Fused Mackey functors

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Abstract: Let G be a finite group. In [6], Hambleton, Taylor and Williams have considered the question of comparing Mackey functors for G and biset functors defined on subgroups of G and bifree bisets as morphisms.
This paper proposes a different approach to this problem, from the point of view of various categories of G-sets. In particular, the category G-$\underline{\mathsf{set}}$ of *fused G-sets* is introduced, as well as the category $\underline{\mathbf{S}}(G)$ of spans in G-$\underline{\mathsf{set}}$. The *fused Mackey functors* for G over a commutative ring R are defined as R-linear functors from $R\underline{\mathbf{S}}(G)$ to R-modules. They form an abelian subcategory $\mathsf{Mack}_R^f(G)$ of the category of Mackey functors for G over R. The category $\mathsf{Mack}_{\mathbb{Z}}^f(G)$ is equivalent to the category of conjugation Mackey functors of [6]. The category $\mathsf{Mack}_R^f(G)$ is also equivalent to the category of modules over the *fused Mackey algebra* $\mu_R^f(G)$, which is a quotient of the usual Mackey algebra $\mu_R(G)$ of G over R.

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1. Introduction

1.1. This note is devoted to the frequently asked question of comparing Mackey functors for a single finite group G (cf. [9]) with biset functors (cf. [3]) defined only on subgroups of G and left-right free bisets as morphisms. More precisely, let F be a biset functor defined on the category $\mathcal{D}$ of all finite groups, where morphisms are given by Grothendieck groups of left-right free bisets. Let moreover G be a fixed finite group. Now:

- when H is a subgroup of G, set $M(H) = F(H)$.
- when $H \leq K$ are subgroups of G, define a restriction map
$$r_H^K : M(K) \to M(H)$$
by $r_H^K = F({}_HK_K)$, where ${}_HK_K$ is the set K, viewed and an (H, K)-biset by left and right multiplication, hence also as a morphism from K to H in the category $\mathcal{D}$.
- similarly, define a transfer map
$$t_H^K : M(H) \to M(K)$$
by $t_H^K = F({}_KK_H)$, where ${}_KK_H$ is the set K, viewed as a (K, H)-biset.

- finally, when $x \in G$, and $H \leq G$, define a conjugation map

$$c_{x,H} : M(H) \to M({}^xH)$$

by $c_{x,H} = F({}_{{}^xH}\, x H_H)$, where ${}_{{}^xH}\, x H_H$ is the coset xH, wiewed as an $({}^xH, H)$-biset.

One checks easily that this yields a Mackey functor M for G.

1.2. The question is now to characterize those Mackey functors for G for which the restriction maps r_H^K, the transfer maps t_H^K, and the conjugation maps $c_{x,H}$ only depend on the above bisets ${}_H K_K$, ${}_K K_H$, and ${}_{{}^xH}\, x H_H$, respectively. Equivalently, to characterize those Mackey functors for G which can be viewed as additive functors on the (non full) subcategory of $\mathcal{D}$ consisting of subgroups of G, where morphisms from H to K are linear combinations of isomorphism classes of (K, H)-bisets obtained by composition of the above three types of bisets.

These bisets have been called *conjugation bisets* by Hambleton, Taylor and Williams, who answered first the above question ([6]): the Mackey functors in question are the *conjugation invariant Mackey functors*, namely the Mackey functors M for G such that for any subgroup H of G, the centralizer $C_G(H)$ acts trivially on $M(H)$. However, the proof of this characterization given in [6] is rather computational and non canonical (in particular, in Section 7, the definition of the functor $j_\bullet$ requires the choice of sets of representatives of orbits of any finite G-set).

The present paper makes a systematic use of Dress definition ([4]) and Lindner definition ([7]) of Mackey functors, to avoid these non canonical choices. This leads to the definition of the category of *fused G-sets* (Section 3), and the category of *fused Mackey functors* (Section 4) for a finite group G, which is equivalent to the category of "conjugation invariant Mackey functors" of [6]. This category is also equivalent to the category of modules over the *fused Mackey algebra*, introduced in Section 5.

2. Conjugation bisets revisited

2.1. First a notation : when G is a finite group, and X is a finite G-set, let $G\text{-}\mathsf{set}{\downarrow_X}$ denote the category of (finite) G-sets over X: its objects are pairs (Y, b) consisting of a finite G-set Y, and a morphism of G-sets $b : Y \to X$. A morphism $f : (Y, b) \to (Z, c)$ in $G\text{-}\mathsf{set}{\downarrow_X}$ is a morphism of G-sets $f : Y \to Z$ such that $c \circ f = b$.

There is an obvious notion of disjoint union in $G\text{-}\mathsf{set}{\downarrow_X}$, and the corresponding Grothendieck group is called the Burnside group over X. It will be denoted by $\mathcal{B}({}_G X)$, or $\mathcal{B}(X)$ when G is clear from the context.

Similarly, when G and H are finite groups, and U is a (G,H)-biset, one can define the category (G,H)-$\mathsf{biset}{\downarrow_U}$ of (G,H)-bisets over U, and the Burnside group $\mathcal{B}({}_GU_H)$ of (G,H)-bisets over U.

2.2. When H is a subgroup of G, and Y is an H-set, induction from H-sets to G-sets is an equivalence of categories from H-$\mathsf{set}{\downarrow_Y}$ to G-$\mathsf{set}{\downarrow_{\mathrm{Ind}_H^G Y}}$. A quasi-inverse equivalence is the functor sending the G-set (X,a) over $\mathrm{Ind}_H^G Y$ to the H-set $a^{-1}(1\times_H Y)$ (cf. [2] Lemma 2.4.1). In particular $\mathcal{B}({}_HY)\cong\mathcal{B}({}_G\mathrm{Ind}_H^G Y)$.

2.3. Now an observation: when H and K are subgroups of G, the conjugation (K,H)-bisets defined in Section 6 of [6] are exactly those over the biset ${}_KG_H$ (the set G on which K and H act by multiplication), i.e. the (K,H)-bisets U for which there exists a biset morphism $U\to {}_KG_H$.

