \cong P'$, so $\mathcal{C}_1 - \mathcal{C}_2 = \{[P]\}$. Now

$$F_1 = \Theta(\mathcal{C}_1) = \Theta(\mathcal{C}_2) + \langle k_P \rangle = F_2 + \langle k_P \rangle.$$

It follows that $F_1/F_2 \cong \langle k_P \rangle / (\langle k_P \rangle \cap F_2)$ is a simple quotient of $\langle k_P \rangle$, so $F_1/F_2 \cong S_P$. The uniqueness of P (up to isomorphism) with this property follows from Lemma 5.3.

2. Choose an enumeration $P_1, P_2, \ldots, P_n, \ldots$ of $\mathcal{P}$ such that for any indices i and j, $P_i \hookrightarrow P_j$ implies $i \leq j$. This can be achieved starting with $P_1 = \mathbf{1}$, $P_2 = C_p$, and then enumerating all the p-groups of order p^2, then the groups of order p^3, and so on. With such a numbering, set $\mathcal{C}_0 = \emptyset$ and $\mathcal{C}_n = \cup_{i \leq n}]., P_i]$ for $n > 0$, and then set $F_n = \Theta(\mathcal{C}_n)$ for $n \in \mathbb{N}$. In particular $F_0 = 0$.

Since $\mathcal{C}_n = \mathcal{C}_{n-1} \cup]., P_n]$ for $n > 0$, the sequence $(\mathcal{C}_n)_{n \in \mathbb{N}}$ is increasing. Moreover $P_n \in \mathcal{C}_n - \mathcal{C}_{n-1}$, for otherwise $P_n \in]., P_i]$ for some $i < n$, meaning that $P_n \hookrightarrow P_i$, which implies $n \leq i$, a contradiction. So the sequence $(\mathcal{C}_n)_{n \in \mathbb{N}}$ is stricly increasing, and its union is the whole of $\mathcal{P}$. In other words, the sequence $(F_n)_{n \in \mathbb{N}}$ is stricly increasing and its union is equal to kR_k, which proves (a).

Since $\mathcal{C}_i = \mathcal{C}_{i-1} \cup]., P_i]$ for $i > 0$, and since $]., P_i[\subseteq \mathcal{C}_{i-1}$ by our choice of the numbering of $\mathcal{P}$, it follows that $\mathcal{C}_i = \mathcal{C}_{i-1} \cup \{P_i\}$, and then $F_i/F_{i-1} \cong S_{P_i}$ as in the proof of Assertion 1. This proves (b). Now (c) is clear, since any finite p-group P appears exactly once in our enumeration. □

Remark 5.5: Theorem 5.4 shows that kR_k admits a "composition series", where the "composition factors" are the simple functors S_P, for $P \in \mathcal{P}$, and each simple functor S_P appears exactly once. The quote signs in the previous sentence indicate that one should be careful with the notions of composition factors and composition series for diagonal p-permutation functors.

Remark 5.6: Proposition 5.1 and Lemma 5.3 show that P is a minimal group for S_P: More precisely $S_P(P)$ is one dimensional, and $S_P(G) = \{0\}$ for any group G of order smaller than $|P|$. Moreover P is unique (up to isomorphism) with this property. In addition $S_P(P)$ is generated by the image of the trivial module k_P, and in particular the group $\mathrm{Out}(P)$ of outer automorphisms of P acts trivially on $S_P(P)$. These two facts show that with the notation of Theorem 5.25 of [4], the functor S_P is isomorphic to the simple functor $\mathsf{S}_{P,1,k}$.

6 The simple functor S_1

We consider first the case where the p-group P is trivial[2]. In this case $S_P = S_1 = \langle k_1 \rangle \leq kR_k$, so for a finite group G

$$S_1(G) = kT^\Delta(G,1)(k_1).$$

Now $T^\Delta(G,1)$ is the group $P_k(G)$ of projective kG-modules, and we set set $kP_k(G) = k \otimes_{\mathbb{Z}} P_k(G)$. So $S_1(G) = \langle k_1 \rangle(G)$ is equal to the image of the map $k\mathsf{c}^G : kP_k(G) \to kR_k(G)$ linearly extending the Cartan map $\mathsf{c}^G : P_k(G) \to R_k(G)$ sending a projective kG-module to the sum of its composition factors.

Remark 6.1: The functor S_1 is the only simple subfunctor of kR_k: indeed, any non-empty closed subset of $\mathcal{P}$ must contain the trivial group, so any non-zero subfunctor of kR_k must contain $\langle k_1 \rangle = S_1$.

