Idempotents of double Burnside algebras, L-enriched bisets, and decomposition of p-biset functors

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Abstract: Let R be a (unital) commutative ring, and G be a finite group with order invertible in R. We introduce new idempotents $\epsilon_{T,S}^G$ in the double Burnside algebra $RB(G,G)$ of G over R, indexed by conjugacy classes of minimal sections (T,S) of G (i.e. sections such that $S \leq \Phi(T)$). These idempotents are orthogonal, and their sum is equal to the identity. It follows that for any biset functor F over R, the evaluation $F(G)$ splits as a direct sum of specific R-modules indexed by minimal sections of G, up to conjugation.
The restriction of these constructions to the biset category of p-groups, where p is a prime number invertible in R, leads to a decomposition of the category of p-biset functors over R as a direct product of categories $\mathcal{F}_L$ indexed by *atoric* p-groups L up to isomorphism.
We next introduce the notions of *L-enriched biset* and *L-enriched biset functor* for an arbitrary finite group L, and show that for an atoric p-group L, the category $\mathcal{F}_L$ is equivalent to the category of L-enriched biset functors defined over elementary abelian p-groups.
Finally, the notion of *vertex* of an indecomposable p-biset functor is introduced (when $p \in R^\times$), and when R is a field of characteristic different from p, the objects of the category $\mathcal{F}_L$ are characterized in terms of vertices of their composition factors.

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1. Introduction

Let R denote throughout a commutative ring (with identity element). For a finite group G, we consider the double Burnside algebra $RB(G,G)$ of a G over R. In the case where the order of G is invertible in R, we introduce idempotents $\epsilon_{T,S}^G$ in $RB(G,G)$, indexed by the set $\mathcal{M}(G)$ of minimal sections of G, i.e. the set of pairs (T,S) of subgroups of G with $S \trianglelefteq T$ and $S \leq \Phi(T)$, where $\Phi(T)$ is the Frattini subgroup of G (such sections have been considered in Section 5 of [9]). The idempotent $\epsilon_{T,S}^G$ only depends of the conjugacy class of (T,S) in G. Moreover, the idempotents $\epsilon_{T,S}^G$, where (T,S) runs through a set $[\mathcal{M}(G)]$ of representatives of orbits of G acting on $\mathcal{M}(G)$ by conjugation, are orthogonal, and their sum is equal to the identity element of $RB(G,G)$.

The idempotents $\epsilon_{G,\mathbf{1}}^G$ plays a special role in our construction, and it is denoted by $\varphi_{\mathbf{1}}^G$. In particular, when F is a biset functor over R (and the order of G is invertible in R), we set $\delta_\Phi F(G) = \varphi_{\mathbf{1}}^G F(G)$. We show that $\delta_\Phi F(G)$ consists of those elements $u \in F(G)$ such that $\mathrm{Res}_H^G u = 0$ whenever H is a proper subgroup of G, and $\mathrm{Def}_{G/N}^G u = 0$ whenever N is a non-trivial normal subgroup of G contained in $\Phi(G)$. This yields moreover a decomposition

$$F(G) \cong \big(\bigoplus_{(T,S)\in\mathcal{M}(G)} \delta_\Phi F(T/S)\big)^G \cong \bigoplus_{(T,S)\in[\mathcal{M}(G)]} \delta_\Phi F(T/S)^{N_G(T,S)/T} \;\;.$$

Restricting these constructions to the biset category $R\mathcal{C}_p$ of p-groups with coefficients in R, where p is a prime invertible in R, we get orthogonal idempotents b_L in the *center* of $R\mathcal{C}_p$, indexed by *atoric* p-groups, i.e. finite p-groups which cannot be split as a direct product $C_p \times Q$, for some p-group Q. We show next that every finite p-group P admits a unique largest atoric quotient $P^@$, well defined up to isomorphism, and that there exists an elementary abelian p-subgroup E of P (non unique in general) such that $P \cong E \times P^@$. For a given atoric p-group L, we introduce a category $R\mathcal{C}_p^{\sharp L}$, defined as a quotient of the subcategory of $R\mathcal{C}_p$ consisting of p-groups P such that $P^@ \cong L$. This leads to a decomposition of the category $\mathcal{F}_{p,R}$ of p-biset functors over R as a direct product

$$\mathcal{F}_{p,R} \cong \prod_{L\in[\mathcal{A}t_p]} \mathsf{Fun}_R\big(R\mathcal{C}_p^{\sharp L}, R\text{-}\mathsf{Mod}\big)$$

of categories of representations of $R\mathcal{C}_p^{\sharp L}$ over R, where L runs through a set $[\mathcal{A}t_p]$ of isomorphism classes of atoric p-groups. Similar questions on idempotents in double Burnside algebras and decomposition of biset functors categories have been considered by L. Barker ([1]), R. Boltje and S. Danz ([2], [3]), R. Boltje and B. Külshammer ([4]), and P. Webb ([16]).

In particular, via the above decomposition, to any indecomposable p-biset functor F is associated a unique atoric p-group, called the *vertex* of F. We show that this vertex is isomorphic to $Q^@$, for any p-group Q such that $F(Q) \neq \{0\}$ but F vanishes on any proper subquotient of Q.

Going back to arbitrary finite groups, we next introduce the notions of *L-enriched biset* and *L-enriched biset functor*, and show that when L is an atoric p-group, the abelian category $\mathsf{Fun}_R\big(R\mathcal{C}_p^{\sharp L}, R\text{-}\mathsf{Mod}\big)$ is equivalent to the category of L-enriched biset functors from elementary abelian p-groups to R-modules.

The paper is organized as follows: Section 2 is a review of definitions and basic results on Burnside rings and biset functors. Section 3 is concerned with the algebra $\mathcal{E}(G)$ obtained by "cutting" the double Burnside algebra $RB(G,G)$ of a finite group G by the idempotent $\widetilde{e_G^G}$ corresponding to the "top" idempotent e_G^G of the Burnside algebra $RB(G)$. Orthogonal idempotents φ_N^G of $\mathcal{E}(G)$ are introduced, indexed by normal subgroups N of G contained in $\Phi(G)$. It is shown moreover that if G is nilpotent, then $\varphi_{\mathbf{1}}^G$ is central in $\mathcal{E}(G)$. In Section 4, the idempotents $\epsilon_{T,S}^G$ of $RB(G,G)$ are introduced, leading in Section 5 to the corresponding direct sum decomposition of the evaluation at G of any biset functor over R. In Section 6, atoric p-groups are introduced, and their main properties are stated. In Section 7, the biset category of p-groups over R is considered, leading to a splitting of the category $\mathcal{F}_{p,R}$ of p-biset functors over R as a direct product of abelian categories $\mathcal{F}_L = \mathsf{Fun}_R\big(R\mathcal{C}_p^{\sharp L}, R\text{-}\mathsf{Mod}\big)$ indexed by atoric p-groups L up to isomorphism. In Section 8, for an arbitrary finite group L, the notions of L-enriched biset and L-enriched biset functor are introduced, and it is shown that when L is an atoric p-group, the category $\mathcal{F}_L$ is equivalent to the category of L-enriched biset functors on elementary abelian p-groups. Finally, in Section 9, for a given atoric p-group L, and when p is invertible in R, the structure of the category $\mathcal{F}_L$ is considered, and the notion of vertex of an indecomposable p-biset functor over R is introduced. In particular, when R is a field of characteristic different from p, it is shown that the objects of $\mathcal{F}_L$ are those p-biset functors all composition factors of which have vertex L.

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2. Review of Burnside rings and biset functors

2.1. Let G be a finite group, let s_G denote the set of subgroups of G, let $\overline{s_G}$ denote the set of conjugacy classes of subgroups of G, and let $[s_G]$ denote a set of representatives of $\overline{s_G}$.

Let $B(G)$ denote the Burnside ring of G, i.e. the Grothendieck ring of the category of finite G-sets. It is a commutative ring, with an identity element, equal to the class of a G-set of cardinality 1. The additive group $B(G)$ is a free abelian group on the set $\{[G/H] \mid H \in [s_G]\}$ of isomorphism classes of transitive G-sets.

2.2. • When G and H are finite groups, and L is a subgroup of $G \times H$, set

$$\begin{aligned} p_1(L) &= \{g \in G \mid \exists h \in H,\ (g,h) \in L\} \ , \\ p_2(L) &= \{h \in H \mid \exists g \in G,\ (g,h) \in L\} \ , \\ k_1(L) &= \{g \in G \mid (g,1) \in L\} \ , \\ k_2(L) &= \{h \in H \mid (1,h) \in L\} \ . \end{aligned}$$

Recall that $k_i(L) \trianglelefteq p_i(L)$, for $i \in \{1,2\}$, that $\big(k_1(L) \times k_2(L)\big) \trianglelefteq L$, and that there are canonical isomorphisms

$$p_1(L)/k_1(L) \cong L/\big(k_1(L) \times k_2(L)\big) \cong p_2(L)/k_2(L) \ .$$

Set moreover $q(L) = L/\big(k_1(L) \times k_2(L)\big)$.

• When Z is a subgroup of G, set

$$\Delta(Z) = \{(z,z) \mid z \in Z\} \leq (G \times G) \ .$$

When N is a normal subgroup of G, set

$$\Delta_N(G) = \{(a,b) \in G \times G \mid ab^{-1} \in N\} \ .$$

It is a subgroup of $G \times G$.

• When G, H, and K are groups, when $L \leq (G \times H)$ and $M \leq (H \times K)$, set

$$L * M = \{(g,k) \in (G \times K) \mid \exists h \in H,\ (g,h) \in L \text{ and } (h,k) \in K\} \ .$$

It is a subgroup of $(G \times K)$.

2.3. When G and H are finite groups, a (G,H)-*biset* U is a set endowed with a left action of G and a right action of H which commute. In other words U is a $G \times H^{op}$-set, where H^{op} is the opposite group of H. The *opposite biset* U^{op} is the (H, G)-biset equal to U as a set, with actions defined for $h \in H$, $u \in U$ and $g \in G$ by $h \cdot u \cdot g$ (in U^{op}) $= g^{-1}uh^{-1}$ (in U).

The Burnside group $B(G, H)$ is the Grothendieck group of the category of finite (G, H)-bisets. It is a free abelian group on the set of isomorphism classes $[(G \times H)/L]$, for $L \in [s_{G\times H}]$, where the (G, H)-biset structure on $(G \times H)/L$ is given by

$$\forall a, g \in G,\ \forall b, h \in H,\ \ a \cdot (g, h)L \cdot b = (ag, b^{-1}h)L \ \ .$$

When G, H, and K are finite groups, there is a unique bilinear product

$$\times_H : B(G, H) \times B(H, K) \to B(G, K)$$

induced by the usual product $(U, V) \mapsto U \times_H V = (U \times V)/H$ of bisets, where the right action of H on $U \times V$ is defined for $u \in U$, $v \in V$ and $h \in H$ by $(u, v)\cdot h = (uh, h^{-1}v)$. This product will also be denoted as a composition $(\alpha, \beta) \mapsto \alpha \circ \beta$ or as a product $(\alpha, \beta) \mapsto \alpha\beta$.

This leads to the following definitions:

2.4. Definition: *The* biset category of finite groups $\mathcal{C}$ *is defined as follows:*

- *The objects of $\mathcal{C}$ are the finite groups.*
- *When G and H are finite groups,*

$$\mathrm{Hom}_{\mathcal{C}}(G, H) = B(H, G) \ \ .$$

- *When G, H, and K are finite groups, the composition*

$$\circ : \mathrm{Hom}_{\mathcal{C}}(H, K) \times \mathrm{Hom}_{\mathcal{C}}(G, H) \to \mathrm{Hom}_{\mathcal{C}}(G, K)$$

  is the product

$$\times_H : B(K, H) \times B(H, G) \to B(K, G) \ \ .$$

- *The identity morphism of the group G is the class of the set G, viewed as a (G, G)-biset by left and right multiplication.*

A biset functor *is an additive functor from $\mathcal{C}$ to the category of abelian groups.*

When R is a commutative (unital) ring, the category $R\mathcal{C}$ is defined similarly by extending coefficients to R, i.e. by setting

$$\mathrm{Hom}_{R\mathcal{C}}(G,H) = R\otimes_{\mathbb{Z}} B(H,G) \ \ ,$$

which will be simply denoted by $RB(H,G)$. A *biset functor over* R is an R-linear functor from $R\mathcal{C}$ to the category R-Mod of R-modules. The category of biset functors over R (where morphisms are natural transformations of functors) is denoted by $\mathcal{F}_R$.

The correspondence sending a (G,H)-biset U to its opposite U^{op} extends to an isomorphism of R-modules $RB(G,H)\to RB(H,G)$. These isomorphisms give an equivalence of R-linear categories from $R\mathcal{C}$ to its opposite category, which is the identity on objects.

2.5. Let G and H be finite groups, and F be a biset functor (with values in R-Mod). For any finite (H,G)-biset U, the isomorphism class $[U]$ of U belongs to $B(H,G)$, and it yields an R-linear map $F([U]) : F(G)\to F(H)$, simply denoted by $F(U)$, or even $f\in F(G)\mapsto U(f)\in F(H)$. In particular:

- When H is a subgroup of G, denote by Ind_H^G the set G, viewed as a (G,H)-biset for left and right multiplication, and by Res_H^G the same set, viewed as an (H,G)-biset. This gives a map $\mathrm{Ind}_H^G : F(H)\to F(G)$, called induction, and a map $\mathrm{Res}_H^G : F(G)\to F(H)$, called restriction.
- When N is a normal subgroup of G, let $\mathrm{Inf}_{G/N}^G$ denote the set G/N, viewed as a $(G,G/N)$-biset for the left action of G, and right action of G/N by multiplication. Also let $\mathrm{Def}_{G/N}^G$ denote the set G/N, viewed as a $(G/N,G)$-biset. This gives a map $\mathrm{Inf}_{G/N}^G : F(G/N)\to F(G)$, called inflation, and a map $\mathrm{Def}_{G/N}^G : F(G)\to F(G/N)$, called deflation.
- Finally, when $f:G\to G'$ is a group isomorphism, denote by $\mathrm{Iso}(f)$ the set G', viewed as a (G',G)-biset for left multiplication in G', and right action of G given by multiplication by the image under f. This gives a map $\mathrm{Iso}(f) : F(G)\to F(G')$, called transport by isomorphism.

When G and H are finite groups, any (G,H)-biset is a disjoint union of transitive ones. It follows that any element of $B(G,H)$ is a linear combination of morphisms of the form $[(G\times H)/L]$, where $L\in s_{G\times H}$. Moreover, any such morphism factors as

$$(\mathbf{2.6})\quad [(G\times H)/L] = \mathrm{Ind}_{p_1(L)}^G\circ\mathrm{Inf}_{p_1(L)/k_1(L)}^{p_1(L)}\circ\mathrm{Iso}(f)\circ\mathrm{Def}_{p_2(L)/k_2(L)}^{p_2(L)}\circ\mathrm{Res}_{p_2(L)}^H \ \ ,$$

where $f: p_2(L)/k_2(L)\to p_1(L)/k_1(L)$ is the canonical group isomorphism.

In particular, for $N\trianglelefteq G$,

$$(\mathbf{2.7})\qquad [(G\times G)/\Delta_N(G)] = \mathrm{Inf}_{G/N}^G\circ\mathrm{Def}_{G/N}^G \ \ .$$

For finite groups G, H, K, for $L \leq (G \times H)$ and $M \leq (H \times K)$, one has that

$$(\mathbf{2.8})\ \ [(G\times H)/L]\times_H [(H\times K)/M] = \sum_{h\in p_2(L)\backslash H/p_1(M)} [(G\times K)/(L*^{(h,1)}M)$$

in $B(G, K)$.

2.9. When G is a finite group, the group $B(G, G)$ is the ring of endomorphisms of G in the category $\mathcal{C}$. This ring is called the double Burnside ring of G. It is a non-commutative ring (if G is non trivial), with identity element equal to the class of the set G, viewed as a (G, G)-biset for left and right multiplication.

There is a unitary ring homomorphism $\alpha \mapsto \widetilde{\alpha}$ from $B(G)$ to $B(G, G)$, induced by the functor $X \mapsto \widetilde{X}$ from G-sets to (G, G)-bisets, where $\widetilde{X} = G \times X$, with (G, G)-biset structure given by

$$\forall a, b, g \in G,\ \forall x \in X,\ a(g, x)b = (agb, ax)\ \ .$$

This construction has in particular the following properties ([7], Corollary 2.5.12):

2.10. Lemma: *Let G be a finite group.*

1. *If H is a subgroup of G, and X is a finite G-set, then there is an isomorphism of (G, H)-bisets*

$$\widetilde{X} \times_G \mathrm{Ind}_H^G \cong \mathrm{Ind}_H^G \times_H \widetilde{\mathrm{Res}_H^G X}\ \ ,$$

and an isomorphism of (H, G)-bisets

$$\mathrm{Res}_H^G \times_G \widetilde{X} \cong \widetilde{\mathrm{Res}_H^G X} \times_H \mathrm{Res}_H^G\ \ .$$

2. *If H is a subgroup of G, and Y is a finite H-set, then there is an isomorphism of (G, G)-bisets*

$$\mathrm{Ind}_H^G \times_H \widetilde{Y} \times_H \mathrm{Res}_H^G \cong \widetilde{\mathrm{Ind}_H^G Y}\ \ .$$

3. *If N is a normal subgroup of G, and X is a finite G/N-set, then there is an isomorphism of $(G/N, G)$-bisets*

$$\widetilde{X} \times_{G/N} \mathrm{Def}_{G/N}^G \cong \mathrm{Def}_{G/N}^G \times_G \widetilde{\mathrm{Inf}_{G/N}^G X}\ \ .$$

4. *If N is a normal subgroup of G, and X is a finite G-set, then there is an isomorphism of $(G/N,G/N)$-bisets*

$$\mathrm{Def}^G_{G/N}\times_G X\times_G \mathrm{Inf}^G_{G/N}\cong \widetilde{\mathrm{Def}^G_{G/N}X} \quad .$$

2.11. Let $RB(G)$ denote the R-algebra $R\otimes_{\mathbb{Z}} B(G)$. Assume moreover that the order of G is invertible in R. Then for $H\leq G$, let $e^G_H\in RB(G)$ be defined by

$$e^G_H=\frac{1}{|N_G(H)|}\sum_{K\leq H}|K|\mu(K,H)\;[G/K] \quad , \tag{2.12}$$

where μ is the Möbius function of the poset of subgroups of G. The elements e^G_H, for $H\in[s_G]$, are orthogonal idempotents of $RB(G)$, and their sum is equal to the identity element of $RB(G)$. It follows that the elements $\widetilde{e^G_H}$, for $H\in[s_G]$, are orthogonal idempotents of the R-algebra $RB(G,G)=R\otimes_{\mathbb{Z}} B(G,G)$, and the sum of these idempotents is equal to the identity element of $RB(G,G)$. The idempotents $\widetilde{e^G_G}$ play a special role, due to the following lemma:

2.13. Lemma: *Let R be a commutative ring, and G be a finite group with order invertible in R.*

1. *Let H be a proper subgroup of G. Then*

$$\mathrm{Res}^G_H\circ\widetilde{e^G_G}=0 \;\; \textit{and} \;\; \widetilde{e^G_G}\circ\mathrm{Ind}^G_H=0 \quad .$$

2. *When $N\trianglelefteq G$, let $m_{G,N}\in R$ be defined by*

$$m_{G,N}=\frac{1}{|G|}\sum_{\substack{X\in s_G\\ XN=G}}|X|\mu(X,G) \quad .$$

Then

$$\mathrm{Def}^G_{G/N}\circ\widetilde{e^G_G}\circ\mathrm{Inf}^G_{G/N}=m_{G,N}\widetilde{e^{G/N}_{G/N}} \quad .$$

3. *Let $N\trianglelefteq G$, and suppose that N is contained in the Frattini subgroup*

$\Phi(G)$ *of* G. *Then*

$$\widetilde{e_{G/N}^{G/N}} \circ \mathrm{Def}_{G/N}^G = \mathrm{Def}_{G/N}^G \circ \widetilde{e_G^G} \quad \textit{and} \quad \mathrm{Inf}_{G/N}^G \circ \widetilde{e_{G/N}^{G/N}} = \widetilde{e_G^G} \circ \mathrm{Inf}_{G/N}^G \quad .$$

Proof : Assertion 1 follows from Lemma 2.10 and Assertion 1 of Theorem 5.2.4. of [7].