Indeed, a conjugation (K,H)-biset U is a bifree (K,H)-biset isomorphic to a disjoint union of bisets of the form $(K\times H)/S$, where S is a subgroup of $K\times H$ of the form

$$S_{g,A}=\{({}^gx,x)\mid x\in A\}$$

where A is a subgroup of H, and g is an element of G such that ${}^gA\leq K$. For such a transitive biset $(K\times H)/S$, the map

$$\forall (k,h)S\in (K\times H)/S,\ (k,h)S\mapsto kgh^{-1}$$

is a morphism of (K,H)-bisets.

Conversely, let U be a (K,H)-biset for which there exists a biset morphism $\alpha: U\to {}_KG_H$. Then for any $u\in U$, the stabilizer S_u of u in $K\times H$ is the subgroup

$$S_u=\{(k,h)\in K\times H\mid k\cdot u\cdot h^{-1}=u\}$$

of $K\times H$. Then if $(k,h)\in S_u$,

$$\alpha(k\cdot u)=k\alpha(u)=\alpha(u\cdot h)=\alpha(u)h\ .$$

Let A_u denote the projection of S_u into H, and set $g_u=\alpha(u)$. It follows that $S_u\subseteq S_{g_u,A_u}$.

Conversely, if $(k,h)\in S_{g_u,A_u}$, then $k={}^{g_u}h$, and there exists some $x\in K$ such that $(x,h)\in S_u$, since $h\in A_u$. Thus $x\cdot u\cdot h^{-1}=u$, from which follows that

$$\alpha(x\cdot u)=xg_u=\alpha(u\cdot h)=g_uh\ ,$$

hence $x={}^{g_u}h=k$, and $S_u=S_{g_u,A_u}$. Observation 2.3 follows.

2.4. In other words, conjugation (K,H)-bisets form a category $\mathsf{Conj}_{K,H}^G$, and

there is a forgetful functor $\Phi : (K,H)\text{-}\mathsf{biset}{\downarrow}_{{}_K G_H} \to \mathsf{Conj}^G_{K,H}$ sending (U,a) to U. This functor preserves disjoint unions, and it induces a surjection on the corresponding sets of isomorphism classes. This means that Φ induces a surjective group homomorphism (still denoted by Φ) from $\mathcal{B}({}_K G_H)$ to the Grothendieck group $\mathcal{B}^G_{K,H}$ of conjugation (K,H)-bisets.

2.5. If H, K and L are subgroups of G, if (U,a) is a (K,H)-biset over ${}_K G_H$ and (V,b) is an (L,K)-biset over ${}_L G_K$, the composition $(V,b)\circ(U,a)$ is the (L,H)-biset over ${}_L G_H$ defined by the following diagram:

$$\begin{array}{ccc} V & & U \\ \downarrow{\scriptstyle b} & \circ & \downarrow{\scriptstyle a} \\ {}_L G_K & & {}_K G_H \end{array} \quad = \quad \begin{array}{c} V \times_K U \\ \downarrow{\scriptstyle b\times_K a} \\ G \times_K G \\ \downarrow{\scriptstyle \mu} \\ {}_L G_H \end{array}$$

where μ is multiplication in G. This composition is associative, and additive with respect to disjoint unions. Hence it induces a composition

$$\widehat{\circ} : \mathcal{B}({}_L G_K) \times \mathcal{B}({}_K G_H) \to \mathcal{B}({}_L G_H) \ .$$

Hence, one can define a category $\widehat{\mathbf{B}}(G)$ whose objects are the subgroups of G, and such that $\mathrm{Hom}_{\widehat{\mathbf{B}}(G)}(H,K) = \mathcal{B}({}_K G_H)$, for subgroups H and K of G. Composition is given by $\widehat{\circ}$, and the identity morphism of the subgroup H of G in the category $\widehat{\mathbf{B}}(G)$ is the class of the biset $({}_H H_H, i_H)$, where $i_H : {}_H H_H \to {}_H G_H$ is the inclusion map from H to G.

Since the functor Φ maps the composition $\widehat{\circ}$ to the composition of bisets, and the identity morphism of H in $\widehat{\mathbf{B}}(G)$ to the identity biset ${}_H H_H$, one can extend Φ to a functor $\widehat{\mathbf{B}}(G) \to \mathbf{B}(G)$, which is the identity on objects, where $\mathbf{B}(G)$ is the category introduced in Section 3 of [6]: its objects are the subgroups of G, and $\mathrm{Hom}_{\mathbf{B}(G)}(H,K) = \mathcal{B}^G_{K,H}$ for any subgroups H and K of G, the composition of morphisms being given by linear extension of the composition of bisets.

More precisely, the category $\mathbf{B}(G)$ is the quotient of the category $\widehat{\mathbf{B}}(G)$ obtained by identifying morphisms which have the same image by Φ.

2.6. By the above Remark 2.2, when H and K are subgroups of G, there is a group isomorphism

$$\mathcal{B}({}_K G_H) \cong \mathcal{B}\big(\mathrm{Ind}^{G\times G}_{K\times H}({}_K G_H)\big) \ ,$$

(with the usual identification of (K,H)-bisets with $(K\times H)$-sets). Now the biset ${}_K G_H$ is actually the restriction to $(K\times H)$ of the (G,G)-biset G. By

Frobenius reciprocity, it follows that

$$\mathrm{Ind}_{K\times H}^{G\times G}({}_KG_H) \cong \mathrm{Ind}_{K\times H}^{G\times G}\mathrm{Res}_{K\times H}^{G\times G}({}_GG_G) \cong \big(\mathrm{Ind}_{K\times H}^{G\times G}\bullet\big) \times {}_GG_G \ ,$$

where $\bullet$ is a set of cardinality 1. Since $\mathrm{Ind}_{K\times H}^{G\times G}\bullet \cong (G/K)\times(G/H)$, it follows (after switching G/H and G) that

$$\mathrm{Ind}_{K\times H}^{G\times G}({}_KG_H) \cong (G/K)\times G\times (G/H) \ ,$$

where the (G,G)-biset structure of the right hand side is given by

$$\forall (a,b,x,y,g)\in G^5,\ \ a\cdot(xK,g,yH)\cdot b = (axK, agb, b^{-1}yH) \ .$$

2.7. It should now be clear that the additive completion $\widehat{\mathbf{B}}_\bullet(G)$ is equivalent to the category whose objects are finite G-sets, where for any two finite G-sets X and Y