As there are non-empty closed subsets of $\mathcal{P}$ different from $\{1\}$, this shows that S_1 is a proper subfunctor of kR_k, so kR_k is not a semisimple diagonal p-permutation functor. It follows that in contrast to [3], the category of diagonal p-permutation functors over a field k *of characteristic* p is *not semisimple.*

[2]The content of the present section 6 is essentially the same as Subsection 6.5 of [4]. We include it here for the reader's convenience.

We choose a p-modular system $(K, \mathcal{O}, k)$, and we assume that K is big enough for the group G. If S is a simple kG-module, we denote by P_S its projective cover as kG-module, and by $\Phi_S : G_{p'} \to \mathcal{O}$ the modular character of P_S, where $G_{p'}$ is the set of p-regular elements of G. If $v = \sum\limits_{S \in \mathrm{Irr}_k(G)} \omega_S \mathsf{P}_S$, where $\omega_S \in \mathcal{O}$, is an element of $\mathcal{O}P_k(G) := \mathcal{O} \otimes_{\mathbb{Z}} P_k(G)$, we denote by Φ^v the map $\sum\limits_{S \in \mathrm{Irr}_k(G)} \omega_S \Phi_S$ from $G_{p'}$ to $\mathcal{O}$, and we call Φ^v the modular character of v.

Then for a simple kG-module T, the multiplicity of T as a composition factor of P_S is equal to the Cartan coefficient

$$\mathsf{c}^G_{T,S} = \dim_k \mathrm{Hom}_{kG}(\mathsf{P}_T, \mathsf{P}_S) = \frac{1}{|G|} \sum_{x \in G_{p'}} \Phi_T(x) \Phi_S(x^{-1}).$$

In order to describe the image of the map $k\mathsf{c}^G$, we want to evaluate the image of this integer under the projection map $\rho : \mathcal{O} \to k$. For this, we denote by $[G_{p'}]$ a set of representatives of conjugacy classes of $G_{p'}$, and we observe that in the field K, we have

$$\begin{aligned} \mathsf{c}^G_{T,S} &= \frac{1}{|G|} \sum_{x \in [G_{p'}]} \frac{|G|}{|C_G(x)|} \Phi_T(x) \Phi_S(x^{-1}) \\ &= \sum_{x \in [G_{p'}]} \frac{1}{|C_G(x)|} \frac{\Phi_T(x)}{|C_G(x)|_p} \frac{\Phi_S(x^{-1})}{|C_G(x)|_p} |C_G(x)|_p^2 \\ &= \sum_{x \in [G_{p'}]} \frac{\big(\Phi_T(x)/|C_G(x)|_p\big)\big(\Phi_S(x^{-1})/|C_G(x)|_p\big)}{|C_G(x)_{p'}|} |C_G(x)|_p. \end{aligned} \tag{6.2}$$

But since Φ_S and Φ_T are modular characters of projective kG-modules, and since $C_G(x) = C_G(x^{-1})$, the quotients $\Phi_T(x)/|C_G(x)|_p$ and $\Phi_S(x^{-1})/|C_G(x)|_p$ are in $\mathcal{O}$, so

$$\forall x \in [G_{p'}], \ \frac{\big(\Phi_T(x)/|C_G(x)|_p\big)\big(\Phi_S(x^{-1})/|C_G(x)|_p\big)}{|C_G(x)_{p'}|} \in \mathcal{O}.$$

Now it follows from 6.2 that

$$\rho(\mathsf{c}^G_{T,S}) = \sum_{x \in [G_0]} \rho\left(\frac{\Phi_T(x)\Phi_S(x^{-1})}{|C_G(x)|}\right), \tag{6.3}$$

where $[G_0]$ is a set of representatives of conjugacy classes of the set G_0 of elements of *defect zero* of G, i.e. the set of p-regular elements x such that $C_G(x)$ is a p'-group.

Notation 6.4: *For $x \in G_0$, we set*

$$\Gamma_{G,x} = \sum_{S \in \mathrm{Irr}(kG)} \frac{\Phi_S(x^{-1})}{|C_G(x)|} S \in \mathcal{O}R_k(G),$$

where $\mathrm{Irr}(kG)$ *is a set of representatives of isomorphism classes of simple kG-modules. We also set*

$$\gamma_{G,x} = \sum_{S \in \mathrm{Irr}(kG)} \rho\left(\frac{\Phi_S(x^{-1})}{|C_G(x)|}\right) S \in kR_k(G),$$

Remark 6.5: We note that $\Gamma_{G,x}$ and $\gamma_{G,x}$ only depend on the conjugacy class of x in G, that is $\Gamma_{G,x} = \Gamma_{G,x^g}$ and $\gamma_{G,x} = \gamma_{G,x^g}$ for $g \in G$.