Assertion 2 follows from Lemma 2.10 and Assertion 4 of Theorem 5.2.4. of [7].

Finally the first part of Assertion 3 follows from Lemma 2.10 and Assertion 3 of Theorem 5.2.4. of [7]: indeed $\mathrm{Inf}_{G/N}^G e_{G/N}^{G/N}$ is equal to the sum of the different idempotents e_X^G of $RB(G)$ indexed by subgroups X such that $XN = G$. If $N \leq \Phi(G)$, then $XN = G$ implies $X\Phi(G) = G$, hence $X = G$. The second part of Assertion 3 follows by taking opposite bisets, since $\widetilde{e_G^G}$ and $\widetilde{e_{G/N}^{G/N}}$ are equal to their opposite bisets, and since $(\mathrm{Def}_{G/N}^G)^{op} \cong \mathrm{Inf}_{G/N}^G$. □

2.14. Remark: For the same reason, if $N \leq \Phi(G)$, then $m_{G,N} = 1$.

2.15. Remark: It follows from Assertion 1 and Remark 2.6 that if G and H are finite groups and if $L \leq (G \times H)$, then $\widetilde{e_G^G}[(G \times H)/L] = 0$ if $p_1(L) \neq G$, and $[(G \times H)/L]\widetilde{e_H^H} = 0$ if $p_2(L) \neq H$.

3. Idempotents in $\mathcal{E}(G)$

3.1. Notation: *When G is a finite group with order invertible in R, denote by $\mathcal{E}(G)$ the R-algebra defined by*

$$\mathcal{E}(G) = \widetilde{e_G^G} RB(G,G) \widetilde{e_G^G} \quad .$$

Set

$$\Sigma(G,G) = \{M \in s_{G \times G} \mid p_1(L) = p_2(L) = G\} \quad ,$$

and for $L \in s_{G \times G}$, set

$$Y_L = \widetilde{e_G^G} \left[(G \times G)/L\right] \widetilde{e_G^G} \in \mathcal{E}(G) \quad .$$

The R-algebra $\mathcal{E}(G)$ has been considered in [5], Section 9, in the case R is a field of characteristic 0. The extension of the results proved there to the case where R is a commutative ring in which the order of G is invertible is straightforward. In particular:

3.2. Proposition: *Let G be a finite group with order invertible in R.*

1. *If $L \in s_{G\times G} - \Sigma(G,G)$, then $Y_L = 0$.*
2. *The elements Y_L, for L in a set of representatives of $(G\times G)$-conjugacy classes on $\Sigma(G,G)$, form a R-basis of $\mathcal{E}(G)$.*
3. *For $L, M \in \Sigma(G,G)$*

$$Y_L Y_M = \frac{m_{G,k_2(L)\cap k_1(M)}}{|G|} \sum_{\substack{Z\leq G\\ Zk_2(L)=Zk_1(M)=G\\ Z\geq k_2(L)\cap k_1(M)}} |Z|\mu(Z,G)\, Y_{L*\Delta(Z)*M}$$

in $\mathcal{E}(G)$.

3.3. Corollary: *Let $L, M \in \Sigma(G,G)$. If one of the groups $k_2(L)$ or $k_1(M)$ is contained in $\Phi(G)$, then*

$$Y_L Y_M = Y_{L*M} \quad .$$

Proof : Indeed if $k_2(L) \leq \Phi(G)$, then $Zk_2(L) = G$ implies $Z\Phi(G) = G$, hence $Z = G$. Similarly, if $k_1(M) \leq \Phi(G)$, then $Zk_1(M) = G$ implies $Z = G$. In each case, Proposition 3.2 then gives

$$Y_L Y_M = m_{G,k_2(L)\cap k_1(M)} Y_{L*M} \quad ,$$

and moreover $m_{G,k_2(L)\cap k_1(M)} = 1$ since $k_2(L) \cap k_1(M) \leq \Phi(G)$, by Remark 2.14. □

3.4. Notation: *For a normal subgroup N of G such that $N \leq \Phi(G)$, set*

$$\varphi_N^G = \sum_{\substack{M \trianglelefteq G\\ N\leq M\leq \Phi(G)}} \mu_{\trianglelefteq G}(N,M) Y_{\Delta_M(G)} \quad ,$$

where $\mu_{\trianglelefteq G}$ is the Möbius function of the poset of normal subgroups of G.

3.5. Proposition: *Let $N \trianglelefteq G$ with $N \leq \Phi(G)$. Then*

$$\varphi_N^G = \mathrm{Inf}_{G/N}^G \varphi_{\mathbf{1}}^{G/N} \mathrm{Def}_{G/N}^G \ \ .$$

Proof : Indeed if $N \leq M \trianglelefteq G$, then $\mu_{\trianglelefteq G}(N,M) = \mu_{\trianglelefteq G/N}(\mathbf{1}, M/N)$. Since moreover $N \leq \Phi(G)$, setting $\overline{G} = G/N$ and $\overline{M} = M/N$, we have by Lemma 2.13

$$\begin{aligned}
\mathrm{Inf}_{G/N}^G Y_{\Delta_{\overline{G}}(\overline{M})} \mathrm{Def}_{G/N}^G &= \mathrm{Inf}_{G/N}^G \circ \widetilde{e_{\overline{G}}^{\overline{G}}} \big((\overline{G} \times \overline{G})/\Delta_{\overline{G}}(\overline{M})\big) \widetilde{e_{\overline{G}}^{\overline{G}}} \circ \mathrm{Def}_{G/N}^G \\
&= \widetilde{e_G^G} \circ \mathrm{Inf}_{G/N}^G \big((\overline{G} \times \overline{G})/\Delta_{\overline{G}}(\overline{M})\big) \mathrm{Def}_{G/N}^G \circ \widetilde{e_G^G} \\
&= \widetilde{e_G^G} \big((G \times G)/\Delta_M(G)\big) \widetilde{e_G^G} \\
&= Y_{\Delta_M(G)} \ \ ,
\end{aligned}$$

since $\mathrm{Inf}_{G/N}^G \big((\overline{G} \times \overline{G}))/\Delta_{\overline{G}}(\overline{M})\big) \mathrm{Def}_{G/N}^G = (G \times G)/\Delta_M(G)$. □

3.6. Proposition:

1. *Let $N \trianglelefteq G$, with $N \leq \Phi(G)$. Then*

$$\begin{aligned}
\varphi_N^G &= \widetilde{e_G^G} \times_G \Big(\sum_{\substack{M \trianglelefteq G \\ N \leq M \leq \Phi(G)}} \mu_{\trianglelefteq G}(N,M)[(G \times G)/\Delta_M(G)] \Big) \\
&= \Big(\sum_{\substack{M \trianglelefteq G \\ N \leq M \leq \Phi(G)}} \mu_{\trianglelefteq G}(N,M)[(G \times G)/\Delta_M(G)] \Big) \times_G \widetilde{e_G^G} \ \ .
\end{aligned}$$

2. *In particular*

$$\varphi_{\mathbf{1}}^G = \frac{1}{|G|} \sum_{\substack{X \leq G, M \trianglelefteq G \\ M \leq \Phi(G) \leq X \leq G}} |X| \mu(X,G) \mu_{\trianglelefteq G}(\mathbf{1}, M) \, \mathrm{Indinf}_{X/M}^G \circ \mathrm{Defres}_{X/M}^G \ \ .$$

Proof : For Assertion 1, by definition

$$\varphi_N^G = \sum_{\substack{M \trianglelefteq G \\ N \leq M \leq \Phi(G)}} \mu_{\trianglelefteq G}(N,M) \widetilde{e_G^G}[(G \times G)/\Delta_M(G)] \times_G \sum_{X \leq G} \tfrac{|X|}{|G|} \mu(X,G)[(G \times G)/\Delta(X)].$$

Moreover $[(G \times G)/\Delta_M(G)] \times_G [(G \times G)/\Delta(X)] = [(G \times G)/\big(\Delta_M(G) * \Delta(X)\big)]$, by (2.8), and $\Delta_M(G) * \Delta(X) = \{(xm, x) \mid x \in X, m \in M\}$. The first projection of this group is equal to XM, hence it is equal to G if and only if $X = G$, since $M \leq \Phi(G)$. The first equality of Assertion 1 follows, by Remark 2.15. The second one follows by taking opposite bisets, since $\widetilde{e_G^G}$ and $[(G \times G)/\Delta_M(G)]$ are equal to their opposite.

Assertion 2 follows in the special case where $N = 1$, expanding $\widetilde{e_G^G}$ as

$$\widetilde{e_G^G} = \frac{1}{|G|} \sum_{X \leq G} |X| \mu(X, G)[(G \times G)/\Delta(X)] \ ,$$

observing that $\mu(X, G) = 0$ unless $X \geq \Phi(G)$, and that if $X \geq \Phi(G) \geq M$, then

$$[(G \times G)/\Delta(X)] \circ [(G \times G)/\Delta_M(G)] = [(G \times G)/\Delta_M(X)] \ ,$$

which is equal to $\mathrm{Indinf}_{X/M}^G \circ \mathrm{Defres}_{X/M}^G$. □

3.7. Corollary:

1. *Let $H < G$. Then* $\mathrm{Res}_H^G \varphi_N^G = 0$ *and* $\varphi_N^G \mathrm{Ind}_H^G = 0$.
2. *Let $M \trianglelefteq G$. If $M \cap \Phi(G) \not\leq N$, then* $\mathrm{Def}_{G/M}^G \varphi_N^G = 0$ *and* $\varphi_N^G \mathrm{Inf}_{G/M}^G = 0$.

Proof : The first part of Assertion 1 follows from Lemma 2.13, since

$$\mathrm{Res}_H^G \varphi_N^G = \mathrm{Res}_H^G \widetilde{e_G^G} \varphi_N^G = 0 \ .$$

The second part follows by taking opposite bisets.

For Assertion 2, let $P = M \cap \Phi(G)$. Since $\mathrm{Def}_{G/M}^G = \mathrm{Def}_{G/M}^{G/P} \circ \mathrm{Def}_{G/P}^G$, it suffices to consider the case $M = P$, i.e. the case where $M \leq \Phi(G)$. Then, since $[(G \times G)/\Delta_M(G)] = \mathrm{Inf}_{G/M}^G \mathrm{Def}_{G/M}^G$ for any $M \trianglelefteq G$, by 2.7, and since $\mathrm{Def}_{G/M}^G \mathrm{Inf}_{G/Q}^G = \mathrm{Inf}_{G/MQ}^{G/M} \mathrm{Def}_{G/MQ}^{G/Q}$ for any $M, Q \trianglelefteq G$,

$$\begin{aligned}
\mathrm{Def}_{G/M}^G \varphi_N^G &= \mathrm{Def}_{G/M}^G \sum_{\substack{Q \trianglelefteq G \\ N \leq Q \leq \Phi(G)}} \mu_{\trianglelefteq G}(N, Q) \mathrm{Inf}_{G/Q}^G \mathrm{Def}_{G/Q}^G \widetilde{e_G^G} \\
&= \sum_{\substack{Q \trianglelefteq G \\ N \leq Q \leq \Phi(G)}} \mu_{\trianglelefteq G}(N, Q) \mathrm{Inf}_{G/MQ}^G \mathrm{Def}_{G/MQ}^G \widetilde{e_G^G} \\
&= \sum_{\substack{P \trianglelefteq G \\ N \leq P \leq \Phi(G)}} \big(\sum_{\substack{Q \trianglelefteq G \\ N \leq Q \leq \Phi(G) \\ QM = P}} \mu_{\trianglelefteq G}(N, Q) \big) \mathrm{Inf}_{G/P}^G \mathrm{Def}_{G/P}^G \widetilde{e_G^G} \ .
\end{aligned}$$

Now for a given $P \trianglelefteq G$ with $P \subseteq \Phi(G)$, the sum $\sum_{\substack{Q \trianglelefteq G \\ N \leq Q \leq \Phi(G) \\ QM=P}} \mu_{\trianglelefteq G}(N,Q)$ is equal to zero unless $NM = N$, that is $M \leq N$, by classical properties of the Möbius function ([15] Corollary 3.9.3). This proves the first part of Assertion 2, and the second part follows by taking opposite bisets. □

3.8. Theorem: *Let G be a finite group with order invertible in R.*

1. *The elements φ_N^G, for $N \trianglelefteq G$ with $N \leq \Phi(G)$, form a set of orthogonal idempotents in the algebra $\mathcal{E}(G)$, and their sum is equal to the identity element $\widetilde{e_G^G}$ of $\mathcal{E}(G)$.*
2. *Let $N \trianglelefteq G$ with $N \leq \Phi(G)$, and let H be a finite group.*
   (a) *If $L \leq (G \times H)$, then $\varphi_N^G \times_G [(G \times H)/L] = 0$ unless $p_1(L) = G$ and $k_1(L) \cap \Phi(G) \leq N$.*
   (b) *If $L' \leq (H \times G)$, then $[(H \times G)/L'] \times_G \varphi_N^G = 0$ unless $p_2(L') = G$ and $k_2(L') \cap \Phi(G) \leq N$.*

Proof : For $N \trianglelefteq G$, set $u_N^G = Y_{\Delta_N(G)}$. Since $\Delta_N(G) * \Delta_M(G) = \Delta_{NM}(G)$ for any normal subgroups N and M of G, it follows from Corollary 3.3 that if either N or M is contained in $\Phi(G)$, then $u_N^G u_M^G = u_{NM}^G$.

Now Assertion 1 follows from the following general procedure for building orthogonal idempotents (see [13] Theorem 10.1 for details): we have a finite lattice P (here P is the lattice of normal subgroups of G contained in $\Phi(G)$), and a set of elements g_x of a ring A, for $x \in P$ (here $A = \mathcal{E}(G)$ and $g_N = u_N^G$), with the property that $g_x g_y = g_{x \vee y}$ for any $x, y \in P$, and $g_0 = 1$, where 0 is the smallest element of P (here this element is the trivial subgroup of G, and $u_{\mathbf{1}}^G = Y_{\Delta_{\mathbf{1}}(G)} = \widetilde{e_G^G}$). The the elements f_x defined for $x \in P$ by

$$f_x = \sum_{\substack{y \in P \\ x \leq y}} \mu(x,y) g_y \quad ,$$

where μ is the Möbius function of P, are orthogonal idempotents of A, and their sum is equal to the identity element of A. This is exactly Assertion 1 (since $f_x = \varphi_N^G$ here, for $x = N \in P$).

Let $L \leq (G \times H)$, then by 2.6

$$\varphi_N^G \times_G [(G \times H)/L] = \varphi_N^G \circ \mathrm{Ind}_{p_1(L)}^G \circ [(p_1(L) \times H)/L] = 0$$

unless $p_1(L) = G$, by Corollary 3.7. And if $p_1(L) = G$, then by 2.6

$$\varphi_N^G \times_G [(G \times H)/L] = \varphi_N^G \circ \mathrm{Inf}_{G/k_1(L)}^G \circ [(G/k_1(L) \times H)/L_1 \quad ,$$

for some subgroup L_1 of $(G/k_1(L) \times H)$. Again, by Corollary 3.7 this is equal to 0 unless $k_1(L) \cap \Phi(G) \leq N$. The proof of Assertion (b) is similar. Alternatively, one can take opposite bisets in (a). □

3.9. Proposition: *Let G be a finite group with order invertible in R.*

1. *Let $L \in \Sigma(G,G)$. Then*

$$\varphi_{\mathbf{1}}^G Y_L = \sum_{\substack{N \trianglelefteq G \\ N \leq \Phi(G)}} \mu_{\trianglelefteq G}(\mathbf{1}, N) Y_{(N\times\mathbf{1})L} \enspace .$$

This is non zero if and only if $k_1(L) \cap \Phi(G) = \mathbf{1}$. Similarly

$$Y_L \varphi_{\mathbf{1}}^G = \sum_{\substack{N \trianglelefteq G \\ N \leq \Phi(G)}} \mu_{\trianglelefteq G}(\mathbf{1}, N) Y_{L(\mathbf{1}\times N)} \enspace ,$$

and $Y_L\varphi_1^G \neq 0$ if and only if $k_2(L) \cap \Phi(G) = \mathbf{1}$.

2. *The elements $\varphi_{\mathbf{1}}^G Y_L$ (resp. $Y_L\varphi_{\mathbf{1}}^G$), when L runs through a set of representatives of conjugacy classes of elements of $\Sigma(G,G)$ such that $k_1(L) \cap \Phi(G) = \mathbf{1}$ (resp $k_2(L) \cap \Phi(G) = \mathbf{1}$), form an R-basis of the right ideal $\varphi_{\mathbf{1}}^G \mathcal{E}(G)$ (resp. the left ideal $\mathcal{E}(G)\varphi_{\mathbf{1}}^G$) of $\mathcal{E}(G)$.*

Proof : Let $L \in \Sigma(G,G)$. By Proposition 3.8, we have

$$\begin{aligned}
\varphi_{\mathbf{1}}^G Y_L &= \widetilde{e_G^G} \times_G \Big(\sum_{\substack{N \trianglelefteq G \\ N \leq \Phi(G)}} \mu_{\trianglelefteq G}(\mathbf{1}, N) [(G \times G)/\Delta_N(G)] \Big) \times_G [(G\times G)/L] \times_G \widetilde{e_G^G} \\
&= \widetilde{e_G^G} \times_G \Big(\sum_{\substack{N \trianglelefteq G \\ N \leq \Phi(G)}} \mu_{\trianglelefteq G}(\mathbf{1}, N) [(G \times G)/\Delta_N(G) * L] \Big) \times_G \widetilde{e_G^G} \\
&= \widetilde{e_G^G} \times_G \Big(\sum_{\substack{N \trianglelefteq G \\ N \leq \Phi(G)}} \mu_{\trianglelefteq G}(\mathbf{1}, N) [(G \times G)/(N\times\mathbf{1})L] \Big) \times_G \widetilde{e_G^G} \enspace . \\
&= \sum_{\substack{N \trianglelefteq G \\ N \leq \Phi(G)}} \mu_{\trianglelefteq G}(\mathbf{1}, N) Y_{(N\times\mathbf{1})L} \enspace .
\end{aligned}$$

Set $M = k_1(L) \cap \Phi(G)$. Then $M \trianglelefteq G$, and $(N \times \mathbf{1})L = (NM \times \mathbf{1})L$ for any normal subgroup N of G contained in $\varphi(G)$. Thus

$$(\mathbf{3.10}) \qquad \varphi_{\mathbf{1}}^G Y_L = \sum_{\substack{P \trianglelefteq G\\ M\leq P\leq \Phi(G)}} \Big(\sum_{\substack{N\trianglelefteq G\\ NM=P}} \mu_{\trianglelefteq G}(\mathbf{1},N)\Big) Y_{(P\times\mathbf{1})L} \;\;.$$

If $M \neq \mathbf{1}$, then $\Big(\sum_{\substack{N\trianglelefteq G\\ NM=P}} \mu_{\trianglelefteq G}(\mathbf{1},N)\Big) = 0$ for any $P \trianglelefteq G$ with $M \leq P \leq \Phi(G)$. Hence $\varphi_{\mathbf{1}}^G Y_L = 0$ in this case. And if $M = \mathbf{1}$, Equation (3.10) reads

$$\varphi_{\mathbf{1}}^G Y_L = \sum_{\substack{P \trianglelefteq G\\ P\leq \Phi(G)}} \mu_{\trianglelefteq G}(\mathbf{1},P) Y_{(P\times\mathbf{1})L} \;\;.$$

The element $Y_{(P\times\mathbf{1})L}$ is equal to Y_L if and only if $(P\times\mathbf{1})L$ is conjugate to L. This implies that $k_1\big((P\times\mathbf{1})L\big)$ is conjugate to (hence equal to) $k_1(L)$. Thus $P \leq k_1\big((P\times\mathbf{1})L\big) \leq k_1(L)\cap\Phi(G)$, hence $P = \mathbf{1}$. So the coefficient of Y_L in $\varphi_{\mathbf{1}}^G Y_L$ is equal to 1, hence $\varphi_{\mathbf{1}}^G Y_L \neq 0$. The remaining statements of Assertion 1 follow by taking opposite bisets.