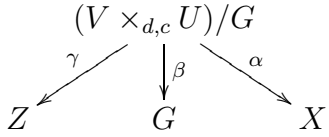
$$\mathrm{Hom}_{\widehat{\mathbf{B}}_\bullet(G)}(X,Y) = \mathcal{B}\big({}_G(Y\times G\times X)_G\big) \ ,$$

the (G,G)-biset structure on $(Y\times G\times X)$ being given as above by

$$\forall (a,b,g,x,y)\in G^3\times X\times Y,\ \ a\cdot(y,g,x)\cdot b = (ay, agb, b^{-1}x) \ .$$

Keeping track of the composition $\widehat{\circ}$ along the above isomorphism shows that the composition in the category $\widehat{\mathbf{B}}_\bullet(G)$ can be defined by linearity from the following: if X, Y, and Z are finite G-sets, if

are (G,G)-bisets over $(Z\times G\times Y)$ and $(Y\times G\times X)$, respectively, their composition is given by the following (G,G)-biset over $(Z\times G\times X)$



where $V\times_{d,c}U$ is the pullback of V and U over Y, i.e. the set of pairs $(v,u)\in V\times U$ with $d(v)=c(u)$, and $(V\times_{d,c}U)/G$ the set of orbits of G on it for the action given by $(v,u)\cdot g = (vg, g^{-1}u)$. This makes sense because $d(v\cdot g) = g^{-1}d(v) = g^{-1}c(u) = c(g^{-1}\cdot u)$ if $d(v)=c(u)$. The map (γ,β,α) is given by

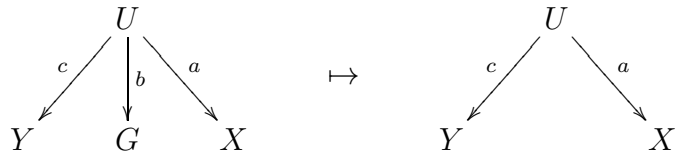
$$(\gamma,\beta,\alpha)\big((v,u)G\big) = \big(f(v), e(v)b(u), a(u)\big) \ .$$

2.8. The functor $\Phi : \widehat{\mathbf{B}}(G)\to\mathbf{B}(G)$ extends uniquely to an additive functor

$\Phi_\bullet : \widehat{\mathbf{B}}_\bullet(G) \to \mathbf{B}_\bullet(G)$, and the category $\mathbf{B}_\bullet(G)$ is the quotient of $\widehat{\mathbf{B}}_\bullet(G)$ obtained by identifying morphisms which have the same image by $\Phi_\bullet$. Clearly, two morphisms $f, g \in \mathrm{Hom}_{\widehat{\mathbf{B}}_\bullet(G)}(X, Y)$ are identified if and only if $f - g$ is in the kernel of the group homomorphism

$$\phi : \mathcal{B}\big({}_G(Y \times G \times X)_G\big) \to \mathcal{B}\big({}_G(Y \times X)_G\big)$$

induced by the correspondence



on bisets. In other words, a morphism f in $\widehat{\mathbf{B}}_\bullet(G)$ gives the zero morphism in $\mathbf{B}_\bullet(G)$ if and only if it belongs to $\mathrm{Ker}\,\phi$.

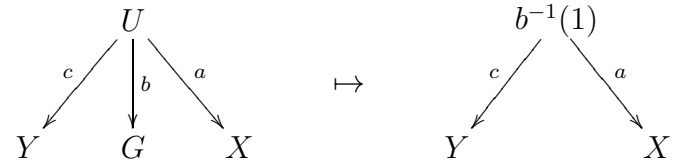
2.9. Now the (G, G)-biset ${}_GG_G$ is isomorphic to $\mathrm{Ind}_{\Delta(G)}^{G\times G}\bullet$, where $\Delta(G)$ is the diagonal subgroup of $G \times G$. It follows that there is an isomorphism of (G, G)-bisets

$$Y \times G \times X \cong \mathrm{Ind}_{\Delta(G)}^{G\times G}(Y \times X) \ .$$

Hence, by Remark 2.2 again, since $\Delta(G) \cong G$,

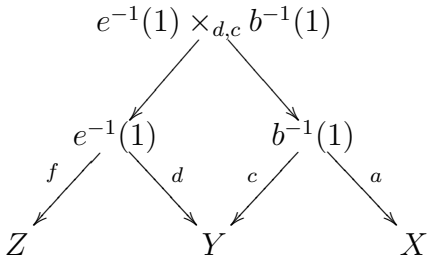
$$\mathcal{B}\big({}_G(Y \times G \times X)_G\big) \cong \mathcal{B}\big({}_G(Y \times X)\big) \ ,$$

where ${}_G(Y \times X)$ is the usual cartesian product with diagonal G-action. More precisely, this isomorphism is induced by the correspondence



It is then easy to check that the composition of

corresponds to the usual pullback diagram



In other words, the category $\widehat{\mathbf{B}}_\bullet(G)$ is equivalent to the category $\mathbf{S}(G)$ whose objects are the finite G-sets, where

$$\mathrm{Hom}_{\mathbf{S}(G)}(X,Y) = \mathcal{B}\big({}_G(Y \times X)\big) \ ,$$

and composition is induced by pullback. It has been shown by Lindner ([7], see also [2]) that the additive functors on this category are precisely the Mackey functors for G.

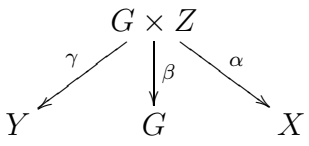
2.10. With this equivalence of categories $\widehat{\mathbf{B}}_\bullet(G) \cong \mathbf{S}(G)$, the main result of [6] can be viewed as a characterization of those Mackey functors for G, wiewed as additive functors on $\widehat{\mathbf{B}}_\bullet(G)$, which factor through the functor $\Phi_\bullet : \widehat{\mathbf{B}}_\bullet(G) \to \mathbf{B}_\bullet(G)$.