By Section III.16 of [5] (see also Theorem 3.6.32 of [7]), the elementary divisors of the Cartan matrix of G are equal to $|C_G(x)|_p$, for $x \in [G_{p'}]$. It follows that the rank of the Cartan matrix modulo p, is equal to the number of conjugacy classes of elements of defect 0 of G, i.e. the cardinality of $[G_0]$. The following can be viewed as an explicit form of this result:

Proposition 6.6:

1. *Let T be a simple kG-module. Then, in $kR_k(G)$,*

$$k\mathsf{c}^G(\mathsf{P}_T) = \sum_{x \in [G_0]} \rho\big(\Phi_T(x)\big)\, \gamma_{G,x}.$$

2. *The elements $\gamma_{G,x}$, for $x \in [G_0]$, form a basis of $S_1(G) \leq kR_k(G)$.*

Proof: Throughout the proof, we simply write γ_x instead of $\gamma_{G,x}$.

1. By 6.3, we have

$$\begin{aligned}
k\mathsf{c}^G(\mathsf{P}_T) &= \sum_{S \in \mathrm{Irr}(kG)} \rho(\mathsf{c}^G_{T,S}) S = \sum_{S \in \mathrm{Irr}(kG)} \sum_{x \in [G_0]} \rho\left(\frac{\Phi_T(x)\Phi_S(x^{-1})}{|C_G(x)|}\right) S \\
&= \sum_{x \in [G_0]} \sum_{S \in \mathrm{Irr}(kG)} \rho\left(\frac{\Phi_T(x)\Phi_S(x^{-1})}{|C_G(x)|}\right) S \\
&= \sum_{x \in [G_0]} \rho\big(\Phi_T(x)\big) \sum_{S \in \mathrm{Irr}(kG)} \rho\left(\frac{\Phi_S(x^{-1})}{|C_G(x)|}\right) S \\
&= \sum_{x \in [G_0]} \rho\big(\Phi_T(x)\big)\, \gamma_x.
\end{aligned}$$

2. We first prove that γ_x lies in the image of $k\mathsf{c}^G$, for any $x \in G_0$. So let $x \in G_0$, and $1_x : \langle x\rangle \to \mathcal{O}$ be the map with value 1 at x and 0 elsewhere. Then $|x|1_x = \sum_\zeta \zeta(x^{-1})\zeta$, where ζ runs through the modular characters of the simple $k\langle x\rangle$-modules, i.e. the group homomorphisms $\langle x\rangle \to \mathcal{O}^\times$, is an element of $\mathcal{O}P_k(\langle x\rangle) = \mathcal{O}R_k(\langle x\rangle)$. Let $v_x = \mathrm{Ind}_{\langle x\rangle}^G(|x|1_x)$. Then $v_x \in \mathcal{O}P_k(G)$, and its modular character evaluated at $g \in G$ is equal to

$$\begin{aligned}\Phi^{v_x}(g) &= \frac{1}{|x|}\sum_{\substack{h\in G\\ g^h\in\langle x\rangle}} \Phi^{|x|1_x}(g^h)\\ &= \frac{1}{|x|}\sum_{\substack{h\in G\\ g^h=x}} |x| = \begin{cases} |C_G(x)| & \text{if } g =_G x\\ 0 & \text{otherwise,}\end{cases}\end{aligned} \tag{6.7}$$

where $g =_G x$ means that g is conjugate to x in G. Now from Assertion 1, we get that

$$k\mathsf{c}^G(v_x) = \sum_{y\in[G_0]} \rho\big(\Phi^{v_x}(y)\big)\gamma_y = |C_G(x)|\gamma_x, \tag{6.8}$$

so γ_x is in the image of $k\mathsf{c}^G$, since $|C_G(x)| \neq 0$ in k.