Assertion 2 follows from Proposition 3.2, and from the fact that the coefficient of Y_L in $\varphi_{\mathbf{1}}^G Y_L$ is equal to 1 when $k_1(L)\cap\Phi(G) = \mathbf{1}$. □

3.11. Corollary: *Let G be a finite group of order invertible in R. If every minimal (non-trivial) normal subgroup of G is contained in $\Phi(G)$, then $\varphi_{\mathbf{1}}^G$ is central in $\mathcal{E}(G)$, and the algebra $\varphi_{\mathbf{1}}^G\mathcal{E}(G)$ is isomorphic to $R\mathrm{Out}(G)$.*

Proof : Indeed if $L \in \Sigma(L,L)$ and $\varphi_{\mathbf{1}}^G Y_L \neq 0$, then $k_1(L)\cap\Phi(G) = \mathbf{1}$. It follows that $k_1(L)$ contains no minimal normal subgroup of G, and then $k_1(L) = \mathbf{1}$. Equivalently $q(L) \cong p_1(L)/k_1(L) \cong G \cong p_2(L)/k_2(L)$, i.e. $k_2(L) = G$ also, or equivalently $k_2(L)\cap\Phi(G) = \mathbf{1}$. Hence $\varphi_{\mathbf{1}}^G Y_L \neq 0$ if and only if $Y_L\varphi_{\mathbf{1}}^G \neq 0$, and in this case, there exists an automorphism θ of G such that

$$L = \Delta_\theta(G) = \{\big(\theta(x),x\big) \mid x \in G\} \;\;.$$

In this case for any normal subgroup N of G contained in $\Phi(G)$

$$\begin{aligned}(N\times\mathbf{1})L &= \{(a,b)\in G\times G \mid a\theta(b)^{-1}\in N\}\\ &= \{(a,b)\in G\times G \mid a^{-1}\theta(b)\in N\}\\ &= L\big(\mathbf{1}\times\theta^{-1}(N)\big) \;\;.\end{aligned}$$

Now $N \mapsto \theta^{-1}(N)$ is a permutation of the set of normal subgroups of G contained in $\Phi(G)$. Moreover $\mu_{\trianglelefteq G}(\mathbf{1},N) = \mu_{\trianglelefteq G}\big(\mathbf{1},\theta^{-1}(N)\big)$.

It follows that $\varphi_{\mathbf{1}}^G Y_L = Y_L \varphi_{\mathbf{1}}^G$, so $\varphi_{\mathbf{1}}^G$ is central in $\mathcal{E}(G)$. Moreover the map $\theta \in \mathrm{Aut}(G) \mapsto \varphi_{\mathbf{1}}^G Y_{\Delta_\theta(G)}$ clearly induces an algebra isomorphism $R\mathrm{Out}(G) \to \varphi_{\mathbf{1}}^G \mathcal{E}(G)$. □

3.12. Theorem: *Let G be a finite group with order invertible in R. If G is nilpotent, then $\varphi_{\mathbf{1}}^G$ is a central idempotent of $\mathcal{E}(G)$.*

Proof : Let $L \in \Sigma(G,G)$. Setting $Q = q(L)$, there are two surjective group homomorphisms $s,t : G \to Q$ such that $L = \{(x,y) \in G \times G \mid s(x) = t(y)\}$. Then $k_1(L) = \mathrm{Ker}\, s$ and $k_2(L) = \mathrm{Ker}\, t$. Now by Proposition 3.9

$$\varphi_{\mathbf{1}}^G Y_L = \sum_{\substack{N \trianglelefteq G \\ N \leq \Phi(G)}} \mu_{\trianglelefteq G}(\mathbf{1}, N) Y_{(N \times \mathbf{1})L} \quad ,$$

and this is non zero if and only if $\mathrm{Ker}\, s \cap \Phi(G) = \mathbf{1}$. Now $s\big(\Phi(G)\big)$ is equal to $\Phi(Q)$ since G is nilpotent: indeed $G = \prod_p G_p$ (resp. $Q = \prod_p Q_p$) is the direct product of its p-Sylow subgroups G_p (resp. Q_p), and s induces a surjective group homomorphism $G_p \to Q_p$, for any prime p. Moreover $\Phi(G) = \prod_p \Phi(G_p)$ (resp. $\Phi(Q) = \prod_p \Phi(Q_p)$). Finally $\Phi(G_p)$ is the subgroup of G_p generated by commutators and p-powers of elements of G_p, hence it maps by s *onto* the subgroup of Q_p generated by commutators and p-powers of elements of Q_p, that is $\Phi(Q_p)$. Similarly $t\big(\Phi(G)\big) = \Phi(Q)$.

If $\mathrm{Ker}\, s \cap \Phi(G) = \mathbf{1}$, it follows that s induces an isomorphism from $\Phi(G)$ to $\Phi(Q)$. Then the surjective homomorphism $\Phi(G) \to \Phi(Q)$ induced by t is also an isomorphism, and in particular $\mathrm{Ker}\, t \cap \Phi(G) = \mathbf{1}$.

Let $D = L \cap \big(\Phi(G) \times \Phi(G)\big)$. Then $k_1(D) \subseteq k_1(L) \cap \Phi(G) = \mathrm{Ker}\, s \cap \Phi(G)$, hence $k_1(D) = \mathbf{1}$. Similarly $k_2(L) \subseteq k_2(L) \cap \Phi(G) = \mathrm{Ker}\, t \cap \Phi(G) = \mathbf{1}$, hence $k_2(D) = \mathbf{1}$. Since $s\big(\Phi(G)\big) = \Phi(Q) = t\big(\Phi(G)\big)$, we have moreover $p_1(D) = \Phi(G) = p_2(D)$. It follows that there is an automorphism α of $\Phi(G)$ such that $D = \{\big(x, \alpha(x)\big) \mid x \in \Phi(G)\}$.

Moreover for any $y \in G$, there exists $z \in G$ such that $(y,z) \in L$. It follows that $\big(x^y, \alpha(x)^z\big) \in D$ for any $x \in \Phi(G)$, that is $\alpha(x^y) = \alpha(x)^z$. In particular if N is a normal subgroup of G contained in $\Phi(G)$, then so is $\alpha(N)$. Hence α induces an automorphism of the poset of normal subgroups of G contained in $\Phi(G)$. In particular $\mu_{\trianglelefteq G}(\mathbf{1}, N) = \mu_{\trianglelefteq G}\big(\mathbf{1}, \alpha(N)\big)$.

Moreover for $n \in N$ and $(y,z) \in L$, we have

$$(n,1)(y,z) = (y,z)(n^y,1) = (y,z)\big(n^y, \alpha(n^y)\big)\big(1, \alpha(n^y)^{-1}\big) \quad .$$

Since $\big(n^y, \alpha(n^y)\big) \in D \leq L$, we have $(N \times \mathbf{1})L = L\big(\mathbf{1} \times \alpha(N)\big)$. It follows that

$$\begin{aligned}
\varphi_{\mathbf{1}}^G Y_L &= \sum_{\substack{N \trianglelefteq G \\ N \leq \Phi(G)}} \mu_{\trianglelefteq G}(\mathbf{1}, N) Y_{(N\times\mathbf{1})L} = \sum_{\substack{N \trianglelefteq G \\ N \leq \Phi(G)}} \mu_{\trianglelefteq G}(\mathbf{1}, N) Y_{L(\mathbf{1}\times\alpha(N))} \\
&= \sum_{\substack{N \trianglelefteq G \\ N \leq \Phi(G)}} \mu_{\trianglelefteq G}\big(\mathbf{1}, \alpha(N)\big) Y_{L(\mathbf{1}\times\alpha(N))} = \sum_{\substack{N \trianglelefteq G \\ N \leq \Phi(G)}} \mu_{\trianglelefteq G}\big(\mathbf{1}, N\big) Y_{L(\mathbf{1}\times N)} \\
&= Y_L \varphi_{\mathbf{1}}^G \ ,
\end{aligned}$$

as was to be shown. □

3.13. Remark: When G is not nilpotent, it is not true in general that $\varphi_{\mathbf{1}}^G$ is central in $\mathcal{E}(G)$. This is because $t\big(\Phi(G)\big)$ need not be equal to $\Phi(Q)$ for a surjective group homomorphism $t : G \to Q$. For example, there is a surjection t from the group $G = C_4 \times (C_5 \rtimes C_4)$ to $Q = C_4$ with kernel $C_4 \times C_5$ *containing* $\Phi(G) = C_2 \times \mathbf{1}$, and another surjection $s : G \to Q$ with kernel $\mathbf{1} \times (C_5 \rtimes C_4)$ intersecting trivially $\Phi(G)$. In this case, the group $L = \{(x, y) \in G \times G \mid s(x) = t(y)\}$ is in $\Sigma(G, G)$, and $k_1(L) \cap \Phi(G) = \mathbf{1}$, but $k_2(L) \cap \Phi(G) = \Phi(G) \neq \mathbf{1}$. By Proposition 3.9, we have $\varphi_{\mathbf{1}}^G Y_L \neq 0$ and $Y_L \varphi_{\mathbf{1}}^G = 0$, so $\varphi_{\mathbf{1}}^G$ is not central in $\mathcal{E}(G)$.

4. Idempotents in $RB(G, G)$

4.1. Definition: *When G is a finite group, a* section (T, S) *of G is a pair of subgroups of G such that $S \trianglelefteq T$.*

A section (T, S) is called minimal *(cf. [9]) if $S \leq \Phi(T)$. Let $\mathcal{M}(G)$ denote the set of minimal sections of G.*

A group H is called a subquotient of G (notation $H \sqsubseteq G$) if there exists a section (T, S) of G such that $T/S \cong H$.

A section (T, S) is minimal if and only if the only subgroup H of T such that $H/(H \cap S) \cong T/S$ is T itself.

4.2. Notation: *Let G be a finite group, and let (T, S) be a section of G.*

1. *Let* $\mathrm{Indinf}_{T/S}^G \in B(G, T/S)$ *denote (the isomorphism class of) the $(G, T/S)$-biset G/S, and let* $\mathrm{Defres}_{T/S}^G \in B(T/S, G)$ *denote (the isomorphism class of) the $(T/S, G)$-biset $S\backslash G$.*
2. *Let R be a commutative ring in which the order of G is invertible. Let*

$u^G_{T,S} \in RB(G, T/S)$ *be defined by*

$$u^G_{T,S} = \mathrm{Indinf}^G_{T/S}\varphi_{\mathbf{1}}^{T/S} \quad ,$$

and let $v^G_{T,S} \in RB(T/S, G)$ *be defined by*

$$v^G_{T,S} = \varphi_{\mathbf{1}}^{T/S}\mathrm{Defres}^G_{T/S} \quad .$$

4.3. Remark: Observe that $v^G_{T,S} = (u^G_{T,S})^{op}$: indeed $(G/S)^{op} \cong S\backslash G$, and $(\varphi_{\mathbf{1}}^{T/S})^{op} = \varphi_{\mathbf{1}}^{T/S}$.

4.4. Theorem: *Let G be a finite group with order invertible in R.*

1. *If (T, S) and (T', S') are minimal sections of G, then*

$$v^G_{T',S'}u^G_{T,S} = 0$$

   unless (T, S) and (T', S') are conjugate in G.

2. *If (T, S) is a minimal section of G, then*

$$v^G_{T,S}u^G_{T,S} = \varphi_{\mathbf{1}}^{T/S}\Big(\sum_{g\in N_G(T,S)/T} \mathrm{Iso}(c_g)\Big) \quad ,$$

   where $N_G(T, S) = N_G(T) \cap N_G(S)$, and c_g is the automorphism of T/S induced by conjugation by g.

Proof : Indeed $(S'\backslash G) \times_G (G/S) \cong S'\backslash G/S$ as a $(T'/S', T/S)$-biset. Hence

$$v^G_{T',S'}u^G_{T,S} = \varphi_{\mathbf{1}}^{T'/S'}\Big(\sum_{g\in T'\backslash G/T} S'\backslash T'gT/S\Big)\varphi_{\mathbf{1}}^{T/S} \quad .$$

For any $g \in G$, the $(T'/S', T/S)$-biset $S'\backslash T'gT/S$ is transitive, isomorphic to $\big((T'/S') \times (T/S)\big)/L_g$, where

$$L_g = \{(t'S', tS) \in (T'/S') \times (T/S) \mid t'gt^{-1} \in S'gS\} \quad .$$

Then $t'S' \in p_1(L_g)$ if and only if $t' \in S' \cdot gTg^{-1} \cap T'$. Hence

$$p_1(L_g) = ({}^gT \cap T')S'/S' \quad .$$

Similarly $p_2(L_g) = (T'^g \cap T)S/S$. In particular $p_1(L_g) = T'/S'$ if and only if $({}^gT \cap T')S' = T'$, i.e. ${}^gT \cap T' = T'$, since $S' \leq \Phi(T')$. Thus $p_1(L_g) = T'/S'$

if and only if $T' \leq {}^gT$. Similarly $p_2(L_g) = T/S$ if and only if $T \leq T'^g$. By Theorem 3.8, it follows that $\varphi_{\mathbf{1}}^{T'/S'}(S'\backslash T'gT/S)\varphi_{\mathbf{1}}^{T/S} = 0$ unless $T' = {}^gT$.

Assume now that $T' = {}^gT$. Then $t'S' \in k_1(L_G)$ if and only if t' lies in $S' \cdot gSg^{-1} \cap T'$. Hence

$$k_1(L_g) = ({}^gS \cap T')S'/S' \ ,$$

and similarly $k_2(L_g) = (S'^g \cap T)S/S$. But since $S \leq \Phi(T)$ and $S \trianglelefteq T$, it follows that ${}^gS \trianglelefteq {}^gT = T'$ and ${}^gS \leq {}^g\Phi(T) = \Phi(T')$. Hence ${}^gS \cdot S'/S'$ is contained in $k_1(L_g) \cap \Phi(T')/S'$. Moreover $\Phi(T')/S' = \Phi(T'/S')$, as

$$\Phi(T'/S') = \bigcap_{S' \leq M' < T'} (M'/S') = \bigcap_{M' < T'} (M'/S') = (\bigcap_{M' < T'} M')/S' = \Phi(T')/S' \ ,$$

where M' runs through maximal subgroups of T', which all contain S' since $S' \leq \Phi(T')$.

It follows that if $k_1(L_g) \cap \Phi(T'/S') = \mathbf{1}$, then ${}^gS \cdot S' = S'$, that is ${}^gS \leq S'$. Similarly if $k_2(L_g) \cap \Phi(T/S) = \mathbf{1}$, then $S'^g \leq S$. By Theorem 3.8, it follows that $\varphi_{\mathbf{1}}^{T'/S'}(S'\backslash T'gT/S)\varphi_{\mathbf{1}}^{T/S} = 0$ unless $T' = {}^gT$ and $S' = {}^gS$. This proves Assertion 1.

For Assertion 2, the same computation shows that

$$v_{T,S}^G u_{T,S}^G = \sum_{g \in N_G(T,S)/T} \varphi_{\mathbf{1}}^{T/S}(S\backslash TgT/S)\varphi_{\mathbf{1}}^{T/S} \ .$$

But $S\backslash TgT/S = gT/S$ if $g \in N_G(T,S)$, and this $(T/S, T/S)$-biset is isomorphic to $\mathrm{Iso}(c_g)$. Assertion 2 follows, since moreover $\varphi_{\mathbf{1}}^{T/S}$ commutes with any biset of the form $\mathrm{Iso}(\theta)$, where θ is an automorphism of T/S. □

4.5. Notation: *For a minimal section (T, S) of the group G, set*

$$\epsilon_{T,S}^G = \frac{1}{|N_G(T,S):T|} u_{T,S}^G v_{T,S}^G = \frac{1}{|N_G(T,S):T|} \mathrm{Indinf}_{T/S}^G \varphi_{\mathbf{1}}^G \mathrm{Defres}_{T/S}^G \in RB(G,G) \ .$$

Note that $\epsilon_{T,S}^G = \epsilon_{{}^gT,{}^gS}^G$ for any $g \in G$, and that $\epsilon_{G,N}^G = \varphi_N^G$ when $N \trianglelefteq G$ and $N \leq \Phi(G)$, by Proposition 3.5.

4.6. Proposition: *Let (T, S) be a minimal section of G. Then*

$$\epsilon_{T,S}^G = \frac{1}{|N_G(T,S)|} \sum_{\substack{X \leq T, M \trianglelefteq T \\ S \leq M \leq \Phi(T) \leq X \leq T}} |X|\mu(X,T)\mu_{\trianglelefteq T}(S,M)\, \mathrm{Indinf}_{X/M}^G \circ \mathrm{Defres}_{X/M}^G \ .$$

Proof : This is a straightforward consequence of the above definition of $\epsilon^G_{T,S}$, and from Assertion 2 of Proposition 3.6. □

4.7. Theorem: *Let G be a finite group with order invertible in R, let $[\mathcal{M}(G)]$ be a set of representatives of conjugacy classes of minimal sections of G. Then the elements $\epsilon^G_{T,S}$, for $(T,S) \in [\mathcal{M}(G)]$, are orthogonal idempotents of $RB(G,G)$, and their sum is equal to the identity element of $RB(G,G)$.*

Proof : Let (T,S) and (T',S') be distinct elements of $[\mathcal{M}(G)]$. Then

$$\epsilon^G_{T',S'}\epsilon^G_{T,S} = \frac{1}{|N_G(T',S'):T'|}\frac{1}{|N_G(T,S):T|}u^G_{T',S'}v^G_{T',S'}u^G_{T,S}v^G_{T,S} = 0 \ ,$$

since $v^G_{T',S'}u^G_{T,S} = 0$ by Theorem 4.4. Moreover:

$$\begin{aligned}\sum_{(T,S)\in[\mathcal{M}(G)]}\epsilon^G_{T,S} &= \sum_{(T,S)\in[\mathcal{M}(G)]}\frac{1}{|N_G(T,S):T|}u^G_{T,S}v^G_{T,S}\\ &= \sum_{(T,S)\in\mathcal{M}(G)}\frac{1}{|G:T|}u^G_{T,S}v^G_{T,S}\\ &= \sum_{(T,S)\in\mathcal{M}(G)}\frac{1}{|G:T|}\mathrm{Indinf}^G_{T/S}\varphi^{T/S}_{\mathbf{1}}\mathrm{Defres}^G_{T/S}\end{aligned}$$

Now $\varphi^{T/S}_{\mathbf{1}} = \widetilde{e^{T/S}_{T/S}}f_{T/S}$ by Proposition 3.6, where

$$f_{T/S} = \sum_{\substack{N/S\,\trianglelefteq(T/S)\\ N/S\leq\Phi(T/S)}}\mu_{\trianglelefteq G}(\mathbf{1},N/S)\big[\big((T/S)\times(T/S)\big)/\Delta_{N/S}(T/S)\big] \ .$$

Hence $\varphi^{T/S}_{\mathbf{1}} = \widetilde{e^{T/S}_{T/S}}\mathrm{Def}^T_{T/S}\mathrm{Inf}^T_{T/S}f_{T/S}$, and

$$\sum_{(T,S)\in[\mathcal{M}(G)]}\epsilon^G_{T,S} = \sum_{(T,S)\in\mathcal{M}(G)}\frac{1}{|G:T|}\mathrm{Ind}^G_T\mathrm{Inf}^T_{T/S}\widetilde{e^{T/S}_{T/S}}\mathrm{Def}^T_{T/S}\mathrm{Inf}^T_{T/S}f_{T/S}\mathrm{Def}^T_{T/S}\mathrm{Res}^G_T \ .$$