This characterization amounts to a precise description of the identifications effected by Φ on morphisms: starting with $f \in \mathrm{Hom}_{\mathbf{S}(G)}(X,Y)$, one can lift it to

$$f^+ \in \mathrm{Hom}_{\widehat{\mathbf{B}}_\bullet(G)}(X,Y) = \mathcal{B}\big({}_G(Y \times G \times X)_G\big) \ ,$$

and see when f^+ lies in $\mathrm{Ker}\,\phi$. Now f is represented by a difference of two G-sets over ${}_G(Y \times X)$ of the form

By induction from $\Delta(G)$ to $G \times G$, the G-set on the left hand side lifts to the following $(G \times G)$-set over ${}_{(G\times G)}(Y \times G \times X)$

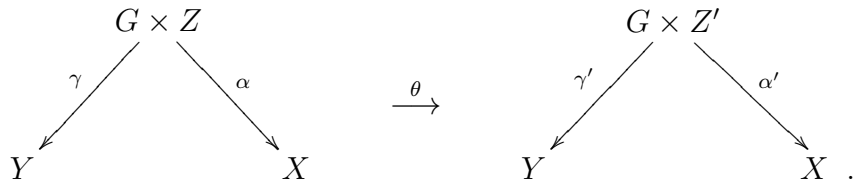


where the $(G \times G)$-actions on $G \times Z$ and $Y \times G \times X$ are given respectively by $(s,t) \cdot (g,z) = (sgt^{-1}, tz)$ and $(s,t) \cdot (y,g,x) = (sy, sgt^{-1}, tx)$, and where

$$(\gamma, \beta, \alpha)(g,z) = \big(gb(z), g, a(z)\big) \ .$$

Similarly the G-set $\big(Z', (b', a')\big)$ lifts to $\big(G \times Z', (\gamma', \beta', \alpha')\big)$.

Now f^+ is in $\operatorname{Ker} \phi$ if and only if there is an isomorphism



of $(G \times G)$-sets over $Y \times X$. Since $(g,z) = g \cdot (1,z)$ for any $(g,z) \in G \times Z$, it follows that θ is a map from $G \times Z$ to $G \times Z'$ of the form

$$(g,z) \mapsto \big(gu(z), v(z)\big) \ ,$$

where u is a map from Z to G and v is a map from Z to Z'. Now for any $(s,t) \in G \times G$, the equality

$$\theta\big((s,t) \cdot (g,z)\big) = (s,t) \cdot \theta\big((g,z)\big)$$

gives

$$\big(sgt^{-1}u(tz), v(tz)\big) = \big(sgu(z)t^{-1}, tv(z)\big) \ .$$

This is equivalent to

$$u(tz) = {}^t u(z) \ \text{ and } \ v(tz) = tv(z) \ .$$

This means that u is a morphism of G-sets from Z to G^c, which is the set G with G-action by conjugation, and v is a morphism of G-sets.

Moreover θ is a bijection if and only if v is.

Finally θ is a morphism of (G,G)-bisets *over* $Y \times X$ if and only if $\alpha' \circ \theta = a$ and $\gamma' \circ \theta = \gamma$, i.e. equivalently if

$$a' \circ v = a \ \text{ and } \ gu(z) \cdot b' \circ v(z) = g \cdot b(z)$$

for any $(g,z) \in G \times Z$. In other words

$$a = a' \circ v \ \text{ and } \ b = u * (b' \circ v) \ ,$$

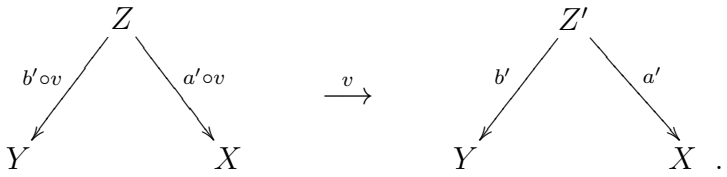
where, for any map $w : Z \to Y$, the map $u * w : Z \to Y$ is defined by $(u * w)(z) = u(z) \cdot w(z)$. The map $u * w$ is a map of G-sets if $u : Z \to G^c$

and $w : Z \to Y$ are. Note that $w' = u * w$ if and only if $w = \bar{u} * w'$, where $\bar{u} : Z \to G^c$ is defined by $\bar{u}(z) = u(z)^{-1}$.

It follows that f maps to the zero morphism in $\mathbf{B}(G)$ if and only if there exists $u : Z \to G^c$ and an isomorphism $v : Z \to Z'$ such that

$$a' \circ v = a \text{ and } b' \circ v = u * b \ ,$$

But then v is an isomorphism



of G-sets over $Y \times X$, and f is also represented by the difference

since $a' \circ v = a$ and $b' \circ v = u * b$. These are the morphisms in the category $\mathbf{S}(G)$ that vanish in $\mathbf{B}_\bullet(G)$. In other words:

2.11. Theorem : *Let G be a finite group. Let $\underline{\mathbf{S}}(G)$ denote the quotient category of $\mathbf{S}(G)$ defined by setting, for any two finite G-sets Y and Y*

$$\mathrm{Hom}_{\underline{\mathbf{S}}(G)}(X, Y) = \mathcal{B}\big({}_G(Y \times X)\big)/K(Y, X) \ ,$$

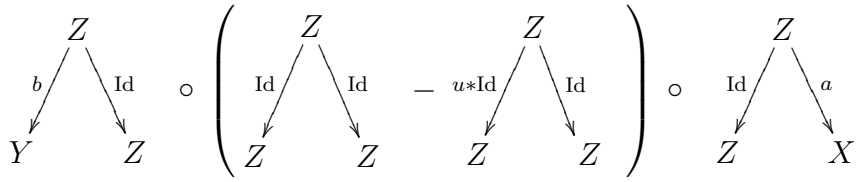
where $K(Y, X)$ is the subgroup generated by the differences

(**2.12**)

where $a : Z \to X$, $b : Z \to Y$, and $u : Z \to G^c$ are morphisms of G-sets.

Then the functor $\Phi_\bullet$ induces an equivalence of categories $\underline{\mathbf{S}}(G) \cong \mathbf{B}_\bullet(G)$.

Since the difference 2.12 factors as



the morphisms vanishing in $\underline{\mathbf{S}}(G)$ are generated in the category $\mathbf{S}(G)$ by the morphisms of the form

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2.13. It follows that the additive functors from $\underline{\mathbf{S}}(G)$ to the category of abelian groups are exactly those Mackey functors (in the sense of Dress) such that for any G-set Z and any $u : Z \to G^c$, the morphism $M_*(u * \mathrm{Id})$ is equal to the identity map of $M(Z)$.