Now by Assertion 1, the elements γ_x, for $x \in [G_0]$, generate the image of $k\mathsf{c}^G$, i.e. $S_1(G)$. They are moreover linearly independent: Suppose indeed that some linear combination $\sum\limits_{x\in[G_0]} \lambda_x\gamma_x$, where $\lambda_x \in k$, is equal to 0. For all $x \in [G_0]$, choose $\widetilde{\lambda}_x \in \mathcal{O}$ such that $\rho(\widetilde{\lambda}_x) = \lambda_x$. By (6.8), we get an element $\sum\limits_{x\in[G_0]} \widetilde{\lambda}_x \frac{v_x}{|C_G(x)|}$ of $\mathcal{O}P_k(G)$ whose modular character has values in the maximal ideal $J(\mathcal{O})$ of $\mathcal{O}$. But by (6.7), the value at $g \in G_{p'}$ of this modular character is equal to

$$\sum_{x\in[G_0]} \widetilde{\lambda}_x \frac{\Phi^{v_x}(g)}{|C_G(x)|},$$

which is equal to 0 if $g \notin G_0$, and to $\widetilde{\lambda}_x$ if g is conjugate to $x \in [G_0]$ in G. It follows that $\widetilde{\lambda}_x \in J(\mathcal{O})$, hence $\lambda_x = \rho(\widetilde{\lambda}_x) = 0$. Since $g \in G_{p'}$ was arbitrary, we get that $\lambda_x = 0$ for any $x \in [G_0]$, so the elements γ_x, for $x \in [G_0]$, are linearly independent. This completes the proof of Proposition 6.6. □

7 The simple functors S_P

In this section, we generalize the results of Section 6 to the functor S_P, for an arbitrary finite p-group P.

Theorem 7.1: *Let P be a finite p-group. Then for any finite group G, the evaluation $\langle k_P\rangle(G)$ is generated by the elements of $kR_k(G)$ of the form*

$$\mathrm{Ind}_{RC_G(R)}^{G}\mathrm{Inf}_{RC_G(R)/R}^{RC_G(R)} k\mathsf{c}^{RC_G(R)/R}(F),$$

where R is a subgroup of G such that $R \hookrightarrow P$, and F is an indecomposable projective $kRC_G(R)/R$-module.

Proof: By definition $\langle k_P\rangle(G) = kT^{\Delta}(G,P)(k_P)$, and $T^{\Delta}(G,P)$ is generated by the bimodules of the form $\mathrm{Ind}_{N_{R,\pi,Q}}^{G\times P}\mathrm{Inf}_{\overline{N}_{R,\pi,Q}}^{N_{R,\pi,Q}} E$, where:

- $N_{R,\pi,Q}$ is the normalizer in $G\times P$ of a diagonal p-subgroup $\Delta(R,\pi,Q) = \{(\pi(q),q) \mid q \in Q\}$, where $\pi : Q \to R$ is a group isomorphism from a subgroup Q of P to a subgroup R of G.
- $\overline{N}_{R,\pi,Q} = N_{R,\pi,Q}/\Delta(R,\pi,Q)$.
- E is an indecomposable projective $k\overline{N}_{R,\pi,Q}$-module.

Then $N_{R,\pi,Q} = \big\{(a,b) \in G\times P \mid \forall q \in Q,\ {}^{a}\pi(q) = \pi({}^{b}q)\big\}$. Let $\hat{Q} \leq P$ denote the second projection of $N_{R,\pi,Q}$. Then $\overline{N}_{R,\pi,Q}$ fits in a short exact sequence

$$1 \longrightarrow C_G(R) \xrightarrow{\ i\ } \overline{N}_{R,\pi,Q} \xrightarrow{\ s\ } \hat{Q}/Q \longrightarrow 1,$$

where i is the map $x \mapsto (x,1)\Delta(R,\pi,Q)$ from $C_G(R)$ to $\overline{N}_{R,\pi,Q}$, and s maps $(a,b)\Delta(R,\pi,Q) \in \overline{N}_{R,\pi,Q}$ to $bQ \in \hat{Q}/Q$. Since $\hat{Q}/Q$ is a p-group and k is algebraically closed, it follows from Corollary 5.12.4 of [6] that there exists an indecomposable projective $kC_G(R)$-module M such that