Now $\mathrm{Inf}^T_{T/S}\widetilde{e^{T/S}_{T/S}}\mathrm{Def}^T_{T/S} = \widetilde{\mathrm{Inf}^T_{T/S}e^{T/S}_{T/S}}$, and $\mathrm{Inf}^T_{T/S}e^{T/S}_{T/S}$ is equal to the sum over subgroups X or T such that $XS = T$, up to conjugation, of the idempotents e^T_X. Since $S \leq \Phi(T)$, the only subgroup X of T such that $XS = T$ is T itself. Hence

$$\mathrm{Inf}^T_{T/S}\widetilde{e^{T/S}_{T/S}}\mathrm{Def}^T_{T/S} = \widetilde{e^T_T} \ .$$

On the other hand

$$\mathrm{Inf}_{T/S}^T[\big((T/S)\times(T/S)\big)/\Delta_{N/S}(T/S)]\mathrm{Def}_{T/S}^T=[(T\times T)/\Delta_N(T)]\ \ .$$

It follows that the sum $\Sigma=\sum_{(T,S)\in[\mathcal{M}(G)]}\epsilon_{T,S}^G$ is equal to

$$\begin{aligned}
\Sigma &= \sum_{(T,S)\in\mathcal{M}(G)}\tfrac{1}{|G:T|}\mathrm{Ind}_T^G\widetilde{e_T^T}\sum_{\substack{N\trianglelefteq T\\ S\leq N\leq\Phi(T)}}\mu_{\trianglelefteq T}(S,N)[(T\times T)/\Delta_N(T)]\mathrm{Res}_T^G\\
&= \sum_{(T,S)\in\mathcal{M}(G)}\tfrac{1}{|G:T|}\mathrm{Ind}_T^G\widetilde{e_T^T}\varphi_S^T\mathrm{Res}_T^G\ \ \text{[by definition of } \varphi_S^T]\\
&= \sum_{T\leq G}\tfrac{1}{|G:T|}\mathrm{Ind}_T^G\widetilde{e_T^T}\sum_{\substack{S\trianglelefteq T\\ S\leq\Phi(T)}}\varphi_S^T\mathrm{Res}_T^G\\
&= \sum_{T\leq G}\tfrac{1}{|G:T|}\mathrm{Ind}_T^G\widetilde{e_T^T}\mathrm{Res}_T^G\ \ \text{[by Theorem 3.8]}\\
&= \sum_{T\leq G}\tfrac{1}{|G:T|}\widetilde{\mathrm{Ind}_T^G e_T^T}\ \ \text{[by Lemma 2.10]}\\
&= \sum_{T\leq G}\tfrac{1}{|G:N_G(T)|}\widetilde{e_T^G}\ \ \text{[by (2.12)]}\\
&= \sum_{T\in[s_G]}\widetilde{e_T^G}=\widetilde{G/G}=[(G\times G)/\Delta(G)]\ \ .
\end{aligned}$$

So the sum Σ is equal to the identity of $RB(G,G)$. Since $\epsilon_{T,S}^G\epsilon_{T',S'}^G=0$ if (T,S) and (T',S') are distinct elements of $[\mathcal{M}(G)]$, it follows that for any $(T,S)\in[\mathcal{M}(G)]$

$$\epsilon_{T,S}^G=\epsilon_{T,S}^G\Sigma=(\epsilon_{T,S}^G)^2\ \ ,$$

which completes the proof of the theorem. □

5. Application to biset functors

5.1. Notation: *Let F be a biset functor over R. When G is a finite group with order invertible in R, we set*

$$\delta_\Phi F(G)=\varphi_{\mathbf{1}}^G F(G)$$

5.2. Proposition: *Let F be a biset functor over R. Then for any finite group G with order invertible in R, the R-submodule $\delta_\Phi F(G)$ of $F(G)$ is the set of elements $u \in F(G)$ such that*

$$\begin{cases} \mathrm{Res}^G_H u = 0 & \forall H < G \\ \mathrm{Def}^G_{G/N} u = 0 & \forall N \trianglelefteq G,\ N \cap \Phi(G) \neq \mathbf{1} \end{cases} .$$

Proof : If $u \in \delta_\Phi F(G) = \varphi_{\mathbf{1}}^G F(G)$, then $\mathrm{Res}^G_H u = 0$ for any proper subgroup H of G, and $\mathrm{Def}^G_{G/N} u = 0$ for any $N \trianglelefteq G$ such that $N \cap \Phi(G) \neq \mathbf{1}$, by Corollary 3.7.

Conversely, if $u \in F(G)$ fulfills the two conditions of the proposition, then $\widetilde{e_G^G} u = u$, because $\widetilde{e_G^G}$ is equal to the identity element $[(G \times G)/\Delta(G)]$ of $RB(G,G)$, plus a linear combination of elements of the form $[(G \times G)/\Delta(H)] = \mathrm{Ind}_H^G \circ \mathrm{Res}^G_H$, for proper subgroups H of G. Similarly $\mathrm{Inf}^G_{G/N} \mathrm{Def}^G_{G/N} u = 0$ for any non-trivial normal subgroup of G contained in $\Phi(G)$, thus $\varphi_{\mathbf{1}}^G u = u$. □

5.3. Remark: Since $\mathrm{Def}^G_{G/N} = \mathrm{Def}^{G/M}_{G/N} \circ \mathrm{Def}^G_{G/M}$, where $M = N \cap \Phi(G)$, saying that $\mathrm{Def}^G_{G/N} u = 0$ for any $N \trianglelefteq G$ with $N \cap \Phi(G) \neq \mathbf{1}$ is equivalent to saying that $\mathrm{Def}^G_{G/N} u = 0$ for any non trivial normal subgroup N of G contained in $\Phi(G)$.

5.4. Theorem: *Let F be a biset functor over R. Then for any finite group G with order invertible in R, the maps*

$$F(G) \rightleftarrows \bigoplus_{(T,S) \in [\mathcal{M}(G)]} \big(\delta_\Phi F(T/S)\big)^{N_G(T,S)/T}$$

$$w \stackrel{V}{\longmapsto} \bigoplus_{(T,S)} \frac{1}{|N_G(T,S):T|} v^G_{T,S} w$$

$$\sum_{(T,S)} u^G_{T,S} w_{T,S} \stackrel{U}{\longleftarrow\!\!\shortmid} \bigoplus_{(T,S)} w_{T,S}$$

are well defined isomorphisms of R-modules, inverse to one other.

Proof : We have first to check that if $w \in F(G)$, then the element $v^G_{T,S} w$ of $\varphi_{\mathbf{1}}^{T/S} F(T/S) = \delta_\Phi F(T/S)$ is invariant under the action of $N_G(T,S)/T$. But for any $g \in N_G(T/S)$

$$\mathrm{Iso}(c_g) v^G_{T,S} = v^G_{{}^gT,{}^gS} \mathrm{Iso}(c_g) = v^G_{T,S} \mathrm{Iso}(c_g) \ ,$$

where $\mathrm{Iso}(c_g) : F(G) \to F(G)$ on the right hand side is conjugation by g, that is an inner automorphism, hence the identity map, for $g \in G$.

Now for $w \in F(G)$

$$\begin{aligned} UV(w) &= \sum_{(T,S)\in[\mathcal{M}(G)]} \tfrac{1}{|N_G(T,S):T|} u_{T,S}^G v_{T,S}^G w \\ &= \sum_{(T,S)\in[\mathcal{M}(G)]} \epsilon_{T,S}^G w = w \quad , \end{aligned}$$

so UV is the identity map of $F(G)$.

Conversely, if $w_{T,S} \in \big(\delta_\Phi F(T/S)\big)^{N_G(T,S)/T}$, for $(T,S) \in [\mathcal{M}(G)]$, then

$$\begin{aligned} VU\big(\bigoplus_{(T,S)\in[\mathcal{M}(G)]} w_{T,S}\big) &= \bigoplus_{(T,S)\in[\mathcal{M}(G)]} \sum_{(T',S')\in[\mathcal{M}(G)]} \tfrac{1}{|N_G(T,S):T|} v_{T,S}^G u_{T',S'}^G w_{T',S'} \\ &= \bigoplus_{(T,S)\in[\mathcal{M}(G)]} \tfrac{1}{|N_G(T,S):T|} v_{T,S}^G u_{T,S}^G w_{T,S} \\ &= \bigoplus_{(T,S)\in[\mathcal{M}(G)]} \tfrac{1}{|N_G(T,S):T|} \sum_{g\in N_G(T,S)/T} \mathrm{Iso}(c_g) w_{T,S} \\ &= \bigoplus_{(T,S)\in[\mathcal{M}(G)]} w_{T,S} \quad , \end{aligned}$$

so VU is also equal to the identity map. □

6. Atoric p-groups

For the remainder of the paper, we denote by p a (fixed) prime number.

6.1. Notation and Definition:

- *If P is a finite p-group, let $\Omega_1 P$ denote the subgroup of P generated by the elements of order p.*
- *A finite p-group P is called* atoric *if it does not admit any decomposition $P = E \times Q$, where E is a non-trivial elementary abelian p-group. Let $\mathcal{A}t_p$ denote the class of atoric p-groups, and let $[\mathcal{A}t_p]$ denote a set of representatives of isomorphism classes in $\mathcal{A}t_p$.*

The terminology "atoric" is inspired by [14], where elementary abelian p-groups are called *p-tori*. Atoric p-groups have been considered (without naming them) in [6], Example 5.8.

6.2. Lemma: *Let P be a finite p-group, and N be a normal subgroup of P. The following conditions are equivalent:*

1. $N \cap \Phi(P) = \mathbf{1}$

2. *N is elementary abelian and central in P, and admits a complement in P.*

3. *N is elementary abelian and there exists a subgroup Q of P such that $P = N \times Q$.*

Proof :

$\boxed{1 \Rightarrow 3}$ Let $N \trianglelefteq P$ with $N \cap \Phi(P) = \mathbf{1}$. Then N maps injectively in the elementary abelian p-group $P/\Phi(P)$, so N is elementary abelian. Let $Q/\Phi(P)$ be a complement of $N\Phi(P)/\Phi(P)$ in $P/\Phi(P)$. Then $Q \geq \Phi(P) \geq [P, P]$, so Q is normal in P. Moreover $Q \cdot N = P$ and $Q \cap N\Phi(P) = (Q \cap N)\Phi(P) = \Phi(P)$, thus $Q \cap N \leq \Phi(P) \cap N = \mathbf{1}$. Now N and Q are normal subgroups of P which intersect trivially, hence they centralize each other. It follows that $P = N \times Q$.

$\boxed{3 \Rightarrow 2}$ This is clear.

$\boxed{2 \Rightarrow 1}$ If $P = N \cdot Q$ for some subgroup Q of P, and if N is central in P, then $P = N \times Q$. Thus $\Phi(P) = \mathbf{1} \times \Phi(Q)$, as N is elementary abelian. Then $N \cap \Phi(P) \leq N \cap Q = \mathbf{1}$. □

6.3. Lemma: *Let P be a finite p-group. The following conditions are equivalent:*

1. *P is atoric.*

2. *If $N \trianglelefteq P$ and $N \cap \Phi(P) = \mathbf{1}$, then $N = \mathbf{1}$.*

3. *$\Omega_1 Z(P) \leq \Phi(P)$.*

Proof :

$\boxed{1 \Rightarrow 2}$ Suppose that P is atoric. Let $N \trianglelefteq P$ with $N \cap \Phi(P) = \mathbf{1}$. Then by Lemma 6.2, the group N is elementary abelian and there exists a subgroup Q of P such that $P = N \times Q$. Hence $N = \mathbf{1}$.

$\boxed{2 \Rightarrow 3}$ Suppose now that Assertion 2 holds. If x is a central element of order p of P, then the subgroup N of P generated by x is normal in P, and non trivial. Then $N \cap \Phi(P) \neq \mathbf{1}$, hence $N \leq \Phi(P)$ since N has order p, thus $x \in \Phi(P)$.

$\boxed{3 \Rightarrow 1}$ Finally, if Assertion 3 holds, and if $P = E \times Q$ for some subgroups E and Q of P with E elementary abelian, then $\Phi(P) = \mathbf{1} \times \Phi(Q)$. Moreover $E \leq \Omega_1 Z(P) \leq \Phi(P) \leq Q$, so $E = E \cap Q = \mathbf{1}$, and P is atoric. □

6.4. Proposition: *Let P be a finite p-group, and N be a maximal normal subgroup of P such that $N \cap \Phi(P) = \mathbf{1}$. Then:*

1. *The group N is elementary abelian and there exists a subgroup T of P such that $P = N \times T$.*
2. *The group $P/N \cong T$ is atoric.*
3. *If Q is an atoric p-group and $s : P \twoheadrightarrow Q$ is a surjective group homomorphism, then $s(T) = Q$. In particular Q is isomorphic to a quotient of T.*

Proof : (1) This follows from Lemma 6.2.

(2) By (1), there exists $T \leq P$ such that $P = N \times T$. In particular $P/N \cong T$. Now if $T = E \times S$, for some subgroups E and S of T with E elementary abelian, then $P \cong P_1 = (N \times E) \times S$, and $N \times E$ is an elementary abelian normal subgroup of P_1 which intersects trivially $\Phi(P_1) = \Phi(S)$. By maximality of N, it follows that $E = \mathbf{1}$, so $T \cong P/N$ is atoric.

(3) Let $s : P \twoheadrightarrow Q$ be a surjective group homomorphism, where Q is atoric. By (1), the group N is elementary abelian, and there exists a subgroup T of P such that $P = N \times T$. Then $T \cong P^{@}$, and $\Phi(P) = \Phi(T)$. Moreover $s\big(\Phi(P)\big) = \Phi(Q)$ as P is a p-group, and $s\big(Z(P)\big) \leq Z(Q)$ as s is surjective. It follows that $s(N)$ is an elementary abelian central subgroup of Q, so $s(N) \leq \Phi(Q)$ since Q is atoric, by Lemma 6.3. Now $s(P) = Q = s(N)s(T)$, thus $Q = \Phi(Q)s(T)$, and $s(T) = Q$, as was to be shown. □

6.5. Notation: *When P is a finite p-group, and N is a maximal normal subgroup of P such that $N \cap \Phi(P) = \mathbf{1}$, we set $P^{@} = P/N$.*

By Proposition 6.4, the group $P^{@}$ does not depend on the choice of N, up to isomorphism: it is the greatest atoric quotient of P, in the sense that any atoric quotient of P is isomorphic to a quotient of $P^{@}$. In particular $P^{@}$ is trivial if and only if P is elementary abelian.

6.6. Proposition: *Let $s : P \twoheadrightarrow Q$ be a surjective group homomorphism. Then $P^{@} \cong Q^{@}$ if and only if $\mathrm{Ker}(s) \cap \Phi(P) = \mathbf{1}$.*

Proof : Let E be a maximal normal subgroup of P such that $E \cap \Phi(P) = \mathbf{1}$, and T be a subgroup of P such that $P = E \times T$. Then E is elementary abelian, and $\Phi(P) = \Phi(T)$. Let $\pi : Q \to Q^{@}$ be the canonical projection. By definition, we have $T \cong P^{@}$, and by Proposition 6.4, we have $\pi \circ s(T) = Q^{@}$.

Hence $Q^@$ is a quotient of $P^@$, and $P^@ \cong Q^@$ if and only if the map $\pi \circ s$ induces an isomorphism from T to $Q^@$, that is if $\mathrm{Ker}(\pi \circ s) \cap T = \mathbf{1}$. This implies $\mathrm{Ker}(s) \cap T = \mathbf{1}$, hence $\mathrm{Ker}(s) \cap \Phi(P) = \mathbf{1}$.

Conversely, if $\mathrm{Ker}(s) \cap \Phi(P) = \mathbf{1}$, then $\mathrm{Ker}(s) \cap \Phi(T) = \mathbf{1}$. Now the group $M = \mathrm{Ker}(s) \cap T$ is a normal subgroup of T such that $M \cap \Phi(T) = \mathbf{1}$. Since T is atoric, it follows from Lemma 6.3 that $M = \mathbf{1}$, hence $s(T) \cong T$. Now $Q = s(E)s(T)$, and $s(E)$ is a central elementary abelian subgroup of Q, since s is surjective. Let F be a complement of $G = s(E) \cap s(T)$ in $s(E)$. Then $Q = (F \cdot G)s(T) = F \cdot s(T)$, thus $Q = F \times s(T)$ since F is central in Q. It follows that $s(T)$ is a quotient of Q. Since $s(T) \cong T \cong P^@$ is atoric, the group $P^@$ is isomorphic to a quotient of $Q^@$, thus $P^@ \cong Q^@$. □

6.7. Proposition: *Let P be a finite p-group, and Q be a subquotient of P. Then $Q^@$ is a subquotient of $P^@$.*

Proof : Let (V, U) be a section of P such that $V/U \cong Q$. Then $Q^@$ is isomorphic to a quotient of $V^@$, by Lemma 6.4. Hence it suffices to prove that $V^@$ is a subquotient of $P^@$.

Let E be a maximal normal subgroup of P such that $E \cap \Phi(P) = \mathbf{1}$, and T be a subgroup of P such that $P = E \times T$. Then $V \leq E \times T$, so there exist a subgroup F of E, a subgroup X of T, a group Y, and surjective group homomorphisms $\alpha : F \to Y$ and $\beta : X \to Y$ such that

$$V = \{(f, x) \in F \times X \mid \alpha(f) = \beta(x)\} \ .$$

Now $F \leq E$ is elementary abelian. If $(f, x), (f', x') \in V$, then $[(f, x), (f', x')] = (1, [x, x'])$, so $[V, V] \leq \mathbf{1} \times [X, X]$. Conversely if $x, x' \in X$, then there exist $f, f' \in F$ such that $\alpha(f) = \beta(x)$ and $\alpha(f') = \beta(x')$, i.e. $(f, x), (f', x') \in V$. Then $[(f, x), (f', x')] = (1, [x, x'])$, and it follows that $[V, V] = \mathbf{1} \times [X, X]$. Similarly, if $(f, x) \in V$, then $(f, x)^p = (1, x^p)$. Conversely, if $x \in X$, then there exists $f \in F$ such that $\alpha(f) = \beta(x)$, i.e. $(f, x) \in V$, and $(1, x^p) = (f, x)^p$. It follows that $\Phi(V) = \mathbf{1} \times \Phi(X)$.

Now $N = \mathrm{Ker}(\alpha) \times \mathbf{1}$ is a normal subgroup of V, and $N \cap \Phi(V) = \mathbf{1}$. By Proposition 6.6, it follows that $V^@ \cong (V/N)^@$. Moreover the group homomorphism $(f, x) \in V \mapsto x \in X$ is surjective with kernel N, hence $V/N \cong X$. It follows that $V^@ \cong X^@$ is a isomorphic to a quotient of the subgroup X of $T \cong P^@$. Hence $V^@$ is a subquotient of $P^@$, as was to be shown. □

6.8. Proposition: *Let P be a finite p-group, let N be a normal subgroup of P such that $P/N \cong P^@$, and let Q be a subgroup of P. The following are* *equivalent:*

1. $Q^{@} \cong P^{@}$.
2. $QN = P$.
3. *There exists a central elementary abelian subgroup E of P such that $P = EQ$.*
4. *There exists an elementary abelian subgroup E of P such that $P = E \times Q$.*

Proof : $\boxed{1 \Rightarrow 2}$ Suppose $Q^{@} \cong P^{@}$. We have $N \cap \Phi(T) = \mathbf{1}$, by Proposition 6.6. Moreover $\Phi(Q) \leq \Phi(P)$, as P is a p-group. Setting $M = N \cap Q$, we have $M \cap \Phi(Q) = \mathbf{1}$, so $(Q/M)^{@} \cong Q^{@} \cong P^{@}$. But $\overline{Q} = Q/M \cong QN/N$ is a subgroup of $P/N \cong P^{@}$, and moreover there exists an elementary abelian subgroup E of $\overline{Q}$ such that $\overline{Q} \cong E \times \overline{Q}^{@} \cong E \times P^{@}$. Hence $E = \mathbf{1}$ and $\overline{Q} \cong QN/N \cong P/N$, so $QN = P$, as was to be shown.