This condition is additive with respect to Z, since the map $u * \mathrm{Id}_Z$ maps each G-orbit of Z to itself. Hence these functors are exactly the functors for which the map $M_*(u * \mathrm{Id})$ is the identity map of $M(G/H)$, for any subgroup H of G and any $u : G/H \to G^c$. Such a map is of the form $gH \mapsto {}^g c$, where $c \in C_G(H)$. The map $u * \mathrm{Id} : G/H \to G/H$ is the map $gH \mapsto gcH$.

Translated in terms of the usual definition of Mackey functors, this map expresses the action of c on $M(H) = M(G/H)$. This shows that additive functors from $\underline{\mathbf{S}}(G)$ to abelian groups are exactly the Mackey functors for the group G such that, for any $H \leq G$, the centralizer $C_G(H)$ acts trivially on $M(H)$. These are the "conjugation invariant Mackey functors" introduced in [6].

2.14. Remark : In view of Paragraph 1.1, one might be tempted to believe that such a conjugation invariant Mackey functor for G can always be obtained from a biset functor defined on all finite groups by the restriction procedure to subgroups of G described in Paragraph 1.1, but this is not true: for example, let G be an elementary abelian group of order 4, and let A, B, and C denote its subgroups of order 2. The simple Mackey functor $M = S_{A,\mathbb{F}_2}$ for G over the field with 2 elements has value $\mathbb{F}_2$ at A, and $\{0\}$ elsewhere (cf. [9] Lemma 15.1). The functor M is obviously a conjugation invariant Mackey functor, but if it were the restriction of a biset functor defined over all finite groups, then in particular its values at A, B, and C would be isomorphic to one other, as A, B and C are all isomorphic to C_2.

3. Fused G-sets

Let Z be any (finite) G-set. The multiplication $(u, v) \mapsto u * v$ endows the set $\mathrm{Hom}_{G\text{-}\mathsf{set}}(Z, G^c)$ with a group structure. Moreover, for any finite G-set X, this group acts on the left on the set $\mathrm{Hom}_{G\text{-}\mathsf{set}}(Z, X)$, via $(u, f) \mapsto u * f$. This action is compatible with the composition of morphisms: if Y is a finite G-set, if $u : Z \to G^c$ and $v : Y \to G^c$ are morphisms of G-sets, then for any morphisms of G-sets $f : Z \to Y$ and $g : Y \to X$, one checks easily that

$$(v * g) \circ (u * f) = \big(u * (v \circ f)\big) * (g \circ f) \quad . \tag{3.1}$$

3.2. Notation : *Let G-$\underline{\mathsf{set}}$ denote the category of* fused G-sets*: its objects are finite G-sets, and for any finite G-sets Z and Y*

$$\mathrm{Hom}_{G\text{-}\underline{\mathsf{set}}}(Z, Y) = \mathrm{Hom}_{G\text{-}\mathsf{set}}(Z, G^c) \backslash \mathrm{Hom}_{G\text{-}\mathsf{set}}(Z, Y) \quad .$$

The composition of morphisms in G-$\underline{\mathsf{set}}$ is induced by the composition of morphisms in G-set.

3.3. Remark : For any G-set Y, set $Y^I = Y \times G^c$. This notation is chosen to evoke a path object in homotopy theory (cf. [5] Section 4.12). There is a natural morphism $p : Y^I \to Y \times Y$, defined by $p(y, g) = (y, gy)$, for $y \in Y$ and $g \in G$, and a morphism $i : Y \to Y^I$ defined by $i(y) = (y, 1)$, for $y \in Y$. The composition $p \circ i$ is equal to the diagonal map $Y \to Y \times Y$.

Two morphisms $a, b : Z \to Y$ in G-set are equal in the category G-$\underline{\mathsf{set}}$ if and only if the morphism $(a, b) : Z \to Y \times Y$ factors as

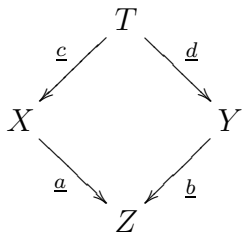
$$\begin{array}{ccc} & & Y^I \\ & \overset{\varphi}{\nearrow} & \big\downarrow p \\ Z & \xrightarrow[(a,b)]{} & Y \times Y \end{array}$$

for some morphism of G-sets $\varphi : Z \to Y^I$.

3.4. Remark : It follows from 3.1 that the map $u \mapsto u * \mathrm{Id}_Z$ is a group antihomomorphism from $\mathrm{Hom}_{G\text{-}\mathsf{set}}(Z, G^c)$ to the group of G-automorphisms of Z. Hence a morphism $\underline{f} : Z \to Y$ in the category G-$\underline{\mathsf{set}}$ is an isomorphism if and only if any of its representatives $f : Z \to Y$ in G-set is an isomorphism.

3.5. Weak pullbacks of fused G-sets. Disjoint union of G-sets is a

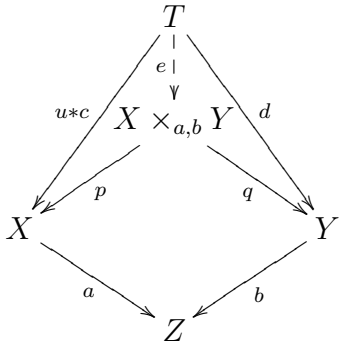
coproduct in G-$\underline{\mathsf{set}}$. There is also a weak version of pullback in G-$\underline{\mathsf{set}}$: let



be a commutative diagram in G-$\underline{\mathsf{set}}$, where underlines denote the images in G-$\underline{\mathsf{set}}$ of morphisms in G-set. This means that $\underline{a} \circ \underline{c} = \underline{b} \circ \underline{d}$, i.e. that there exists $u \in \mathrm{Hom}_{G\text{-}\mathsf{set}}(T, G^c)$ such that

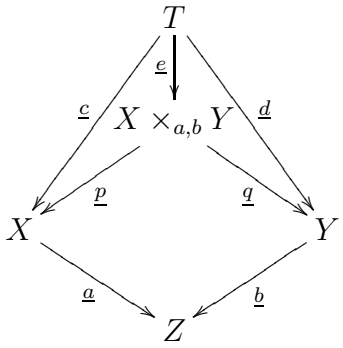
$$b \circ d = u * (a \circ c) \ .$$

But $u * (a \circ c) = a \circ (u * c)$. It follows that there is a unique morphism $e \in \mathrm{Hom}_{G\text{-}\mathsf{set}}(T, X \times_{a,b} Y)$ such that the diagram