$$E \cong \mathrm{Ind}_{C_G(R)}^{\overline{N}_{R,\pi,Q}} M.$$

Then

$$\begin{aligned}\mathrm{Ind}_{N_{R,\pi,Q}}^{G\times P}\mathrm{Inf}_{\overline{N}_{R,\pi,Q}}^{N_{R,\pi,Q}} E &\cong \mathrm{Ind}_{N_{R,\pi,Q}}^{G\times P}\mathrm{Inf}_{\overline{N}_{R,\pi,Q}}^{N_{R,\pi,Q}}\mathrm{Ind}_{C_G(R)}^{\overline{N}_{R,\pi,Q}} M\\ &\cong \mathrm{Ind}_{N_{R,\pi,Q}}^{G\times P}\mathrm{Ind}_{\Delta(R,\pi,Q)(C_G(R)\times 1)}^{N_{R,\pi,Q}}\mathrm{Inf}_{C_G(R)}^{\Delta(R,\pi,Q)(C_G(R)\times 1)} M\\ &\cong \mathrm{Ind}_{\Delta(R,\pi,Q)(C_G(R)\times 1)}^{G\times P}\mathrm{Inf}_{C_G(R)}^{\Delta(R,\pi,Q)(C_G(R)\times 1)} M.\end{aligned}$$

Now the tensor product $T := \Big(\mathrm{Ind}_{N_{R,\pi,Q}}^{G\times P}\mathrm{Inf}_{\overline{N}_{R,\pi,Q}}^{N_{R,\pi,Q}} E\Big) \otimes_{kP} k_P$ can be viewed as

$$\Big(\mathrm{Ind}_{X}^{G\times P}\mathrm{Inf}_{C_G(R)}^{X} M\Big) \otimes_{kP} \Big(\mathrm{Ind}_{Y}^{P\times 1} k_P\Big),$$

where $X = \Delta(R,\pi,Q)(C_G(R)\times 1) \leq G\times P$ and $Y = P\times 1 \leq P\times 1$. It follows from [1] that this tensor product is isomorphic to

$$T \cong \bigoplus_{x \in p_2(X)\backslash P/p_1(Y)} \mathrm{Ind}_{X*^{(x,1)}Y}^{G\times 1}\Big(\big(\mathrm{Inf}_{C_G(R)}^{X} M\big) \otimes_{k[k_2(X)\cap {}^{x}k_1(Y)]} {}^{x}k_P\Big), \quad (7.2)$$

where $p_2(X)$ is the second projection of X, and $p_1(Y)$ the first projection of $Y = P \times 1$. In particular $p_1(Y) = P$, so $p_2(X)\backslash P/p_1(Y)$ consists of a single double coset, and we can assume $x = 1$ in (7.2).

Moreover $k_1(Y) = \{y \in P \mid (y, 1) \in Y\} = P$, and

$$\begin{aligned} k_2(X) &= \{y \in P \mid (1, y) \in X\} = \{q \in Q \mid \pi(q) \in C_G(R)\} \\ &= \pi^{-1}\big(R \cap C_G(R)\big) = Z(Q) \cong Z(R). \end{aligned}$$

Now $X * Y = \{(a, b) \in G \times 1 \mid \exists c \in P,\ (a, c) \in X,\ (x, 1) \in Y\}$ is equal to the first projection of X, that is $RC_G(R)$. We get finally

$$\begin{aligned} T &\cong \mathrm{Ind}_{RC_G(R)\times 1}^{G\times 1}\Big(\big(\mathrm{Inf}_{C_G(R)}^{\Delta(R,\pi,Q)(C_G(R)\times 1)} M\big) \otimes_{kZ(Q)} k_P\Big) \\ &\cong \mathrm{Ind}_{RC_G(R)}^{G}\mathrm{Inf}_{C_G(R)/Z(R)}^{RC_G(R)} M_{Z(R)}, \end{aligned}$$

where $M_{Z(R)} = M \otimes_{kZ(R)} k$, viewed as a $kC_G(R)/Z(R)$-module. The inflation symbol $\mathrm{Inf}_{C_G(R)/Z(R)}^{RC_G(R)}$ stands more precisely for $\mathrm{Inf}_{RG_G(R)/R}^{RC_G(R)}\mathrm{Iso}_{C_G(R)/Z(R)}^{RC_G(R)/R}$.