$\boxed{2 \Rightarrow 3}$ We have $N \cap \Phi(P) = \mathbf{1}$, by Proposition 6.6. Hence N is elementary abelian, and central in P, and 2 implies 3.

$\boxed{2 \Rightarrow 3}$ Let E be an elementary abelian central subgroup of P such that $P = EQ$. Let F be a complement of $E \cap Q$ in E. Then F is elementary abelian and central in P. Moreover $QF = QE = P$, and $Q \cap F = \mathbf{1}$. Hence $P = F \times Q$.

$\boxed{4 \Rightarrow 1}$ If $P = E \times Q$ and E is elementary abelian, then $\Phi(P) = \mathbf{1} \times \Phi(Q)$. Thus $E \cap \Phi(P) = \mathbf{1}$, so $(P/E)^{@} \cong P^{@}$ by Proposition 6.6, and $Q^{@} \cong P^{@}$. □

6.9. Proposition:

1. *Let L be an atoric p-group, let $P = E \times L$ and $Q = F \times L$, where E and F are elementary abelian p-groups, and let $s : P \to Q$ be a group homomorphism. Then s is surjective if and only if there exist a surjective group homomorphism $a : E \to F$, group homomorphisms $b : L \to F$ and $c : E \to \Omega_1 Z(L)$, and an automorphism d of L such that*
$$\forall (e,l) \in E \times L, \quad s(e,l) = \big(a(e)b(l), c(e)d(l)\big) \ .$$
Moreover in this case $b \circ c(e) = 1$ for any $e \in E$, and s is an isomorphism if and only if a is an isomorphism.
2. *Let P be a finite p-group. For a group homomorphism*
$$\lambda : P \to \Omega_1 Z(P) \cap \Phi(P) \ ,$$

let $\alpha_\lambda : P \to P$ be defined by $\alpha(x) = x\lambda(x)$, for $x \in P$. Then α_λ is an automorphism of P.

3. *Let P be a finite p-group, and let $P = E \times Q$, where Q is atoric and E is elementary abelian. Then the correspondence $\lambda \mapsto \alpha_\lambda(E)$ is a bijection from the set of group homomorphisms $\lambda : P \to \Omega_1 Z(P) \cap \Phi(P)$ such that $Q \leq \operatorname{Ker}\lambda$ to the set of subgroups N of P such that $P = N \times Q$.*

Proof : (1) If s is surjective, then $s(E)$ is central in Q, so $s(E) \leq \Omega_1 Z(Q) = F \times \Omega_1 Z(L)$. Hence there exists group homomorphisms $a : E \to F$ and $c : E \to \Omega_1 Z(L)$ such that $s(e,1) = \big(a(e), c(e)\big)$, for any $e \in E$. Let $b : L \to F$ and $d : L \to L$ be the group homomorphisms defined by $s(1,l) = \big(b(l), d(l)\big)$, for $l \in L$. Then $s(e,l) = s(e,1)s(1,l) = \big(a(e)b(l), c(e)d(l)\big)$ for all $(e,l) \in P$. Moreover $b \circ c(e) = 1$ for any $e \in E$, since $c(E) \leq \Omega_1 Z(L) \leq \Phi(L)$, as L is atoric, and $\Phi(L) \leq \operatorname{Ker} b$, as F is elementary abelian.

Now the composition of s with the projection $F \times L \to L$ is surjective, hence $s(\mathbf{1} \times L) = L$ by Proposition 6.4. In other words d is surjective, hence it is an automorphism of L.

Since s is surjective, for any $(f,y) \in Q$, there exists $(e,x) \in P$ such that $a(e)b(x) = f$ and $c(e)d(x) = y$. The latter gives $x = d^{-1}\big(c(e)^{-1}y\big)$. Then $b(x) = bd^{-1}\big(c(e)^{-1}\big)bd^{-1}(y)$, and $bd^{-1}\big(c(e)^{-1}\big) = 1$ since $d^{-1}\big(c(e)^{-1}\big) \in d^{-1}\Omega_1 Z(L) = \Omega_1 Z(L)$, and $\Omega_1 Z(L) \leq \Phi(L) \leq \operatorname{Ker} b$. Then $b(x) = bd^{-1}(y)$, and $f = a(e)bd^{-1}(y)$. In particular, taking $y = 1$, we get that for any $f \in L$, there exists $e \in E$ such that $f = a(e)$. In other words a is surjective.

Conversely, given a surjective group homomorphism $a : E \to F$, a group homomorphism $b : L \to F$, a group homomorphism $c : E \to \Omega_1 Z(L)$, and an automorphism d of L, we can define $s : P \to Q$ by $s(e,x) = \big(a(e)b(x), c(e)d(x)\big)$, for $(e,x) \in P$. This is clearly a group homomorphism, as F is abelian, and the image of c is central in L. We have again $\Omega_1 Z(L) \leq \Phi(L) \leq \operatorname{Ker} b$, since F is elementary abelian. If $(f,y) \in Q$, we can choose an element $e \in E$ such that $f = a(e)bd^{-1}(y)$, and then set $x = d^{-1}\big(c(e)^{-1}y\big)$, i.e. $c(e)d(x) = y$. We also have $b(x) = bd^{-1}(y)$, since $d^{-1}\big(c(e)\big) \in \Omega_1 Z(L)$, so $f = a(e)b(x)$. Hence $s(e,x) = (f,y)$, and s is surjective.

Finally if s is an isomorphism, then $E \cong F$, and then the surjection a is an isomorphism. Conversely, if a is an isomorphism, then $E \cong F$, so $P \cong Q$, and the surjection s is an isomorphism.

(2) Clearly α_λ is a group homomorphism, since $\lambda(P) \leq Z(P)$. Moreover if $x \in \operatorname{Ker}\alpha_\lambda$, then $\lambda(x) = x$, so $x \in \Omega_1 Z(P) \cap \Phi(P) \leq \Phi(P) \leq \operatorname{Ker}\lambda$, since $\Omega_1 Z(P) \cap \Phi(P)$ is elementary abelian. Thus $x = 1$, and α_λ is injective. Hence it is an automorphism.

(3) Since $P = E \times Q$, we have $\Omega_1 Z(P) = E \times \Omega_1 Z(Q)$, and $\Phi(P) = \mathbf{1} \times \Phi(Q)$.

So if λ is a group homomorphism from P to $\Omega_1 Z(P)\cap\Phi(P)$ with $Q\leq \mathrm{Ker}\,\lambda$, we have $\lambda(e,l)=\big(1,\beta(e)\big)$ for some group homomorphism $\beta: E\to\Omega_1 Z(Q)$. Then the group $N=\alpha_\lambda(E)=\{\big(e,\beta(e)\big)\mid e\in E\}$ is central in P. Moreover $N\cap Q=\mathbf{1}$, and $NQ=P$, so $P=N\times Q$. Note that N determines the homomorphism β, hence also the homomorphism λ, so the map $\lambda\mapsto\alpha_\lambda(E)$ is injective.

It is moreover surjective: indeed, if N is a subgroup of $P=E\times Q$ such that $P=N\times Q$, then $N\cong P/Q\cong E$ is elementary abelian, hence central in P. Since $NQ=P$, for any $e\in E$, there exists $(a,b)\in N$ and $q\in Q$ such that $(e,1)=(a,b)(1,q)$, that is $e=a$ and $q=b^{-1}$. In other words $p_1(N)=E$. Moreover $N\cap Q=\mathbf{1}$, so $k_2(N)=\mathbf{1}$. So for $e\in E$, there exists a unique $x\in Q$ such that $(e,x)\in N$. Setting $x=\beta(e)$, we get a group homomorphism $\beta: E\to Q$, such that $N=\big\{\big(e,\beta(e)\big)\mid e\in E\big\}$. Since N is central in P, the image of β is contained in $\Omega_1 Z(Q)\leq\Phi(Q)$. Moreover $\Omega_1 Z(P)=E\times\Omega_1 Z(Q)$, and $\Phi(P)=1\times\Phi(Q)$, so $\big(\mathbf{1}\times\beta(E)\big)\leq\Omega_1 Z(P)\cap\Phi(P)$. Setting $\lambda(e,l)=\big(1,\beta(e)\big)$, we get a group homomorphism from P to $\Omega_1 Z(P)\cap\Phi(P)$, such that $Q\leq\mathrm{Ker}\,\lambda$, and $N=\alpha_\lambda(E)$. □

7. Splitting the biset category of p-groups, when $p\in R^\times$

7.1. Notation and Definition: *Let $R\mathcal{C}_p$ denote the full subcategory of the biset category $R\mathcal{C}$ consisting of finite p-groups. A p-*biset functor *over R is an R-linear functor from $R\mathcal{C}_p$ to the category of R-modules. Let $\mathcal{F}_{p,R}$ denote the full subcategory of $\mathcal{F}_R$ consisting of p-biset functors over R.*

In the statements below, we indicate by $[\ p\in R^\times]$ the assumption that p is invertible in R.

7.2. Theorem: $[\ p\in R^\times]$ *Let P and Q be finite p-groups, let (T,S) be a minimal section of P, and (V,U) be a minimal section of Q. Then*

$$\epsilon_{V,U}^{Q}\,RB(Q,P)\,\epsilon_{T,S}^{P}\neq\{0\}\implies (V/U)^{@}\cong(T/S)^{@}\ .$$

Proof : If $\epsilon_{V,U}^{Q}RB(Q,P)\epsilon_{T,S}^{P}$, there exists $a\in RB(Q,P)$ such that

$$\epsilon_{V,U}^{Q}\,a\,\epsilon_{T,S}^{P}=\mathrm{Indinf}_{V/U}^{Q}\varphi_{\mathbf{1}}^{V/U}\mathrm{Defres}_{V/U}^{Q}\,a\,\mathrm{Indinf}_{T/S}^{P}\varphi_1^{T/S}\mathrm{Defres}_{T/S}^{P}\neq 0\ ,$$

and in particular the element $b=\mathrm{Defres}_{V/U}^{Q}\,a\,\mathrm{Indinf}_{T/S}^{P}$ of $RB(V/U,T/S)$ is such that $\varphi_{\mathbf{1}}^{V/U}\,b\,\varphi_{\mathbf{1}}^{T/S}\neq 0$. It follows that there is a subgroup L of the product $(V/U)\times(T/S)$ such that

$$\varphi_{\mathbf{1}}^{V/U}\big[\big((V/U)\times(T/S)\big)/L\big]\varphi_{\mathbf{1}}^{T/S}\neq 0 \ \ .$$

Then Theorem 3.8 implies that $p_1(L)=V/U$, $k_1(L)\cap\Phi(V/U)=\mathbf{1}$, $p_2(L)=T/S$, and $k_2(L)\cap\Phi(T/S)=\mathbf{1}$. By Proposition 6.6, it follows that

$$(V/U)^{@}\cong\big(p_1(L)/k_1(L)\big)^{@}\cong\big(p_2(L)/k_2(L)\big)^{@}\cong(T/S)^{@} \ \ ,$$

as was to be shown. □

7.3. Notation: [$p\in R^\times$] *Let L be an atoric p-group. If P is a finite p-group, we set*

$$b_L^P=\sum_{\substack{(T,S)\in[\mathcal{M}(G)]\\ (T/S)^{@}\cong L}}\epsilon_{T,S}^P \ \ .$$

7.4. Theorem: [$p\in R^\times$]

1. *Let L be an atoric p-group, and P be a finite p-group. Then $b_L^P\neq 0$ if and only if $L\sqsubseteq P^{@}$.*
2. *Let L and M be atoric p-groups, and let P and Q be finite p-groups. If $b_M^Q RB(Q,P)b_L^P\neq\{0\}$, then $M\cong L$.*
3. *Let L be an atoric p-group, and let P and Q be finite p-groups. Then for any $a\in RB(Q,P)$*
$$b_L^Q\,a=a\,b_L^P \ \ .$$
4. *The family of elements $b_L^P\in RB(P,P)$, for finite p-groups P, is an idempotent endomorphism b_L of the identity functor of the category $R\mathcal{C}_p$ (i.e. an idempotent of the center of $R\mathcal{C}_p$). The idempotents b_L, for $L\in[\mathcal{A}t_p]$, are orthogonal, and their sum is equal to the identity element of the center of $R\mathcal{C}_p$.*
5. *For a given finite p-group P, the elements b_L^P, for $L\in[\mathcal{A}t_p]$ such that $L\sqsubseteq P^{@}$, are non zero orthogonal central idempotents of $RB(P,P)$, and their sum is equal to the identity of $RB(P,P)$.*

Proof : (1) The idempotent b_L^P is non zero if and only if there exists a minimal section (T,S) of P such that $(T/S)^{@}\cong L$. Then $L\sqsubseteq P^{@}$, by Proposition 6.7. Conversely, if $L\sqsubseteq P^{@}$, then $L\sqsubseteq P$, and there exists a minimal section (T,S) of P such that $T/S\cong L$. Then $(T/S)^{@}\cong L^{@}\cong L$, so $\epsilon_{T,S}^P$ appears in the sum defining b_L^p, thus $b_L^P\neq 0$.

(2) If $b_M^Q RB(Q,P) b_L^P \neq \{0\}$, then there exist a minimal section (V,U) of Q with $(V/U)^@ \cong M$ and a minimal section (T,S) of P with $(T/S)^@ \cong L$ such that $\epsilon_{V,U}^Q RB(Q,P) \epsilon_{T,S}^P \neq 0$. Then $(V/U)^@ \cong (T/S)^@$ by Theorem 7.2, that is $M \cong L$.

(3) The identity element of $RB(P,P)$ is equal to the sum of the idempotents $\epsilon_{T,S}^P$, for $(T,S) \in [\mathcal{M}(P)]$. Grouping those idempotents $\epsilon_{T,S}^P$ for which $(T/S)^@$ is isomorphic to a given $L \in [\mathcal{A}t_p]$ shows that the identity element of $RB(P,P)$ is equal to the sum of the idempotents b_L^P, for $L \in [\mathcal{A}t_p]$ (and there are finitely many non zero b_L^P, by (1)). It follows that

$$\begin{aligned} b_M^Q\, a &= b_M^Q\, a \sum_{L\in[\mathcal{A}t_p]} b_L^P = \sum_{L\in[\mathcal{A}t_p]} b_M^Q\, a\, b_L^P \\ &= b_M^Q\, a\, b_M^P \;\; \text{[by (2)]} \\ &= \sum_{L\in[\mathcal{A}t_p]} b_L^Q\, a\, b_M^P \;\; \text{[by (2)]} \\ &= a\, b_M^P \;\;, \end{aligned}$$

since $\sum_{L\in[\mathcal{A}t_p]} b_L^Q$ is the identity element of $RB(Q,Q)$.

It follows that the family b_L^P, where P is a finite p-group, is an element b_L of the center of $R\mathcal{C}_p$. Clearly $b_L^2 = b_L$, and if L and M are non isomorphic atoric p-groups, then $b_L b_M = 0$, by (2). Moreover the infinite sum $\sum_{L\in[\mathcal{A}t_p]} b_L$ is actually locally finite, i.e. for each finite p-group P, the sum $\sum_{L\in[\mathcal{A}t_p]} b_L^P$ has only finitely many non zero terms. The sum $\sum_{L\in[\mathcal{A}t_p]} b_L$ is clearly equal to the identity endomorphism of the identity functor of $R\mathcal{C}_p$.

(4) This is a straightforward consequence of (1) and (3). □

7.5. Corollary: $[\ p \in R^\times]$

1. *Let L be an atoric p-group. For a p-biset functor F, the family of maps $F(b_L^P) : F(P) \to F(P)$, for finite p-groups P, is an endomorphism of F, denoted by $F(b_L)$.*
2. *If $\theta : F \to G$ is a natural transformation of p-biset functors, the diagram*

$$\begin{array}{ccc} F & \xrightarrow{F(b_L)} & F \\ {\scriptstyle\theta}\downarrow & & \downarrow{\scriptstyle\theta} \\ G & \xrightarrow[G(b_L)]{} & G \end{array}$$

is commutative. Hence the family of endomorphisms $F(b_L)$, for p-biset functors F, is an idempotent of the center of the category $\mathcal{F}_{p,R}$, denoted by $\widehat{b}_L$.

3. *The idempotents $\widehat{b}_L$, for $L \in [\mathcal{A}t_p]$, are orthogonal idempotents of the center of $\mathcal{F}_{p,R}$, and their sum is the identity.*
4. *If F is a p-biset functor over R, let $\widehat{b}_L F$ denote the image of the endomorphism $F(b_L)$ of F. Then $F = \bigoplus_{L\in[\mathcal{A}t_p]} \widehat{b}_L F$.*
5. *Let $\widehat{b}_L\mathcal{F}_{p,R}$ denote the full subcategory of $\mathcal{F}_{p,R}$ consisting of functors F such that $F = \widehat{b}_L F$. Then $\widehat{b}_L\mathcal{F}_{p,R}$ is an abelian subcategory of $\mathcal{F}_{p,R}$. Moreover the functor*

$$F \in \mathcal{F}_{p,R} \mapsto (\widehat{b}_L F)_{L\in[\mathcal{A}t_p]} \in \prod_{L\in[\mathcal{A}t_p]} \widehat{b}_L\mathcal{F}_{p,R}$$

is an equivalence of categories.

Proof : All assertions are straightforward consequences of Theorem 7.4. □

7.6. Notation: *For an atoric p-group L, let $R\mathcal{C}_p^L$ denote the full subcategory of $R\mathcal{C}_p$ consisting of the class $\mathcal{Y}_L$ of finite p-groups P such that $P^{@} \sqsubseteq L$. When $p \in R^\times$, Let moreover*

$$b_L^+ = \sum_{\substack{H\in[\mathcal{A}t_p]\\ H\sqsubseteq L}} b_H$$

be the sum of the idempotents b_H corresponding to atoric subquotients of L, up to isomorphism.

The class $\mathcal{Y}_L$ is closed under taking subquotients, by Proposition 6.7. It follows that we can apply the results of Section 6 (Appendix) of [12]: if F is a p-biset functor over R, we can restrict F to an R-linear functor from $R\mathcal{C}_p^L$ to R-Mod. This yields a forgetful functor $\mathcal{O}_{\mathcal{Y}_L} : \mathcal{F}_{p,R} \to \mathsf{Fun}_R\big(R\mathcal{C}_p^L, R\text{-}\mathsf{Mod}\big)$. The right adjoint $\mathcal{R}_{\mathcal{Y}_L}$ of this functor is described in full detail in Section 6 of [12], as follows: if G is an R-linear functor from $R\mathcal{C}_p^L$ to R-Mod, and P is a finite p-group, set

$$\mathcal{R}_{\mathcal{Y}_L}(G)(P) = \lim_{\substack{\longleftarrow\\ (X,M)\in\Sigma_L(P)}} G(X/M) \tag{7.7}$$

the inverse limit of modules $G(X/M)$ on the set $\Sigma_L(P)$ of sections (X, M)

of P such that $(X/M)^@ \sqsubseteq L$, i.e. the set of sequences $(l_{X,M})_{(X,M)\in\Sigma_L(P)}$ with the following properties:

1. if $(X,M)\in\Sigma_L(P)$, then $l_{X,M}\in G(X/M)$.
2. if $(X,M),(Y,N)\in\Sigma_L(P)$ and $M\leq N\leq Y\leq X$, then
$$\mathrm{Defres}_{Y/N}^{X/M} l_{X,M} = l_{Y,N} \quad .$$
3. if $x\in P$ and $(X,M)\in\Sigma_L(P)$, then ${}^x l_{X,M} = l_{{}^xX,{}^xM}$.