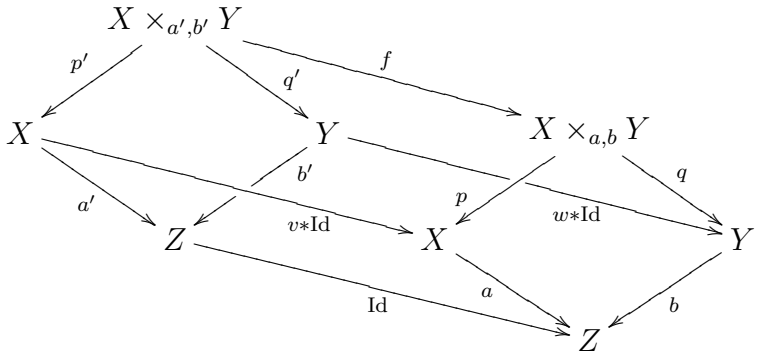
is commutative in G-set, where $p : X \times_{a,b} Y \to X$ and $q : X \times_{a,b} Y \to Y$ are the canonical morphisms from the pullback $X \times_{a,b} Y$. In other words, the diagram

(**3.6**)

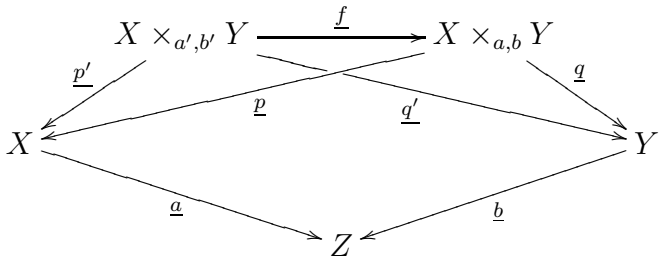


is commutative in G-$\underline{\mathsf{set}}$.

But still $(X \times_{a,b} Y, \underline{p}, \underline{q})$ need not be a pullback in G-$\underline{\mathsf{set}}$, since the morphism $\underline{e}$ making Diagram 3.6 commutative is generally not unique, as e itself depends on the choice of u. Moreover, the lifts a and b of $\underline{a}$ and $\underline{b}$ to G-set are not unique : it should be noted however that if $a' = v * a$ and $b' = w * b$ are other lifts of a and b, respectively, where $v \in \mathrm{Hom}_{G\text{-}\mathsf{set}}(X, G^c)$ and $w \in \mathrm{Hom}_{G\text{-}\mathsf{set}}(Y, G^c)$, then the map $f : (x, y) \mapsto \big(v(x)x, w(y)y\big)$ is an isomorphism of G-sets from $X \times_{a',b'} Y$ to $X \times_{a,b} Y$, such that the diagram

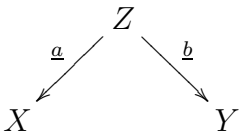


is commutative in G-set. Since $\underline{a'} = \underline{a}$, $\underline{b'} = \underline{b}$, $\underline{v * \mathrm{Id}} = \underline{\mathrm{Id}}$, and $\underline{w * \mathrm{Id}} = \underline{\mathrm{Id}}$, this yields a commutative diagram



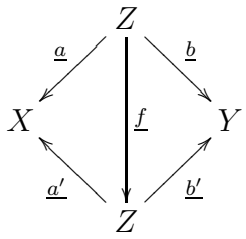
in G-$\underline{\mathsf{set}}$, and $\underline{f}$ is an isomorphism. This shows that the weak pullback $X \times_{a,b} Y$ only depends on $\underline{a}$ and $\underline{b}$ in the category G-$\underline{\mathsf{set}}$. For this reason, it may be denoted by $X \times_{\underline{a},\underline{b}} Y$.

3.7. Spans of fused G-sets. Recall (cf. [10], [1] for the general definition) that if X and Y are finite G-sets, then a *span* $\Lambda_{Z,\underline{a},\underline{b}}$ over X and Y in the category G-$\underline{\mathsf{set}}$ is a diagram of the form



where Z is a finite G-set and $\underline{a}, \underline{b}$ are morphisms in the category G-$\underline{\mathsf{set}}$. Two

spans $\Lambda_{Z,\underline{a},\underline{b}}$ and $\Lambda_{Z',\underline{a}',\underline{b}'}$ over X and Y are equivalent if there exists an isomorphism $\underline{f} : Z \to Z'$ in G-$\underline{\mathsf{set}}$ such that the diagram



is commutative. The set of equivalence classes of spans of fused G-sets over X and Y is an additive monoid, where the addition is defined by disjoint union (i.e. $\Lambda_{Z_1,\underline{a}_1,\underline{b}_1} + \Lambda_{Z_2,\underline{a}_2,\underline{b}_2} = \Lambda_{Z_1 \sqcup Z_2,\underline{a}_1 \sqcup \underline{a}_2,\underline{b}_1 \sqcup \underline{b}_2}$). The corresponding Grothendieck group is isomorphic to $\mathrm{Hom}_{\underline{\mathbf{S}}(G)}(Y, X)$.

It should be noted that even if there is no pullback construction in the category G-$\underline{\mathsf{set}}$, the isomorphism classes of spans in G-$\underline{\mathsf{set}}$ can still be composed by *weak pullback*, and this induces the composition of morphisms in $\underline{\mathbf{S}}(G)$.

4. Fused Mackey functors

4.1. Definition : *Let R be a commutative ring. Let $R\,\mathbf{S}(G)$ (resp. $R\,\underline{\mathbf{S}}(G)$) denote the R-linear extension of the category $\mathbf{S}(G)$ (resp. $\underline{\mathbf{S}}(G)$), defined as follows:*

- *The objects of $R\,\mathbf{S}(G)$ and $R\,\underline{\mathbf{S}}(G)$ are finite G-sets.*
- *For finite G sets X and Y,*

$$\mathrm{Hom}_{R\,\mathbf{S}(G)}(X,Y) = R \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathbf{S}(G)}(X,Y) \ ,$$

$$\mathrm{Hom}_{R\,\underline{\mathbf{S}}(G)}(X,Y) = R \otimes_{\mathbb{Z}} \mathrm{Hom}_{\underline{\mathbf{S}}(G)}(X,Y) \ .$$

- *Composition of morphisms is induced by the pullback in G-set (resp. the weak pullback in G-$\underline{\mathsf{set}}$).*

A Mackey functor for G over R in the sense of Lindner ([7]) is an R-linear functor from $R\,\mathbf{S}(G)$ to the category R-Mod of R-modules.