To complete the proof of Theorem 7.1, it remains to observe that the construction $M \mapsto M_{Z(R)}$ induces a bijection between the set of isomorphism classes of projective indecomposable $kC_G(R)$-modules and the set of isomorphism classes of projective indecomposable $kC_G(R)/Z(R)$-modules. Transporting this module via the isomorphism $C_G(R)/Z(R) \to RC_G(R)/R$ gives the indecomposable projective $kRC_G(R)/R$-module $F := \mathrm{Iso}_{C_G(R)/Z(R)}^{RC_G(R)/R} M_{Z(R)}$. Moreover F has to be viewed as an element of $kR_k\big(RC_G(R)/R\big)$, that is as $k\mathsf{c}^{RC_G(R)/R}(F)$. This completes the proof of Theorem 7.1. □

Definition 7.3: *Let G be a finite group, and x be a p-regular element of G.*

1. *Let P be a p-subgroup of G. Then x is said to* have defect group P *in G if P is conjugate in G to a Sylow p-subgroup of $C_G(x)$. A conjugacy class of p-regular elements of G is said to have defect group P if some of its elements (or equivalently all of its elements) have defect group P.*

2. *Let P be a finite p-group. Then x is said to have* defect group isomorphic to P *if P is isomorphic to a Sylow p-subgroup of $C_G(x)$. A conjugacy class of p-regular elements of G is said to have defect group isomorphic to P if some of its elements (or equivalently all of its elements) have defect group isomorphic to P.*

Remark 7.4: In both cases, if $|P| = p^d$, recall that x (or its conjugacy class in G) is said to have *defect* d in G.

Lemma 7.5: *Let G be a finite group, and R be a p-subgroup of G.*

1. *Let $xR \in \big(RC_G(R)/R\big)_{p'}$. Then xR has defect 0 in $RC_G(R)/R$ if and only if there exists an element $y \in C_G(R)_{p'} \cap xR$ with defect group R in $RC_G(R)$, i.e. such that $RC_G(R,y)/R$ is a p'-group.*

2. *Let $y \in C_G(R)_{p'}$ such that $yR \in \big(RC_G(R)/R\big)_0$. Then*

$$\mathrm{Ind}_{RC_G(R)}^G \mathrm{Inf}_{RC_G(R)/R}^{RC_G(R)} \gamma_{RC_G(R)/R,\, yR} = 0 \in kR_k(G)$$

unless y has defect group R in G.

Proof: 1. It is well known that since R is a p-group, a p' element xR of $RC_G(R)/R$ can be lifted to a p'-element $y \in xR$ of $RC_G(R)$. Moreover $\big(RC_G(R)\big)_{p'} = C_G(R)_{p'}$, so $y \in C_G(R)_{p'}$, and the centralizer H of $yR = xR$ in $RC_G(R)/R$ is the image in $RC_G(R)/R$ of the centralizer of y in $RC_G(R)$. Thus $H = RC_G(R,y)/R$, and H is a p'-group if and only if R is a Sylow subgroup of $RC_G(R,y)$, that is if and only if y has defect group R in $RC_G(R)$.

2. The normalizer $N_G(R)$ normalizes $RC_G(R)$, and it acts on the conjugacy classes of $RC_G(R)/R$ by conjugation. Let T_y be the stabilizer in $N_G(R)$ of the conjugacy class of yR in $RC_G(R)/R$. Then $g \in T_y$ if and only if ${}^gy \in {}^c(yR) = {}^cyR$ for some $c \in RC_G(R)$, that is ${}^gy = {}^cyr$, for some element r of R. But $y \in C_G(R)$, so ${}^cy \in C_G({}^cR) = C_G(R)$, hence the p'-element ${}^cy$ commutes with the p-element r. Then r is the p-part of ${}^cyr = {}^gy$ which is a p'-element. This forces $r = 1$, and ${}^gy = {}^cy$. Then $g \in cC_G(y)$, hence $g \in RC_G(R)C_G(y) \cap N_G(R) = RC_G(R)N_G(R,y)$. Thus $T_y \leq RC_G(R)N_G(R,y)$. Conversely, it is clear that $RC_G(R)N_G(R,y)$ stabilizes the conjugacy class of yR in $RC_G(R)/R$, so $T_y = RC_G(R)N_G(R,y)$.

Let S be a Sylow p-subgroup of T_y containing R. For simplicity, we set $W_y := \mathrm{Inf}_{RC_G(R)/R}^{RC_G(R)} \Gamma_{RC_G(R)/R,yR}$ and $w_y := \mathrm{Inf}_{RC_G(R)/R}^{RC_G(R)} \gamma_{RC_G(R)/R,yR}$. Then

$$\mathrm{Ind}_{RC_G(R)}^G w_y = \mathrm{Ind}_{SC_G(R)}^G \mathrm{Ind}_{RC_G(R)}^{SC_G(R)} w_y,$$

and we want to show that this is equal to zero unless y has defect group R in G. We will show a little more: We claim that $\mathrm{Ind}_{RC_G(R)}^{SC_G(R)} w_y = 0$ in $kR_k\big(SC_G(R)\big)$ if R is not a Sylow p-subgroup of $C_G(y)$.