Recall now that for finite groups P and Q, and for a finite (Q,P)-biset U, for a subgroup T of Q and an element u of U, the subgroup T^u of P is defined by $T^u=\{x\in P \mid \exists t\in T \;\; tu=ux\}$. By Lemma 6.4 of [12], if (T,S) is a section of Q, then (T^u,S^u) is a section of P, and T^u/S^u is a subquotient of T/S.

With this notation, when P and Q are finite p-groups, when U is a finite (Q,P)-biset, and $l=(l_{X,M})_{(X,M)\in\Sigma_L(P)}$ is an element of $\mathcal{R}_{\mathcal{Y}_L}(G)(P)$, we denote by Ul the sequence indexed by $\Sigma_L(Q)$ defined by
$$(Ul)_{Y,N}=\sum_{u\in[Y\backslash U/P]_L}(N\backslash Yu)(l_{Y^u,N^u})$$
where $[Y\backslash U/P]$ is a set of representatives of $(Y\times P)$-orbits on U, and $N\backslash Yu$ is viewed as a $(Y/N,Y^u/N^u)$-biset. It shown in Section 6 of [12] that $Ul\in\mathcal{R}_{\mathcal{Y}_L}(G)(Q)$, and that $\mathcal{R}_{\mathcal{Y}_L}(G)$ becomes a p-biset functor in this way. Moreover[1]:

7.8. Theorem: [[12] Theorem 6.15] *The assignment $G\mapsto\mathcal{R}_{\mathcal{Y}_L}(G)$ is an R-linear functor $\mathcal{R}_{\mathcal{Y}_L}$ from $\mathsf{Fun}_R\big(R\mathcal{C}_p^L, R\text{-}\mathsf{Mod}\big)$ to $\mathcal{F}_{p,R}$, which is right adjoint to the forgetful functor $\mathcal{O}_{\mathcal{Y}_L}$. Moreover the composition $\mathcal{O}_{\mathcal{Y}_L}\circ\mathcal{R}_{\mathcal{Y}_L}$ is isomorphic to the identity functor of $\mathsf{Fun}_R\big(R\mathcal{C}_p^L, R\text{-}\mathsf{Mod}\big)$.*

7.9. Theorem: [$p\in R^\times$] *For an atoric p-group L, let $\widehat{b}_L^+\mathcal{F}_{p,R}$ be the full subcategory of $\mathcal{F}_{p,R}$ consisting of functors F such that $\widehat{b}_L^+F=F$. Then the forgetful functor $\mathcal{O}_{\mathcal{Y}_L}$ and its right adjoint $\mathcal{R}_{\mathcal{Y}_L}$ restrict to quasi-inverse equivalences of categories*
$$\widehat{b}_L^+\mathcal{F}_{p,R}\underset{\mathcal{R}_{\mathcal{Y}_L}}{\overset{\mathcal{O}_{\mathcal{Y}_L}}{\rightleftarrows}}\mathsf{Fun}_R\big(R\mathcal{C}_p^L, R\text{-}\mathsf{Mod}\big) \quad .$$

[1] In Theorem 6.15 of [12], only the case $R=\mathbb{Z}$ is considered, but the proofs extend trivially to the case of an arbitrary commutative ring R

Proof : First step: The first thing to check is that the image of the functor $\mathcal{R}_{\mathcal{Y}_L}$ is contained in $\widehat{b}_L^+\mathcal{F}_{p,R}$. We first prove that if H is an atoric p-group, if $F \in \mathcal{F}_{p,R}$, and if $\mathcal{O}_{\mathcal{Y}_L}(\widehat{b}_H F) \neq 0$, then $H \sqsubseteq L$: indeed in that case, there exists $P \in \mathcal{Y}_L$ such that $b_H^P F(P) \neq 0$. In particular $b_H^P \neq 0$ by Theorem 7.4, hence $H \sqsubseteq P^@$. Since $P^@ \sqsubseteq L$ as $P \in \mathcal{Y}_L$, it follows that $H \sqsubseteq L$, as claimed.

In particular

$$\mathcal{O}_{\mathcal{Y}_L}(F) = \mathcal{O}_{\mathcal{Y}_L}\Big(\sum_{\substack{H\in[\mathcal{A}t_p]\\ H\sqsubseteq L}} \widehat{b}_H F\Big) = \mathcal{O}_{\mathcal{Y}_L}\big(\widehat{b}_L^+ F\big) \ .$$

Set $\mathcal{G}_p^L = \mathsf{Fun}_R\big(R\mathcal{C}_p^L, R\text{-}\mathsf{Mod}\big)$, and let $G \in \mathcal{G}_p^L$. Let H be an atoric p-group such that $H \not\sqsubseteq L$. If $F \in \mathcal{F}_{p,R}$, then

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{F}_{p,R}}\big(F, \widehat{b}_H\mathcal{R}_{\mathcal{Y}_L}(G)\big) &= \mathrm{Hom}_{\mathcal{F}_{p,R}}\big(\widehat{b}_H F, \widehat{b}_H\mathcal{R}_{\mathcal{Y}_L}(G)\big)\\
&= \mathrm{Hom}_{\mathcal{F}_{p,R}}\big(\widehat{b}_H F, \mathcal{R}_{\mathcal{Y}_L}(G)\big)\\
&\cong \mathrm{Hom}_{\mathcal{G}_p^L}\big(\mathcal{O}_{\mathcal{C}_{\mathcal{Y}_L}}\big(\widehat{b}_H F\big), G\big) = \{0\} \ .
\end{aligned}$$

So the functor $F \mapsto \mathrm{Hom}_{\mathcal{F}_{p,R}}\big(F, \widehat{b}_H\mathcal{R}_{\mathcal{Y}_L}(G)\big)$ is the zero functor, and it follows from Yoneda's lemma that $\widehat{b}_H\mathcal{R}_{\mathcal{Y}_L}(G) = 0$ if $H \not\sqsubseteq L$. In other words $\mathcal{R}_{\mathcal{Y}_L}(G) = \widehat{b}_L^+\mathcal{R}_{\mathcal{Y}_L}(G)$, as was to be shown.

Second step: The first step shows that we have adjoint functors

$$\widehat{b}_L^+\mathcal{F}_{p,R} \underset{\mathcal{R}_{\mathcal{Y}_L}}{\overset{\mathcal{O}_{\mathcal{Y}_L}}{\rightleftarrows}} \mathsf{Fun}_R\big(R\mathcal{C}_p^L, R\text{-}\mathsf{Mod}\big) = \mathcal{G}_p^L \ .$$

Moreover, the composition $\mathcal{O}_{\mathcal{Y}_L} \circ \mathcal{R}_{\mathcal{Y}_L}$ is isomorphic to the identity functor, by Theorem 7.8. All we have to show is that the unit of the adjunction is also an isomorphism, in other words, that for any $F \in \widehat{b}_L^+\mathcal{F}_{p,R}$ and any finite p-group P, the natural map

$$\textbf{(7.10)} \qquad \eta_P : F(P) \to \mathcal{R}_{\mathcal{Y}_L}\mathcal{O}_{\mathcal{Y}_L}(F)(P) = \varprojlim_{(X,M)\in\Sigma_L(P)} F(X/M)$$

sending $u \in F(P)$ to the sequence $\big(\mathrm{Defres}_{X/M}^P u\big)_{(X,M)\in\Sigma_L(P)}$, is an isomorphism.

The map η_P is injective: indeed, if $u \in F(P)$, then $u = \sum_{\substack{H\in[\mathcal{A}t_p]\\ H\sqsubseteq L}} b_H^P u$, as $F = \widehat{b}_L^+ F$. If $\mathrm{Defres}_{X/M}^P u = 0$ for any section (X, M) of P with $(X/M)^@ \sqsubseteq L$, then $F(\epsilon^P_{T,S})(u) = 0$ for any section (T,S) of P such that $(T/S)^{@} \sqsubseteq L$, by Proposition 4.6 and Proposition 6.7. In particular $b^P_H u = 0$ for any atoric subquotient H of L, hence $u = 0$.

To prove that η_P is also surjective, we generalize the construction of Theorem A.2 of [11] (which is the case $L = \mathbf{1}$), and we define, for an element $v = (v_{X,M})_{(X,M)\in\Sigma_L(P)}$ in $\mathcal{R}_{\mathcal{Y}_L}\mathcal{O}_{\mathcal{Y}_L}(F)(P)$, an element $u = \iota_P(v)$ of $F(P)$ by

$$u = \frac{1}{|P|}\sum_{\substack{(T,S)\in\mathcal{M}(P)\\ (T/S)^{@}\sqsubseteq L}}\ \sum_{\substack{X\leq T,\, M\trianglelefteq T\\ S\leq M\leq\Phi(T)\leq X\leq T}} |X|\mu(X,T)\mu_{\trianglelefteq T}(S,M)\mathrm{Indinf}^P_{X/M}v_{X,M}\quad .$$

This yields an R-linear map $\iota_P : \mathcal{R}_{\mathcal{Y}_L}\mathcal{O}_{\mathcal{Y}_L}(F)(P) \to F(P)$.

For $(Y,N)\in\Sigma_L(P)$, set $u_{Y,N} = \mathrm{Defres}^P_{Y/N}u$. Then:

$$u_{Y,N} = \sum_{\substack{(T,S)\in\mathcal{M}(P)\\ (T/S)^{@}\sqsubseteq L}}\ \sum_{\substack{X\leq T,\, M\trianglelefteq T\\ S\leq M\leq\Phi(T)\leq X\leq T}} \frac{|X|}{|P|}\mu(X,T)\mu_{\trianglelefteq T}(S,M)\mathrm{Defres}^P_{Y/N}\mathrm{Indinf}^P_{X/M}v_{X,M}.$$

Moreover

$$\mathrm{Defres}^P_{Y/N}\mathrm{Indinf}^P_{X/M}v_{X,M} = \sum_{g\in[Y\backslash P/X]}\mathrm{Indinf}^{Y/N}_{J_g/J'_g}\mathrm{Iso}(\phi_g)\mathrm{Defres}^{{}^gX/{}^gM}_{I_g/I'_g}v_{X,N}\quad ,$$

where $J_g = N(Y\cap{}^gX)$, $J'_g = N(Y\cap{}^gM)$, $I_g = {}^gM(Y\cap{}^gX)$, $I'_g = {}^gM(N\cap{}^gX)$, and ϕ_g is the isomorphism $I_g/I'_g \to J_g/J'_g$ sending xI'_g to xJ'_g, for $x\in Y\cap{}^gX$. Hence

$$\begin{aligned}\mathrm{Defres}^P_{Y/N}\mathrm{Indinf}^P_{X/M}v_{X,M} &= \sum_{g\in[Y\backslash P/X]}\mathrm{Indinf}^{Y/N}_{J_g/J'_g}\mathrm{Iso}(\phi_g)v_{I_g,I'_g}\\ &= \frac{|Y\cap{}^gX|}{|Y||X|}\sum_{g\in P}\mathrm{Indinf}^{Y/N}_{J_g/J'_g}\mathrm{Iso}(\phi_g)v_{I_g,I'_g}\quad .\end{aligned}$$

Thus

$$u_{Y,N} = \sum_{\substack{(T,S)\in\mathcal{M}(P)\\ (T/S)^{@}\sqsubseteq L\\ X\leq T,\, M\trianglelefteq T\\ S\leq M\leq\Phi(T)\leq X\leq T\\ g\in P}} \frac{|Y\cap{}^gX|}{|P||Y|}\mu(X,T)\mu_{\trianglelefteq T}(S,M)\mathrm{Indinf}^{Y/N}_{J_g/J'_g}\mathrm{Iso}(\phi_g)v_{I_g,I'_g}.$$

Now $\mu(X,T) = \mu({}^gX,{}^gT)$ and $\mu_{\trianglelefteq T}(S,M) = \mu_{\trianglelefteq {}^gT}({}^gS,{}^gM)$, so summing over $({}^gT,{}^gS,{}^gX,{}^gM)$ instead of (T,S,X,M) we get

$$u_{Y,N} = \sum_{\substack{(T,S)\in\mathcal{M}(P)\\ (T/S)^{@}\sqsubseteq L\\ X\leq T,\, M\trianglelefteq T\\ S\leq M\leq\Phi(T)\leq X\leq T}} \frac{|Y\cap X|}{|Y|}\mu(X,T)\mu_{\trianglelefteq T}(S,M)\mathrm{Indinf}^{Y/N}_{J_1/J'_1}\mathrm{Iso}(\phi_1)v_{I_1,I'_1}.$$

Setting $W = Y \cap X$, we have $J_1 = NW$, $J_1' = N(W \cap M)$, $I_1 = MW$, $I_1' = M(N \cap W)$, and these four groups only depend on W, once M and N are given. Hence, for given T, S and M, we can group together the terms of the above summation for which $Y \cap X$ is a given subgroup W of $Y \cap T$. This gives

$$u_{Y,N} = \sum_{\substack{(T,S)\in\mathcal{M}(P)\\ (T/S)^@\sqsubseteq L\\ M\trianglelefteq T\\ S\leq M\leq\Phi(T)\\ W\leq Y\cap T}} \Big(\sum_{\substack{\Phi(T)\leq X\leq T\\ X\cap Y=W}} \mu(X,T)\Big)\frac{|W|}{|Y|}\mu_{\trianglelefteq T}(S,M)\mathrm{Indinf}_{J_1/J_1'}^{Y/N}\mathrm{Iso}(\phi_1)v_{I_1,I_1'}.$$

Moreover $\sum\limits_{\substack{\Phi(T)\leq X\leq T\\ X\cap Y=W}} \mu(X,T) = \sum\limits_{\substack{X\leq T\\ X\cap(Y\cap T)=W}} \mu(X,T)$, since $\mu(X,T) = 0$ unless $X \geq \Phi(T)$, and the latter summation vanishes unless $Y \cap T = T$, by classical combinatorial lemmas ([15] Corollary 3.9.3). This gives:

$$u_{Y,N} = \sum_{\substack{(T,S)\in\mathcal{M}(P)\\ (T/S)^@\sqsubseteq L\\ M\trianglelefteq T\\ S\leq M\leq\Phi(T)\leq W\leq T\leq Y}} \frac{|W|}{|Y|}\mu(W,T)\mu_{\trianglelefteq T}(S,M)\mathrm{Indinf}_{J_1/J_1'}^{Y/N}\mathrm{Iso}(\phi_1)v_{I_1,I_1'}.$$

Moreover in this summation $J_1 = NW$, $J_1' = N(W \cap M) = NM$, $I_1 = MW = W$, $I_1' = M(N\cap W) = MN\cap W$. All these groups remain unchanged if we replace M by $M\big(N \cap \Phi(T)\big)$, so for given T, S and W, we can group together those terms for which $M\big(N \cap \Phi(T)\big)$ is a given normal subgroup U of T with $U \leq \Phi(T)$. The sum $\sum\limits_{\substack{S\leq M\trianglelefteq T\\ M(N\cap\Phi(T))=U}} \mu_{\trianglelefteq T}(S,M)$ is equal to 0 (by the same above-mentioned classical combinatorial lemmas) unless $N\cap\Phi(T) \leq S$. Hence

$$u_{Y,N} = \sum_{\substack{(T,S)\in\mathcal{M}(P)\\ (T/S)^@\sqsubseteq L\\ U\trianglelefteq T\\ N\cap\Phi(T)\leq S\leq U\leq\Phi(T)\leq W\leq T\leq Y}} \frac{|W|}{|Y|}\mu(W,T)\mu_{\trianglelefteq T}(S,U)\mathrm{Indinf}_{J_1/J_1'}^{Y/N}\mathrm{Iso}(\phi_1)v_{I_1,I_1'},$$

where $J_1 = NW$, $J_1' = NU$, $I_1 = W$, $I_1' = UN \cap W$.

Now if $N \cap \Phi(T) \leq S \leq \Phi(T) \leq T \leq Y$, then $(TN/N)^@ \sqsubseteq (Y/N)^@$. Moreover the normal subgroup $(N\cap T)/\big(N\cap\Phi(T)\big)$ of $T/\big(N\cap\Phi(T)\big)$ intersects trivially the Frattini subgroup

$$\Phi\Big(T/\big(N\cap\Phi(T)\big)\Big) = \Phi(T)\big(N\cap\Phi(T)\big)/\big(N\cap\Phi(T)\big)\ ,$$

so $\Big(T/\big(N\cap\Phi(T)\big)\Big)^{@} \cong (T/(T\cap N)^{@} \cong (TN/N)^{@}$ by Proposition 6.6. Then $(T/S)^{@} \sqsubseteq \Big(T/\big(N\cap\Phi(T)\big)\Big)^{@} \sqsubseteq (TN/N)^{@} \sqsubseteq (Y/N)^{@}$. As $(Y/N)^{@} \sqsubseteq L$ by assumption, it follows that

$$u_{Y,N} = \sum_{\substack{S\trianglelefteq T\leq Y\\ U\trianglelefteq T\\ N\cap\Phi(T)\leq S\leq U\leq\Phi(T)\leq W\leq T\leq Y}} \frac{|W|}{|Y|}\mu(W,T)\mu_{\trianglelefteq T}(S,U)\mathrm{Indinf}_{J_1/J_1'}^{Y/N}\mathrm{Iso}(\phi_1)v_{I_1,I_1'}.$$

Now the sum $\sum\limits_{\substack{S\trianglelefteq T\\ N\cap\Phi(T)\leq S\leq U}} \mu_{\trianglelefteq T}(S,U)$ is equal to zero unless $U = N\cap\Phi(T)$. Hence

$$u_{Y,N} = \sum_{\Phi(T)\leq W\leq T\leq Y} \frac{|W|}{|Y|}\mu(W,T)\mathrm{Indinf}_{J_1/J_1'}^{Y/N}\mathrm{Iso}(\phi_1)v_{I_1,I_1'}.$$

For a given subgroup W of Y, the sum $\sum\limits_{\Phi(T)\leq W\leq T\leq Y} \mu(W,T)$ is equal to $\sum\limits_{W\leq T\leq Y} \mu(W,T)$ since $\mu(W,T)=0$ unless $W\geq\Phi(T)$, and the latter is equal to zero if $W\neq Y$, and to 1 if $W=Y$. Thus

$$u_{Y,N} = \frac{|Y|}{|Y|}\mathrm{Indinf}_{J_1/J_1'}^{Y/N}\mathrm{Iso}(\phi_1)v_{I_1,I_1'},$$

where $J_1 = NY = Y$, $J_1' = N(Y\cap U) = N$, $I_1 = Y =, I_1' = UN\cap Y = N$. Hence $I_1 = J_1 = Y$ and $I_1' = J_1' = N$, so ϕ_1 is equal to the identity. It follows that $u_{Y,N} = v_{Y,N}$ for any $(Y,N)\in\Sigma_L(P)$, so $\eta_P(u) = v$. This proves that the map η_P is surjective, hence an isomorphism, with inverse ι_P. This completes the proof of Theorem 7.9. □

7.11. Definition: *Let $R\mathcal{C}_p^{\sharp L}$ be the following category:*

- *The objects of $R\mathcal{C}_p^{\sharp L}$ are the finite p-groups P such that $P^{@}\cong L$.*
- *If P and Q are finite p-groups such that $P^{@}\cong Q^{@}\cong L$, then*

$$\mathrm{Hom}_{R\mathcal{C}_p^{\sharp L}}(P,Q) = RB(Q,P)/\sum_{L\not\sqsubseteq S} RB(Q,S)B(S,P)$$

  is the quotient of $RB(Q,P)$ by the R-submodule generated by all morphisms from P to Q in $R\mathcal{C}_p$ which factor through a p-group S which do not admit L as a subquotient.
- *The composition of morphisms in $R\mathcal{C}_p^{\sharp L}$ is induced by the composition of morphisms in $R\mathcal{C}_p$.*

7.12. Remark: Morphisms in $R\mathcal{C}_p$ which factor through a p-group S such that $L \not\sqsubseteq S$ clearly generate a two-sided ideal, so the composition in $R\mathcal{C}_p^{\sharp L}$ is well defined. Moreover the category $R\mathcal{C}_p^{\sharp L}$ is R-linear. Let $\mathsf{Fun}_R\big(R\mathcal{C}_p^{\sharp L}, R\text{-}\mathsf{Mod}\big)$ denote the category of R-linear functors from $R\mathcal{C}_p^{\sharp L}$ to the category R-Mod of R-modules.