Similarly, a fused Mackey functor *for G over R is an R-linear functor from $R\,\underline{\mathbf{S}}(G)$ to R-Mod. A morphism of fused Mackey functors is a natural transformation of functors. Fused Mackey functors for G over R form a category denoted by $\mathsf{Mack}_R^f(G)$.*

The following is an equivalent definition of fused Mackey functors, *à la* Dress:

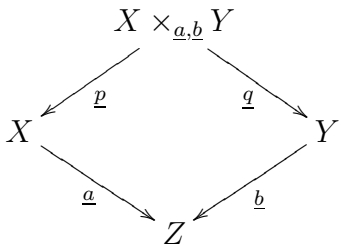
4.2. Definition : *Let R be a commutative ring. A* fused Mackey functor *for the group G over R is a bivariant R-linear functor $M = (M^*, M_*)$ from G-$\underline{\mathsf{set}}$ to R-Mod such that:*

1. *For any finite G-sets X and Y, the maps*

$$M(X) \oplus M(Y) \underset{(M^*(\underline{i}_X), M^*(\underline{i}_Y))}{\overset{(M_*(\underline{i}_X), M_*(\underline{i}_Y))}{\rightleftarrows}} M(X \sqcup Y)$$

   induced by the canonical inclusions $i_X : X \to X \sqcup Y$ and $i_Y : Y \to X \sqcup Y$ are mutual inverse isomorphisms.

2. *If*



   is a weak pullback diagram in G-$\underline{\mathsf{set}}$, then $M^*(\underline{a})M_*(\underline{b}) = M_*(\underline{p})M^*(\underline{q})$.

A morphism of fused Mackey functors is a natural transformation of bivariant functors.

The category $\mathsf{Mack}_R^f(G)$ can be viewed as a full subcategory of the category $\mathsf{Mack}_R(G)$ of Mackey functors for G over R. In the case $R = \mathbb{Z}$, this category is equivalent to the category of conjugation invariant Mackey functors introduced in [6].

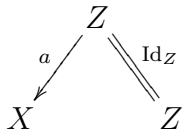
The inclusion functor $\mathsf{Mack}_R^f(G) \hookrightarrow \mathsf{Mack}_R(G)$ has a left adjoint:

4.3. Definition : *Let M be a Mackey functor for G over R, in the sense of Lindner, i.e. an R-linear functor $R\mathbf{S}(G) \to R$-Mod. When X is a finite G-set, set*

$$M^f(X) = M(X) / \sum_{Z,a,u} \mathrm{Im}\big(M(\Lambda_{a,\mathrm{Id}_Z}) - M(\Lambda_{u*a,\mathrm{Id}_Z})\big) \ ,$$

where the summation runs through triples (Z, a, u) consisting of a finite G-set Z, and morphisms of G-sets $a : Z \to X$ and $u : Z \to G^c$, and $\Lambda_{a,\mathrm{Id}_Z}$

denotes the span



of G-sets.

4.4. Proposition : *Let R be a commutative ring, and G be a finite group.*

1. *Let M be a Mackey functor for G over R. The correspondence*

$$X \mapsto M^f(X)$$

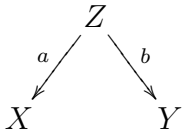
is a fused functor M^f for G over R.

2. *The correspondence $\mathcal{F} : M \mapsto M^f$ is a functor from $\mathsf{Mack}_R(G)$ to $\mathsf{Mack}_R^f(G)$, which is left adjoint to the inclusion functor*

$$\mathcal{I} : \mathsf{Mack}_R^f(G) \hookrightarrow \mathsf{Mack}_R(G) \ .$$

Moreover $\mathcal{F} \circ \mathcal{I}$ is isomorphic to the identity functor of $\mathsf{Mack}_R^f(G)$.

Proof : For Assertion 1, to prove that M^f is a Mackey functor, observe that if $\Lambda_{Z,a,b}$ is a span of finite G-sets of the form



and $u : Z \to G^c$ is a morphism of G-sets, then

$$\Lambda_{Z,a,b} - \Lambda_{Z,u*a,b} = (\Lambda_{Z,a,\mathrm{Id}_Z} - \Lambda_{Z,u*a,\mathrm{Id}_Z}) \circ \Lambda_{Z,\mathrm{Id}_Z,b} \ .$$

It follows that the R-module

$$\sum_{Z,a,u} \mathrm{Im}\big(M(\Lambda_{a,\mathrm{Id}_Z}) - M(\Lambda_{u*a,\mathrm{Id}_Z})\big)$$

is equal to the sum

$$\sum_{Z,a,b,u} \mathrm{Im}\big(M(\Lambda_{a,b}) - M(\Lambda_{u*a,b})\big) \ .$$

In other words, it is equal to the image by M of the R-submodule $K_R(X,Y)$ of $\mathrm{Hom}_{R\mathbf{S}(G)}(Y,X)$ generated by the morphisms $\Lambda_{a,b} - \Lambda_{u*a,b}$, i.e. to the kernel of the quotient morphism

$$\mathrm{Hom}_{R\mathbf{S}(G)}(Y,X) \to \mathrm{Hom}_{R\underline{\mathbf{S}}(G)}(Y,X) \ .$$

This shows that K_R is an ideal in the category $R\mathbf{S}(G)$. So if M is an R-linear functor $R\mathbf{S}(G) \to R$-Mod, the correspondence

$$X \mapsto M^f(X) = M(X)/\sum_{f\in K_R(X,Y)} \mathrm{Im}M(f)$$

is an R-linear functor from the quotient category $R\underline{\mathbf{S}}(G)$ to R-Mod.

Assertion 2 is straightforward: first it is clear that $\mathcal{F}\circ\mathcal{I}$ is isomorphic to the identity functor, since $N^f = N$ when N is a fused Mackey functor. This isomorphism $\mathcal{F}\circ\mathcal{I} \cong \mathrm{Id}_{\mathsf{Mack}_R^f(G)}$ provides the counit of the adjunction. Next for any Mackey functor M, there is a projection morphism $M \to \mathcal{I}\mathcal{F}(M)$, and this yields the unit of the adjunction. □

4.5. Remark : Assertion 2 shows that $\mathsf{Mack}_R^f(G)$ is a *reflective* subcategory of $\mathsf{Mack}_R(G)$ (cf. [8], Chapter IV, Section3).