To prove this claim, we compute the coefficients of $\mathrm{Ind}_{RC_G(R)}^{SC_G(R)} w_y$ in the basis of $kR_k\big(SC_G(R)\big)$ consisting of the simple $k\big(SC_G(R)\big)$-modules. Since w_y is the reduction in k of W_y, we can use the modular character θ_y associated to W_y to do this computation. Let U be a simple $k\big(SC_G(R)\big)$-module. Since $RC_G(R)$ is a normal subgroup of $SC_G(R)$ with p-power index, it follows from Corollary 5.12.4 of [6] that the projective cover of U is isomorphic to $\mathrm{Ind}_{RC_G(R)}^{SC_G(R)} E$, where E is a projective $kRC_G(R)$-module. Let Φ^E

denote the modular character of E. The coefficient of U in the expression of $\mathrm{Ind}_{RC_G(R)}^{SC_G(R)} w_y$ in the basis of simple $kSC_G(R)$-modules is then equal to $\rho(m_U)$, where

$$\begin{aligned} m_U &= \langle \mathrm{Ind}_{RC_G(R)}^{SC_G(R)} \Phi^E, \mathrm{Ind}_{RC_G(R)}^{SC_G(R)} \theta_y \rangle_{SC_G(R)} \\ &= \langle \Phi^E, \mathrm{Res}_{RC_G(R)}^{SC_G(R)} \mathrm{Ind}_{RC_G(R)}^{SC_G(R)} \theta_y \rangle_{RC_G(R)} \\ &= \sum_{g \in SC_G(R)/RC_G(R)} \langle \Phi^E, {}^g\theta_y \rangle_{RC_G(R)} \\ &= |SC_G(R)/RC_G(R)| \langle \Phi^E, \theta_y \rangle_{RC_G(R)}, \end{aligned}$$

since ${}^g\theta_y = \theta_y$ for $g \in SC_G(R)$, as $SC_G(R) \leq T_y$.

Then $\rho(m_U) = 0$ in k if $SC_G(R) \neq RC_G(R)$, or equivalently if p divides $|T_y : RC_G(R)| = |N_G(R, y) : RC_G(R, y)|$, that is, since $RC_G(R, y)/R$ is a p'-group by Assertion 1, if p divides $|N_G(R, y) : R|$. Hence if $\rho(m_U) \neq 0$, then R is a Sylow p-subgroup of $N_G(R, y) = N_{C_G(y)}(R)$, hence R is a Sylow p-subgroup of $C_G(y)$, which proves the claim. This completes the proof of Lemma 7.5. □

Corollary 7.6: *Let G be a finite group, and P be a finite p-group. Let $[G_{\hookrightarrow P}]$ be a set of representatives of conjugacy classes of the set $G_{\hookrightarrow P}$ of p-regular elements of G with defect group $R \hookrightarrow P$. For $x \in G_{\hookrightarrow P}$, let R_x be a chosen Sylow p-subgroup of $C_G(x)$. Then the elements*

$$U_x = \mathrm{Ind}_{R_x C_G(R_x)}^G \mathrm{Inf}_{R_x C_G(R_x)/R_x}^{R_x C_G(R_x)} \gamma_{R_x C_G(R_x)/R_x, xR_x}$$

for $x \in [G_{\hookrightarrow P}]$, form a basis of $\langle k_P \rangle(G)$.

Proof: By Lemma 7.5, Theorem 7.1, and Proposition 6.6, the elements U_x, for $x \in [G_{\hookrightarrow P}]$, generate $\langle k_P \rangle(G)$, and all we have to show is that these elements are linearly independent. Let S be a Sylow p-subgroup of G. If $x \in [G_{\hookrightarrow P}]$ has defect group $R \leq G$, then $R \hookrightarrow S$, so $U_x \in \langle k_S \rangle(G)$. So it is enough to prove that the elements U_x, for $x \in [G_{\hookrightarrow S}]$, are linearly independent. But $\langle k_S \rangle(G) = kR_k(G)$ by Corollary 3.4, and $\big|[G_{\hookrightarrow S}]\big| = \big|[G_{p'}]\big|$ as any element of $G_{p'}$ has defect group R for some $R \hookrightarrow S$. Hence

$$\big|[G_{p'}]\big| = \dim_k kR_k(G) = \dim_k \langle k_S \rangle(G) \leq \big|[G_{\hookrightarrow S}]\big| = \big|[G_{p'}]\big|,$$

so $\dim_k \langle k_S \rangle(G) = \big|[G_{\hookrightarrow S}]\big|$, which completes the proof. □