7.13. Lemma: *Let p be a prime, and L be an atoric p-group. Let P and Q be finite p-groups.*

1. *If $P^@ \cong L$ or $Q^@ \cong L$, and if $M \leq (Q \times P)$, then $q(M)^@ \sqsubseteq L$. Moreover $q(M)^@ \cong L$ if and only if $L \sqsubseteq q(M)$.*
2. *If $P^@ \cong Q^@ \cong L$, then*

$$\mathrm{Hom}_{R\mathcal{C}_p^{\sharp L}}(P,Q) = RB(Q,P) / \sum_{S^@ \sqsubset L} RB(Q,S)B(S,P)$$

   is also the quotient of $RB(Q,P)$ by the R-submodule generated by all morphisms from P to Q in $R\mathcal{C}_p$ which factor through a p-group S such that $S^@$ is a proper subquotient of L.

3. *If $P^@ \cong Q^@ \cong L$, then $\mathrm{Hom}_{R\mathcal{C}_p^{\sharp L}}(P,Q)$ has an R-basis consisting of the (images of the) transitive (Q,P)-bisets $(Q \times P)/M$, where M is a subgroup of $(Q \times P)$ such that $q(M)^@ \cong L$ (up to conjugation).*

Proof : (1) Indeed $q(M)$ is a subquotient of P, and a subquotient of Q. Hence $q(M)^@$ is a subquotient of $P^@$ and a subquotient of $Q^@$, thus $q(M) \sqsubseteq L^@ \cong L$. Now suppose that $q(M)^@ \cong L$. Then L is a quotient of $q(M)$, so $L \sqsubseteq q(M)$. Conversely, if $L \sqsubseteq q(M)$, then $L \cong L^@$ is a subquotient of $q(M)^@$, which is a subquotient of L. So $q(M)^@ \cong L$.

(2) Let S be a finite p-group such that $L \not\sqsubseteq S$, or equivalently $L \not\sqsubseteq S^@$. Any element of $RB(Q,S)B(S,P)$ is a linear combination of (Q,P)-bisets of the form $(Q \times P)/(M * N)$, for $M \leq (Q \times S)$ and $N \leq (S \times P)$. This biset $(Q \times P)/(M * N)$ also factors though $T = q(M * N)$, by 2.6. Moreover T is a subquotient of $q(M)$ and $q(N)$, hence a subquotient of Q, S, and P. Hence $T^@ \sqsubseteq Q^@ \cong L$, and $T^@ \not\cong L$, since $L \not\sqsubseteq S^@$. Hence $T \sqsubset L$.

(3) The (images of the) elements $(Q \times P)/M$, where M is a subgroup of $(Q \times P)$ such that $q(M)^@ \cong L$ (up to conjugation), clearly generate $\mathrm{Hom}_{R\mathcal{C}_p^{\sharp L}}(P,Q)$. Moreover, the proof of (2) shows that they are linearly independent, since any transitive (Q,P)-biset $(Q \times P)/N$ appearing in an

element of the sum $\sum_{S^@ \sqsubset L} RB(Q,S)B(S,P)$ is such that $q(N)^@ \sqsubset L$. . $\square$

7.14. Remark: If G is an R-linear functor from $R\mathcal{C}_p^{\sharp L}$ to the category R-Mod of R-modules, we can extend G to an R-linear functor from $R\mathcal{C}_p^{L}$ to R-Mod by setting $G(P) = \{0\}$ if P is a finite p-group such that $P^@$ is a proper subquotient of L. Conversely, an R-linear functor from $R\mathcal{C}_p^{L}$ to R-Mod which vanishes on p-groups P such that $P^@ \not\cong L$ can be viewed as an R-linear functor from $R\mathcal{C}_p^{\sharp L}$ to R-Mod. In the sequel, we will freely identify those two types of functors, and consider $\mathsf{Fun}_R\big(R\mathcal{C}_p^{\sharp L}, R\text{-}\mathsf{Mod}\big)$ as the full subcategory of $\mathsf{Fun}_R\big(R\mathcal{C}_p^{L}, R\text{-}\mathsf{Mod}\big)$ consisting of functors which vanish on p-groups P such that $P^@ \not\cong L$.

7.15. Theorem: [$p \in R^\times$] *Let L be an atoric p-group.*

1. *If F is a p-biset functor over R such that $F = \widehat{b}_L F$, and P is a finite p-group such that $L \not\sqsubseteq P$, then $F(P) = \{0\}$.*
2. *If G is an R-linear functor from $R\mathcal{C}_p^{\sharp L}$ to R-Mod, then $\widehat{b}_L \mathcal{R}_{\mathcal{Y}_L}(G) = \mathcal{R}_{\mathcal{Y}_L}(G)$.*
3. *The forgetful functor $\mathcal{O}_{\mathcal{Y}_L}$ and its right adjoint $\mathcal{R}_{\mathcal{Y}_L}$ restrict to quasi-inverse equivalences of categories*

$$\widehat{b}_L \mathcal{F}_{p,R} \underset{\mathcal{R}_{\mathcal{Y}_L}}{\overset{\mathcal{O}_{\mathcal{Y}_L}}{\rightleftarrows}} \mathsf{Fun}_R\big(R\mathcal{C}_p^{\sharp L}, R\text{-}\mathsf{Mod}\big) \quad .$$

Proof : (1) Since $\widehat{b}_L F = F$, then in particular $F(b_L^P)F(P) = F(P)$. If $L \not\sqsubseteq P$, then there is no minimal section (T,S) of P with $(T/S)^@ \cong L$, thus $b_L^P = 0$, and $F(P) = \{0\}$.

(2) Let G be an R-linear functor from $R\mathcal{C}_p^{\sharp L}$ to R-Mod, in other words an R-linear functor from $\mathcal{F}\mathcal{C}_p^{L}$ to R-Mod which vanishes on p-groups P such that $P^@$ is a proper subquotient of L. By Theorem 7.9, we have $\widehat{b}_L^+ \mathcal{R}_{\mathcal{Y}_L}(G) = \mathcal{R}_{\mathcal{Y}_L}(G)$. If H is an atoric p-group which is a proper subquotient of L, then G vanishes over any subquotient Q of H, since $Q^@ \sqsubseteq H \sqsubset L$ if $Q \sqsubseteq H$. In particular b_H^P acts by 0 on $\mathcal{R}_{\mathcal{Y}_L}(G)(P)$, for any finite p-group P: indeed b_H^P is a linear combination of terms of the form $\mathrm{Indinf}_{X/M}^P \mathrm{Defres}_{X/M}^P$, where (X,M) is a section of P such that $S \leq M \leq \Phi(T) \leq X \leq T$, for some section (T,S) of P with $(T/S)^@ \cong H$. For such a section (X,M) of P, we have $(X/M)^@ \sqsubseteq (T/S)^@ \sqsubseteq H$, thus G vanishes on any subquotient of X/M, so $\mathcal{R}_{\mathcal{Y}_L}(G)(X/M) = \{0\}$, hence $b_H^P = 0$ on $\mathcal{R}_{\mathcal{Y}_L}(G)(P)$, as claimed. It follows

that $\widehat{b}_H\mathcal{R}_{\mathcal{Y}_L}(G)=0$, hence $\widehat{b}_L^+\mathcal{R}_{\mathcal{Y}_L}(G)=\mathcal{R}_{\mathcal{Y}_L}(G)=\widehat{b}_L\mathcal{R}_{\mathcal{Y}_L}(G)$.

(3) This is a straightforward consequence of (1) and (2), by Theorem 7.9. □

The following proposition gives some detail on the structure of the category $R\mathcal{C}_p^{\sharp L}$:

7.16. Proposition: *Let p be a prime, and L be an atoric p-group.*

1. *Let P be a finite p-group. Then $P^@\cong L$ if and only if there exists an elementary abelian p-group E such that $P\cong E\times L$.*
2. *Let $P=E\times L$ and $Q=F\times L$, where E and F are elementary abelian p-groups. If $M\leq(Q\times P)$, then $q(M)^@\cong L$ if and only if*

$$p_{1,2}(M)=p_{2,2}(M)=L \quad \text{and} \quad k_{1,2}(M)=k_{2,2}(M)=\mathbf{1}\ ,$$

where $p_{1,2}$ and $p_{2,2}$ are the morphisms from $\big((H\times L)\times(G\times L)\big)$ to L defined by $p_{1,2}\big((h,x),(g,y)\big)=x$ and $p_{2,2}\big((h,x),(g,y)\big)=y$, and

$$\begin{aligned} k_{1,2}(M) &= \{x\in L\mid \big((1,x),(1,1)\big)\in M\}\ ,\\ k_{2,2}(M) &= \{x\in L\mid \big((1,1),(1,x)\big)\in M\}\ . \end{aligned}$$

Proof : (1) This follows from Proposition 6.8.

(2) By Lemma 7.13, the R-module $RB(Q,P)$ has a basis consisting of the isomorphism classes of (Q,P)-bisets of the form $(Q\times P)/M$, where M is a subgroup of $(Q\times P)$, up to conjugation, and $q(M)^@\cong L$. If M is such a subgroup, then $L\cong\big(p_1(M)/k_1(M)\big)^@\sqsubseteq\big(p_1(M)\big)^@\sqsubseteq Q^@\cong L$, so $p_1(M)^@\cong L$, and similarly $p_2(M)^@\cong L$. By Proposition 6.8 $p_1(M)^@\cong L$ if and only if $Ep_1(M)=P$, which in turn is equivalent to $p_{1,2}(M)=L$. Similarly $p_2(M)^@\cong L$ if and only if $p_{2,2}(M)=L$.

Then $\big(p_1(M)/k_1(M)\big)^@\cong L$ if and only if $k_1(M)\cap\Phi\big(p_1(M)\big)=\mathbf{1}$, by Proposition 6.6. Moreover $\Phi\big(p_1(M)\big)=\Phi(P)$, as there exists an elementary abelian subgroup E' of P such that $P=E'\times p_1(M)$, by Proposition 6.8 again. Since $\Phi(P)=\mathbf{1}\times\Phi(L)$, it follows that $k_1(M)\cap\big(\mathbf{1}\times\Phi(L)\big)=\mathbf{1}$. Now $N=k_1(L)\cap(\mathbf{1}\times L)$ is a normal subgroup of $(\mathbf{1}\times L)$ (since $p_{1,2}(M)=L$), which intersect trivially $\big(\mathbf{1}\times\Phi(L)\big)$. Since L is atoric, by Lemma 6.3, any central element of order p of $(\mathbf{1}\times L)$ is contained in $\big(\mathbf{1}\times\Phi(L)\big)$, so N contains no non trivial central element of $(\mathbf{1}\times L)$, hence $N=\mathbf{1}$. Thus $k_1(L)\cap(\mathbf{1}\times L)=\mathbf{1}$, or equivalently $k_{1,2}(M)=\mathbf{1}$. Similarly $k_{2,2}(M)=\mathbf{1}$. Hence $q(M)^@\cong L$ if and only if $p_{1,2}(M)=p_{2,2}(M)=L$ and $k_{1,2}(M)=k_{2,2}(M)=\mathbf{1}$. □

8. L-enriched bisets

8.1. Notation: *Let G and H be finite groups. If U is an (H,G)-biset, and $u \in U$, let $(H,G)_u$ denote the stabilizer of u in $(H \times G)$, i.e.*

$$(H,G)_u = \{(h,g) \in (H \times G) \mid hu = ug\} \ .$$

Let $H_u = k_1\big((H,G)_u\big)$ denote the stabilizer of u in H, and ${}_uG = k_2\big((H,G)_u\big)$ denote the stabilizer of u in G. Set moreover

$$q(u) = q\big((H,G)_u\big) = (H,G)_u/(H_u \times {}_uG) \ .$$

8.2. Definition: *Let L be a finite group. For two finite groups G and H, an L-*enriched *(H,G)-biset is a $(H \times L, G \times L)$-biset U such that $L \sqsubseteq q(u)$, for any $u \in U$. A morphism of L-enriched (H,G)-bisets is a morphism of $(H \times L, G \times L)$-bisets.*

The disjoint union of two L-enriched (H,G)-bisets is again an L-enriched (H,G)-biset. Let $B[L](H,G)$ denote the Grothendieck group of finite L-enriched (H,G)-bisets for relations given by disjoint union decompositions. The group $B[L](H,G)$ is called the Burnside group of L-enriched (H,G)-bisets.

8.3. Lemma: *Let G,H,L be finite groups, and U be an $(H \times L, G \times L)$-biset. Let $U^{\sharp L}$ denote the set of elements $u \in U$ such that $L \sqsubseteq q(u)$. Then $U^{\sharp L}$ is the largest sub-L-enriched (H,G)-biset of U.*

Proof : It suffices to show that $U^{\sharp L}$ is a sub-$(H \times L, G \times L)$-biset of U, for then it is clearly the largest sub-L-enriched (H,G)-biset of U. And this is straightforward, since for any $(u,g,h,x,y) \in (U \times G \times H \times L \times L)$, if $v = (h,y)u(g,x)^{-1}$, then

$$(H \times L, G \times L)_v = {}^{((h,y),(g,x))}(H \times L, G \times L)_u \ ,$$

and this conjugation induces a group isomorphism $q(v) \cong q(u)$. □

8.4. Lemma: *Let G,H,L be finite groups.*

1. *Let U be an L-enriched (H,G)-biset. If V is a sub-$(H \times L, G \times L)$-biset of U, then V is an L-enriched (H,G)-biset.*

2. *The group $B[L](H,G)$ has a $\mathbb{Z}$-basis consisting of the transitive bisets $\big((H\times L)\times(G\times L)\big)/M$, where M is a subgroup of $\big((H\times L)\times(G\times L)\big)$ (up to conjugation) such that $L \sqsubseteq q(M)$.*

Proof : (1) This is straightforward.

(2) It follows from (1) that $B[L](H,G)$ has a basis consisting of the isomorphism classes of L-enriched (H,G)-bisets which are transitive $(H\times L, G\times L)$-bisets. These are of the form $U = \big((H\times L)\times(G\times L)\big)/M$, for some subgroup M of $\big((H\times L)\times(G\times L)\big)$. Now if u is the element $\big((1,1),(1,1)\big)M$ of U, the group $(H\times L, G\times L)_u$ is equal to M, hence $q(u)\cong q(M)$. □

8.5. Lemma: *Let G,H,K,L be finite groups.*

1. *For an (H,G)-biset U, endow $U\times L$ with the $(H\times L, G\times L)$-biset structure defined by*

$$\forall h\in H, \forall g\in G, \forall x,y,z\in L, \forall u\in U,\quad (h,x)(u,y)(g,z)=(hug,xyz)\ .$$

   Then $U\times L$ is an L-enriched (H,G)-biset.

2. *In particular, for any finite group G, the identity biset of $G\times L$ is an L-enriched (G,G)-biset.*
3. *If U is an (H,G)-biset and V is a (K,H)-biset, then there is an isomorphism*

$$(V\times L)\times_{(H\times L)}(U\times L)\cong (V\times_H U)\times L$$

   of L-enriched (H,G)-bisets.

Proof : (1) For $u\in U$ and $l\in L$,

$$(H\times L, G\times L)_{(u,l)} = \{\big((h,{}^lx),(g,x)\big)\mid hug=u,\ l\in L\}\cong (H,G)_u\times L\ .$$

In particular $(H\times L)_{(u,l)} = H_u\times \mathbf{1}$ and ${}_{(u,l)}(G\times L) = {}_uG\times\mathbf{1}$, and $q\big((u,l)\big)\cong q(u)\times L$ has a (sub)quotient isomorphic to L.

(2) In particular, if $H=G$ and U is the identity (G,G)-biset, then $U\times L$ is the identity biset of $(G\times L)$.

(3) It is straightforward to check that the maps

$$[(v,x),(u,y)]\in (V\times L)\times_{(H\times L)}(U\times L)\longmapsto ([v,u],xy)\in (V\times_H U)\times L$$

$$[(v,1),(u,l)]\in (V\times L)\times_{(H\times L)}(U\times L)\longleftarrow\!\shortmid ([v,u],l)\in (V\times_H U)\times L$$

are well defined isomorphisms of $(K \times L, G \times L)$-bisets, inverse to one another. $\square$

8.6. Notation: *Let G, H, K, L be finite groups. If U is an L-enriched (H,G)-biset and V is an L-enriched (K,H)-biset, let $V \overset{L}{\times}_H U$ denote the L-enriched (K,G)-biset defined by*

$$V \overset{L}{\times}_H U = \big(V \times_{(H\times L)} U\big)^{\sharp L} \quad .$$

8.7. Lemma: *Let G, H, J, K, L be finite groups.*

1. *If V is a $(K \times L, H \times L)$-biset and U is an $(H \times L, G \times L)$-biset, then*

$$(V \times_{(H\times L)} U)^{\sharp L} = V^{\sharp L} \overset{L}{\times}_H U^{\sharp L} \quad .$$

   In particular, if V and U are L-enriched bisets, so is $V \overset{L}{\times}_H U$.

2. *If U and U' are L-enriched (H,G)-bisets, if V, V' are L-enriched (K,H)-bisets, then there are isomorphisms*

$$\begin{aligned} V \overset{L}{\times}_H (U \sqcup U') &\cong (V \overset{L}{\times}_H U) \sqcup (V \overset{L}{\times}_H U') \\ (V \sqcup V') \overset{L}{\times}_H U &\cong (V \overset{L}{\times}_H U) \sqcup (V' \overset{L}{\times}_H U) \end{aligned}$$

   of L-enriched (K,G)-bisets.

3. *If moreover W is an L-enriched (J,K)-biset, then there is a canonical isomorphism*

$$(W \overset{L}{\times}_K V) \overset{L}{\times}_H U \cong W \overset{L}{\times}_K (V \overset{L}{\times}_H U)$$

   of L-enriched (J,G)-bisets.

Proof : (1) Denote by $[v,u]$ the image in $V \times_{(H\times L)} U$ of a pair $(v,u) \in (V \times U)$. By Lemma 2.3.20 of [7],

$$(K \times L, G \times L)_{[v,u]} = (K \times L, H \times L)_v * (H \times L, G \times L)_u \quad ,$$

so by Lemma 2.3.22 of [7], the group $q\big([v,u]\big)$ is a subquotient of $q(v)$ and $q(u)$. So if $[v,u] \in (V \times_{(H\times L)} U)^{\sharp L}$, then L is a subquotient of $q\big([v,u]\big)$, hence it is a subquotient of $q(v)$ and $q(u)$, that is $v \in V^{\sharp L}$ and $u \in U^{\sharp L}$. Hence

$$(V \times_{(H\times L)} U)^{\sharp L} \subseteq (V^{\sharp L} \times_{(H\times L)} U^{\sharp L})^{\sharp L} = V^{\sharp L} \overset{L}{\times}_H U^{\sharp L} \quad ,$$

and the reverse inclusion $(V^{\sharp L}\times_{(H\times L)}U^{\sharp L})^{\sharp L}\subseteq(V\times_{(H\times L)}U)^{\sharp L}$ is obvious. Hence $(V\times_{(H\times L)}U)^{\sharp L}=V^{\sharp L}\overset{L}{\times}_H U^{\sharp L}$. If V and U are L-enriched bisets, i.e. if $V=V^{\sharp L}$ and $U=U^{\sharp L}$, this gives $(V\times_{(H\times L)}U)^{\sharp L}=V\overset{L}{\times}_H U$, so $V\overset{L}{\times}_H U$ is an L-enriched biset.