4.6. Remark : If the Mackey functor M is given in the sense of Dress, then for any finite G-set X

$$M^f(X) = M(X)/\sum_{\substack{a:Z\to X\\ u:Z\to G^c}} \mathrm{Im}\big(M_*(a) - M_*(u*a)\big) \ ,$$

where Z is a finite G-set, and a, u are morphisms of G-sets.

4.7. Corollary :

1. *If P is a projective Mackey functor, then P^f is projective in the category $\mathsf{Mack}_R^f(G)$.*
2. *The category $\mathsf{Mack}_R^f(G)$ has enough projective objects. More precisely, if N is a fused Mackey functor, and $\theta : P \to \mathcal{I}(N)$ is an epimorphism in $\mathsf{Mack}_R(G)$ from a projective Mackey functor P, then $\mathcal{F}(\theta) : P^f \to N$ is an epimorphism in $\mathsf{Mack}_R^f(G)$.*

Proof : Assertion 1 follows from the fact that $\mathcal{F}$ is left adjoint to the exact functor $\mathcal{I}$. Assertion 2 is then straightforward. □

5. The fused Mackey algebra

When G is a finite group, set $\Omega_G = \bigsqcup_{H \leq G} G/H$, and let RB_{Ω_G} denote the Dress construction for the Burnside functor RB over the ring R. Recall that RB_{Ω_G}, as a Mackey functor in the sense of Dress, is obtained by precomposition of RB with the endofunctor $X \mapsto X \times \Omega_G$ of G-set.

Also recall (cf. [2] Lemma 7.3.2 and Proposition 4.5.1) that the functor RB_{Ω_G} is a progenerator of the category $\mathsf{Mack}_R(G)$, and that the algebra $\mathrm{End}_{\mathsf{Mack}_R(G)}(B_{\Omega_G}) \cong B(\Omega_G^2)$ is isomorphic to the Mackey algebra $\mu_R(G)$ of G over R, introduced by Thévenaz and Webb ([9]).

It follows from Corollary 4.7 that the functor $(RB_{\Omega_G})^f$ is a progenerator in the category $\mathsf{Mack}_R^f(G)$. Hence this category is equivalent to the category of modules over the algebra $\mathrm{End}_{\mathsf{Mack}_R^f(G)}\big((RB_{\Omega_G})^f\big)$.

5.1. Definition : *The* fused Mackey algebra *of G over R is the algebra*

$$\mu_R^f(G) = \mathrm{End}_{\mathsf{Mack}_R^f(G)}\big((RB_{\Omega_G})^f\big) \ .$$

5.2. Lemma : *Let X be a finite G-set. Then $(RB_X)^f$ is isomorphic to the Yoneda functor* $\mathrm{Hom}_{R\underline{\mathbf{S}}(G)}(X, -)$.

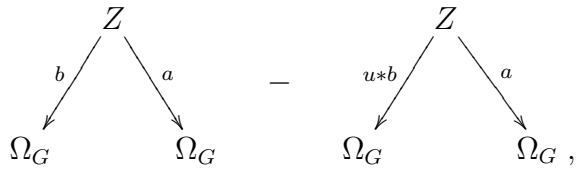
Proof : Denote by $\mathcal{Y}_X$ the Yoneda functor $\mathrm{Hom}_{R\underline{\mathbf{S}}(G)}(X, -)$. For any fused Mackey functor N for G over R

$$\begin{aligned} \mathrm{Hom}_{\mathsf{Mack}_R^f(G)}\big((RB_X)^f, N\big) &\cong \mathrm{Hom}_{\mathsf{Mack}_R(G)}\big(RB_X, \mathcal{I}(N)\big) \\ &\cong \mathcal{I}(N)(X) \cong N(X) \\ &\cong \mathrm{Hom}_{\mathsf{Mack}_R^f(G)}(\mathcal{Y}_X, N) \ . \end{aligned}$$

The lemma follows, since all these isomorphisms are natural. □

5.3. Theorem : *The fused Mackey algebra $\mu_R^f(G)$ is isomorphic to the quotient of the algebra $RB(\Omega_G^2) \cong \mu_R(G)$ by the R-module generated by dif-*

ferences of the form



where $a, b : Z \to \Omega_G$ and $u : Z \to G^c$ are morphisms of G-sets.

Proof : This follows from Lemma 5.2, since the quotient in the theorem is precisely $\mathrm{End}_{R\underline{\mathbf{S}}(G)}(\Omega_G)$. □

5.4. Remark : One can deduce from this theorem that the fused Mackey algebra $\mu_R^f(G)$ is always free of finite rank as an R-module, and this rank does not depend on the commutative ring R. More precisely, Thévenaz and Webb have shown ([9] Proposition 3.2) that the Mackey algebra $\mu_R(G)$ has an R-basis consisting of elements of the form

$$t_K^H \, c_{g,K} \, r_{K^g}^L \quad ,$$

where (H, L, g, K) runs through a set of representatives of 4-tuples consisting of two subgroups H and L of G, and element g of G, and a subgroup K of $H \cap {}^gL$, for the equivalence relation $\equiv$ given by

$$(H, L, g, K) \equiv (H', L', g', K') \Leftrightarrow \begin{cases} H = H', \ L = L', \\ \text{and} \\ \exists h \in H, \ \exists l \in L, \ g' = hgl, \ K' = {}^hK \quad . \end{cases}$$

Similarly, the quotient algebra $\mu_R^f(G)$ of $\mu_R(G)$ has a basis consisting of the images of the elements $t_K^H \, c_{g,K} \, r_{K^g}^L$, where (H, L, g, K) runs through a set of representatives of 4-tuples as above, modulo the relation $\equiv^f$ defined by

$$(H, L, g, K) \equiv^f (H', L', g', K') \Leftrightarrow \begin{cases} H = H', \ L = L', \\ \text{and} \\ \exists h \in H, \ \exists l \in L, \ \exists x \in C_G(K), \\ \quad g' = hxgl, \ K' = {}^hK \quad . \end{cases}$$

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