Remark 7.7: Observe that for $x \in G_{\hookrightarrow P}$, the element U_x does not depend on the choice of the Sylow p-subgroup R_x. Moreover $U_x = U_{x^g}$, for any $g \in G$. So the set $\{U_x \mid x \in [G_{\hookrightarrow P}]\}$ is a canonical basis of $\langle k_P \rangle(G)$.

Theorem 7.8: *Let G be a finite group, and P be a finite p-group. Let $G_{\cong P}$ denote the set of p-regular elements of G with defect group isomorphic to P, and $[G_{\cong P}]$ be a set of representatives of conjugacy classes of elements of $G_{\cong P}$. Then the images of the elements U_x, for $x \in [G_{\cong P}]$, under the projection $\langle k_P\rangle(G) \twoheadrightarrow S_P(G)$, form a basis of $S_P(G)$.*

Proof: Let $x \in G_{\hookrightarrow P}$, and $R = R_x$ be a Sylow subgroup of $C_G(x)$. Then the element U_x defined in Corollary 7.6 lies in $\langle k_R\rangle(G)$, so U_x is sent to 0 under the projection $\langle k_P\rangle(G) \twoheadrightarrow S_P(G)$ if $R \not\cong P$, by Proposition 5.1. It follows that the images of the elements U_x, for $x \in [G_{\cong P}]$, under the projection $\langle k_P\rangle(G) \twoheadrightarrow S_P(G)$, generate $S_P(G)$. It follows that $\dim_k S_P(G) \leq \big|[G_{\cong P}]\big|$. In particular $S_P(G) = 0$ if P is not isomorphic to a subgroup of G (so Theorem 7.8 holds in this case).

Now, as in the proof of Theorem 5.4, we choose an enumeration $P_1, P_2, \ldots$ of $\mathcal{P}$ with the property that $P_i \hookrightarrow P_j$ implies $i \leq j$. This gives a filtration

$$0 = F_0 < F_1 < \ldots < F_n < F_{n+1} < \ldots$$

of kR_k by subfunctors F_i, for $i \in \mathbb{N}$, such that $F_i/F_{i-1} \cong S_{P_i}$ for $i > 0$.

Let n be the unique integer such that P_n is isomorphic to a Sylow p-subgroup of G. If $i > n$, then $S_{P_i}(G) = 0$ by Lemma 5.3, since P_i is not isomorphic to a subgroup of G: Indeed, if it were, then P_i would be isomorphic to a subgroup of P_n, which would imply $i \leq n$. It follows that

$$F_n(G) = F_{n+1}(G) = \ldots = kR_k(G).$$

So we have a filtration of $kR_k(G)$

$$0 \leq F_1(G) \leq \ldots \leq F_{i-1}(G) \leq F_i(G) \leq \ldots \leq F_n(G) = kR_k(G).$$

Moreover, if the quotient $F_i(G)/F_{i-1}(G) \cong S_{P_i}(G)$ is non zero, then $P_i \hookrightarrow G$ by Lemma 5.3, i.e. $P_i \hookrightarrow P_n$, so $i \leq n$. Then

$$\begin{aligned} \big|[G_{p'}]\big| &= \dim_k kR_k(G) = \sum_{i=1}^{n} \dim_k S_{P_i}(G) \\ &= \sum_{R \hookrightarrow P_n} \dim_k S_R(G) \leq \sum_{R \hookrightarrow P_n} \big|[G_{\cong R}]\big| = \big|[G_{p'}]\big|. \end{aligned}$$

Hence all inequalities $\dim_k S_R(G) \leq \big|[G_{\cong R}]\big|$ are equalities, and the theorem follows.

Corollary 7.9: *Let G be a finite group, and P be a finite p-group. Then the dimension of $S_P(G)$ is equal to the number of conjugacy classes of p-regular elements of G with defect group isomorphic to P.*

Remark 7.10: This corollary is consistent with Corollary 6.14 of [4], via Remark 5.6.

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Serge Bouc, CNRS-LAMFA, Université de Picardie, 33 rue St Leu, 80039, Amiens, France.
serge.bouc@u-picardie.fr