(2) This is straightforward.

(3) With the above notation, there is a canonical isomorphism

$$\alpha:(W\times_{(K\times L)}V)\times_{(H\times L)}U\to W\times_{(K\times L)}(V\times_{(H\times L)}U)$$

sending $\big[[w,v],u\big]$ to $\big[w,[v,u]\big]$. Hence

$$\begin{aligned}(W\overset{L}{\times}_K V)\overset{L}{\times}_H U &= \big((W\overset{L}{\times}_K V)\times_{(H\times L)}U\big)^{\sharp L}\\ &= \big((W\times_{(K\times L)}V)^{\sharp L}\times_{(H\times L)}U\big)^{\sharp L}\\ &= \big((W\times_{(K\times L)}V)\times_{(H\times L)}U\big)^{\sharp L}\quad\text{[by (1)]}\end{aligned}$$

Similarly

$$\begin{aligned}W\overset{L}{\times}_K(V\overset{L}{\times}_H U) &= \big(W\times_{(K\times L)}(V\overset{L}{\times}_H U)\big)^{\sharp L}\\ &= \big(W\times_{(K\times L)}(V\times_{(H\times L)}U)^{\sharp L}\big)^{\sharp L}\\ &= \big(W\times_{(K\times L)}(V\times_{(H\times L)}U)\big)^{\sharp L}\quad\text{[by (1)]}\ .\end{aligned}$$

Hence α induces an isomorphism $(W\overset{L}{\times}_K V)\overset{L}{\times}_H U\cong W\overset{L}{\times}_K(V\overset{L}{\times}_H U)$. □

8.8. Definition: *Let L be a finite group, and R be a commutative ring. The* L-enriched biset category *$R\mathcal{C}[L]$ of finite groups over R is defined as follows:*

- *The objects of $R\mathcal{C}[L]$ are the finite groups.*
- *For finite groups G and H,*

$$\mathrm{Hom}_{R\mathcal{C}[L]}(G,H)=R\otimes_{\mathbb{Z}}B[L](H,G)=RB[L](H,G)$$

  is the R-linear extension of the Burnside group of L-enriched (H,G)-bisets.
- *The composition in $R\mathcal{C}[L]$ is the R-linear extension of the product $(V,U)\mapsto V\overset{L}{\times}_H U$ defined in 8.6.*

- *The identity morphism of the group G is (image in $RB[L](G,G)$ of) the identity biset of $G \times L$, viewed as an L-enriched (G,G)-biset.*

The category $R\mathcal{C}[L]$ is R-linear. An L-enriched biset functor *over R is an R-linear functor from $R\mathcal{C}[L]$ to R-*Mod*. The category of L-enriched biset functors over R is denoted by $\mathcal{F}_R[L]$. It is an abelian R-linear category.*

8.9. Theorem: *Let p be a prime number, and R be a commutative ring.*

1. *If L is an atoric p-group, the category $R\mathcal{C}_p^{\sharp L}$ of Definition 7.11 is equivalent to the full subcategory $R\mathcal{E}l_p[L]$ of $R\mathcal{C}[L]$ consisting of elementary abelian p-groups.*
2. *If $p \in \mathcal{F}^\times$, the category $\mathcal{F}_{p,R}$ of p-biset functors over R is equivalent to the direct product of the categories $\mathsf{Fun}_R\big(R\mathcal{E}l_p[L], R\text{-}\mathsf{Mod}\big)$ of R-linear functors from $R\mathcal{E}l_p[L]$ to R-Mod, for $L \in [\mathcal{A}t_p]$.*

Proof : (1) Let E be an elementary abelian p-group. Then $(E \times L)^@ \cong L$, so $E \times L$ is an object of $R\mathcal{C}_p^{\sharp L}$. Set $\mathcal{I}(E) = E \times L$. If E and F are elementary abelian p-groups, and if U is a finite L-enriched (F,E)-biset, then U is in particular an $(F \times L, E \times L)$-biset, an we can consider its image $\mathcal{I}(U)$ in the quotient $\mathrm{Hom}_{R\mathcal{C}_p^{\sharp L}}(E \times L, F \times L)$ of $RB(F \times L, E \times L)$. This yields a unique R-linear map $RB[L](F,E) \to \mathrm{Hom}_{R\mathcal{C}_p^{\sharp L}}(E \times L, F \times L)$, still denoted by $\mathcal{I}$.

We claim that these assignments define a functor $\mathcal{I}$ from $R\mathcal{E}l_p[L]$ to $R\mathcal{C}_p^{\sharp L}$: indeed, the identity $(E \times L, E \times L)$-biset is clearly mapped to the identity morphism of $\mathcal{I}(E)$. Moreover, if G is an elementary abelian p-group, if V is an L-enriched (G,F)-biset and U is an L-enriched (F,E)-biset, it is clear that

$$\mathcal{I}\big(V \overset{L}{\times}_F U\big) = \mathcal{I}(V) \circ \mathcal{I}(U) \ \ ,$$

where the right hand side composition is in the category $R\mathcal{C}_p^{\sharp L}$: indeed, the transitive bisets $(Q \times P)/M$ with $q(M)^@ \sqsubset L$ appearing in the product $V \times_{(F \times L)} U$ are exactly those vanishing in $\mathrm{Hom}_{R\mathcal{C}_p^{\sharp L}}\big(\mathcal{I}(E), \mathcal{I}(F)\big)$, by Lemma 7.13. Hence $\mathcal{I}$ is an isomorphism

$$\mathcal{I} : RB[L](F,E) \to \mathrm{Hom}_{R\mathcal{C}_p^{\sharp L}}\big(\mathcal{I}(E), \mathcal{I}(F)\big) \ \ .$$

In other words $\mathcal{I}$ is a fully faithful functor from $R\mathcal{E}l_p[L]$ to $R\mathcal{C}_p^{\sharp L}$. Moreover, by Proposition 6.8, if P is a finite p-group with $P^@ \cong L$, there exists an elementary abelian p-group E such that P is isomorphic to $E \times L$, hence P is isomorphic to $E \times L$ in the category $R\mathcal{C}_p^{\sharp L}$.

It follows that the functor $\mathcal{I}$ is fully faithful and essentially surjective, so it is an equivalence of categories.

(2) This is a straightforward consequence of (1), Assertion 5 of Corollary 7.5, and Assertion 3 of Theorem 7.15. □

9. The category $\widehat{b}_L\mathcal{F}_{p,R}$, for an atoric p-group L ($p \in R^\times$)

Let L be a fixed atoric p-group. In this section, we give some detail on the structure of the category $\widehat{b}_L\mathcal{F}_{p,R}$ of p-biset functors invariant by the idempotent $\widehat{b}_L$.

We start by straightforward consequences of Theorem 7.15. For a finite p-group P, we denote by $\Sigma_{\sharp L}(P)$ the subset of $\Sigma_L(P)$ consisting of sections (X, M) of P such that $(X/M)^@ \cong L$. When G is an R-linear functor from $R\mathcal{C}_p^{\sharp L}$ to R-Mod, we can compute $\mathcal{R}_{\mathcal{Y}_L}(G)$ at P by restricting the inverse limit of 7.7 to the subset $\Sigma_{\sharp L}(P)$, i.e. by

$$\mathcal{R}_{\mathcal{Y}_L}(G)(P) = \lim_{\substack{\longleftarrow \\ (X,M)\in\Sigma_{\sharp L}(P)}} G(X/M) \ .$$

9.1. Proposition: [$p \in R^\times$] *Let L be an atoric p-group. If F is a p-biset functor in $\widehat{b}_L\mathcal{F}_{p,R}$, and P is a finite p-group, then*

$$\begin{aligned} F(P) &\cong \lim_{\substack{\longleftarrow \\ (X,M)\in\Sigma_{\sharp L}(P)}} F(X/M) \ , \\ &\cong \bigoplus_{\substack{(T,S)\in[\mathcal{M}(P)] \\ (T/S)^@\cong L}} \delta_\Phi F(T/S)^{N_P(T,S)/T} \ . \end{aligned}$$

Proof : The isomorphism $F(P) \cong \lim\limits_{\substack{\longleftarrow \\ (X,M)\in\Sigma_{\sharp L}(P)}} F(X/M)$ is Assertion 3 of Theorem 7.15. The second isomorphism follows from Theorem 5.4, which implies that for $(T, S) \in \mathcal{M}(P)$

$$\delta_\Phi F(T/S)^{N_P(T,S)/T} \cong F(\epsilon_{T,S}^P)\big(F(P)\big) \ .$$

Moreover $F(b_L^P)F(P) = F(P)$ since $F \in \widehat{b}_L\mathcal{F}_{p,R}$, and

$$F(\epsilon_{T,S}^P)F(b_L^P) = F(\epsilon_{T,S}^P b_L^P) = 0$$

unless $(T/S)^@ \cong L$. Thus $\delta_\Phi F(T/S)^{N_P(T,S)/T} = \{0\}$ unless $(T/S)^@ \cong L$, which completes the proof. □

The decomposition of the category $\mathcal{F}_{p,R}$ of p-biset functors stated in Corollary 7.5 leads to the following natural definition:

9.2. Definition: [$p \in R^\times$] *Let F be an indecomposable p-biset functor over R. There exists a unique atoric p-group L (up to isomorphism) such that $F = \widehat{b}_L F$. The group L is called the* vertex *of F.*

9.3. Remark: It follows in particular from this definition that if F and F' are indecomposable p-biset functors over R with non-isomorphic vertices, then $\mathrm{Ext}^*_{\mathcal{F}_{p,R}}(F, F') = \{0\}$.

9.4. Theorem: [$p \in R^\times$] *Let F be an indecomposable p-biset functor over R and let L be a vertex of F. If Q is a finite p-group such that $F(Q) \neq \{0\}$, but F vanishes on any proper subquotient of Q, then $L \cong Q^{@}$.*

Proof : Let Q be a finite p-group such that $F(Q) \neq \{0\}$ and $F(Q') = \{0\}$ for any proper subquotient Q' of Q. By Proposition 4.6, if (T, S) is a minimal section of Q, then

$$\epsilon^Q_{T,S} = \frac{1}{|N_Q(T,S)|} \sum_{\substack{X \le T,\, M \trianglelefteq T \\ S \le M \le \Phi(T) \le X \le T}} |X|\mu(X,T)\mu_{\trianglelefteq T}(S,M)\,\mathrm{Indinf}^Q_{X/M} \circ \mathrm{Defres}^Q_{X/M} \quad .$$

Now if X/M is a proper subquotient of Q, i.e. if $X \neq Q$ or $M \neq \mathbf{1}$, then $F(X/M) = \{0\}$, and $F(\mathrm{Indinf}^Q_{X/M} \circ \mathrm{Defres}^Q_{X/M}) = 0$. Hence $F(\epsilon^Q_{T,S}) = 0$ unless $T = Q$ and $S = \mathbf{1}$, and moreover

$$F(\epsilon^Q_{Q,\mathbf{1}}) = \frac{1}{|Q|}|Q|\mu(Q,Q)\mu_{\trianglelefteq Q}(\mathbf{1},Q)F(\mathrm{Indinf}^Q_{Q/\mathbf{1}}\mathrm{Defres}^Q_{Q/\mathbf{1}}) = \mathrm{Id}_{F(Q)} \quad .$$

If $\widehat{b}_L F = F$, then in particular $F(b^Q_L)$ is equal to the identity map of $F(Q)$. This can only occur if the idempotent $\epsilon^Q_{Q,\mathbf{1}}$ appears in the sum defining b^Q_L, in other words if $(Q/\mathbf{1})^{@} \cong L$, i.e. $Q^{@} \cong L$. Conversely, if $Q^{@} \cong L$, then $F(b^Q_L) = F(\epsilon^Q_{Q,\mathbf{1}}) = \mathrm{Id}_{F(Q)} \neq 0$. It follows that $\widehat{b}_L F \neq 0$, hence $\widehat{b}_L F = F$, since F is indecomposable. Hence $Q^{@}$ is (isomorphic to) the vertex of F, as was to be shown. □

We assume from now on that $R = k$ is a field. Recall ([7] Chapter 4) that the simple p-biset functors over k are indexed by pairs (Q, V) consisting of a p-group Q and a simple $k\mathrm{Out}(Q)$-module V.

9.5. Corollary: *Let k be a field of characteristic different from p.*

1. *If Q is a finite p-group, and V is a simple $k\mathrm{Out}(Q)$-module, then the vertex of the simple p-biset functor $S_{Q,V}$ is isomorphic to $Q^{@}$.*
2. *Let Q (resp. Q') be a finite p-group, and V (resp. V') be a simple $k\mathrm{Out}(Q)$-module (resp. a simple $k\mathrm{Out}(Q')$-module). If $Q^{@} \not\cong Q'^{@}$, then $\mathrm{Ext}^*_{\mathcal{F}_{p,k}}(S_{Q,V}, S_{Q',V'}) = \{0\}$.*

Proof : (1) Indeed Q is a minimal group for $S_{Q,V}$, so $S_{Q,V}(Q) \neq \{0\}$, but $S_{Q,V}$ vanishes on any proper subquotient of Q.

(2) Follows from (1) and Remark 9.3. □

9.6. Definition: *Let F be a p-biset functor. A functor S is a* subquotient *of F (notation $S \sqsubseteq F$) if there exist subfunctors $F_2 < F_1 \leq F$ such that $F_1/F_2 \cong S$. A* composition factor *of F is a simple subquotient of F.*

9.7. Lemma: *Let k be a field, and F be a p-biset functor over k.*

1. *If F is a non zero, then F admits a composition factor.*
2. *If $\mathcal{S}$ is a family of simple p-biset functors over k, there exists a greatest subfunctor of F all composition factors of which belong to $\mathcal{S}$.*

Proof : (1) Let P be a finite p-group such that $F(P) \neq \{0\}$. Then $F(P)$ is a $kB(P,P)$-module. Choose $m \in F(P) - \{0\}$, and consider the $kB(P,P)$-submodule M of $F(P)$ generated by m. Since $kB(P,P)$ is finite dimensional over k, the module M is also finite dimensional over k, hence it contains a simple submodule V. By Proposition 3.1 of [8], there exists a simple p-biset functor S such that $S(P) \cong V$ as $kB(P,P)$-module. Then $S(P)$ is a subquotient of $F(P)$, so by Proposition 3.5 of [8], there exists a subquotient of F isomorphic to S.

(2) Observe first that if M,N are subfunctors of F, then any composition factor of $M+N$ is a composition factor of M or a composition factor of N: indeed, if S is a composition factor of $M+N$, let $F_2 < F_1 \leq M+N$ with $S \cong F_2/F_1$, and consider the images F_1' and F_2' of F_1 and F_2, respectively, in the quotient $(M+N)/N \cong M/(M\cap N)$. If $F_1' \neq F_2'$, that is if $F_1+N \neq F_2+N$, then $F_1'/F_2' \cong (F_1+N)/(F_2+N) \cong F_1/F_2 \cong S$ is a subquotient of $(M+N)/N \cong M/(M\cap N)$, hence S is a subquotient of M. Otherwise $F_1+N = F_2+N$, so $F_1 = F_2+(F_1\cap N)$, hence $S \cong F_1/F_2 \cong (F_1\cap N)/(F_2\cap N)$

is a subquotient of N. It follows by induction that any subquotient S of a finite sum $\sum_{M\in\mathcal{I}} M$ of subfunctors of F is a subquotient of some $M \in \mathcal{I}$.

The latter also holds when $\mathcal{I}$ is infinite: let $\Sigma = \sum_{M\in\mathcal{I}} M$ be an arbitrary sum of subfunctors of F, and S be a composition factor of Σ. Let $F_2 < F_1$ be subfunctors of Σ such that $S \cong F_1/F_2$. If P is a p-group such that $S(P) \cong F_1(P)/F_2(P) \neq 0$, let U be a finite subset of $F_1(P)$ such that $F_1(P)/F_2(P)$ is generated as a $kB(P,P)$-module by the images of the elements of U (such a set exists because $S(P)$ is finite dimensional over k, for any P). If V is the $kB(P,P)$-submodule of $F_1(P)$ generated by U, then V maps surjectively on the module $F_1(P)/F_2(P)$, so there is a $kB(P,P)$-submodule W of V such that $V/W \cong S(P)$. Now since U is finite, there exists a finite subset $\mathcal{J}$ of $\mathcal{I}$ such that $U \subseteq \sum_{M\in\mathcal{J}} M(P)$. Setting $\Sigma_1 = \sum_{M\in\mathcal{J}} M$, it follows that $V/W \cong S(P)$ is a subquotient of $\Sigma_1(P)$, so by Proposition 3.5 of [8], there exists a subquotient of Σ_1 isomorphic to S. By the observation above S is a subquotient of some $M \in \mathcal{J} \subseteq \mathcal{I}$.

Now let $\mathcal{I}$ the set of subfunctors M of F such that all the composition factors of M belong to $\mathcal{S}$, and $N = \sum_{M\in\mathcal{I}} M$. The above discussion shows that $N \in \mathcal{I}$, so N is the greatest element of $\mathcal{I}$. □

9.8. Theorem: *Let k be a field of characteristic different from p, and L be an atoric p-group. Let $\mathcal{F}_{p,k}[L]$ the full subcategory of $\mathcal{F}_{p,k}$ consisting of functors whose composition factors all have vertex L, i.e. are all isomorphic to $S_{P,V}$, for some p-group P such that $P^{@} \cong L$, and some simple $k\mathrm{Out}(P)$-module V.*

1. *If F is a p-biset functor, then $\widehat{b}_L\mathcal{F}_{p,k}$ is the greatest subfunctor of F which belongs to $\mathcal{F}_{p,k}[L]$.*
2. *In particular $\widehat{b}_L\mathcal{F}_{p,k} = \mathcal{F}_{p,k}[L]$.*

Proof : (1) Let F be a p-biset functor over k, and let $F_1 = \widehat{b}_LF$. If S is a composition factor of F_1, then $S = \widehat{b}_LS$, as S is a subquotient of F_1. Hence S has vertex L, by Definition 9.2. It follows that F_1 is contained in the greatest subfunctor F_2 of F which belongs to $\mathcal{F}_{p,k}[L]$ (such a subfunctor exists by Lemma 9.7).

Conversely, we know that $F_2 = \bigoplus_{Q\in[\mathcal{A}t_p]} \widehat{b}_QF_2$. For $Q \in [\mathcal{A}t_p]$, any composition factor S of $\widehat{b}_QF_2$ has vertex Q, by Definition 9.2. But S is also a direct summand of F_2, so $Q \cong L$. It follows that if $Q \ncong L$, then $\widehat{b}_QF_2$ has no composition factor, so $\widehat{b}_Q F_2 = \{0\}$, by Lemma 9.7. In other words $F_2 = \widehat{b}_L F_2$, hence $F_2 \le F_1$, and $F_2 = F_1$, as was to be shown.

(2) Let F be a p-biset functor. Then $F \in \widehat{b}_L \mathcal{F}_{p,k}$ if and only if $F = \widehat{b}_L F$, i.e. by (1) if and only if all the composition factors of F have vertex L. □

9.9. Example: the Burnside functor. Let k be a field of characteristic $q \neq p$ ($q \geq 0$). It was shown in [10] Theorem 8.2 (see also [7] 5.6.9) that the Burnside functor kB is uniserial, hence indecomposable. As $kB(\mathbf{1}) \neq 0$, the vertex of kB is the trivial group, by Theorem 9.4, thus kB is an object of $\widehat{b}_{\mathbf{1}} \mathcal{F}_{p,k} = \mathcal{F}_{p,k}[\mathbf{1}]$. It means that all the composition factors of kB have to be of form $S_{Q,V}$, where $Q^{@} = \mathbf{1}$, i.e. Q is elementary abelian. And indeed by [10] Theorem 8.2, the composition factors of kB are all of the form $S_{Q,k}$, where Q runs through a specific set of elementary abelian p-groups which depends on the order of p modulo q (suitably interpreted when $q = 0$).

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