Functorial equivalence of blocks

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Category: Objects + Morphisms.

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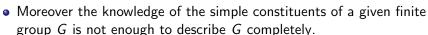
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 - **Examples:** Projective kG-modules, modules inflated from projective k(G/N)-modules $(N \subseteq G)$, modules induced from p-permutation kH-modules $(H \subseteq G)$.

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This is equivalent to saying that there exists $\sigma \in cRT^{\Delta}(H,G)b$ and $\tau \in bRT^{\Delta}(G,H)c$ such that $\sigma \circ \tau = [kHc]$ and $\tau \circ \sigma = [kGb]$.



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$$k(kGb) - l(kGb) = k(kHc) - l(kHc).$$



THANK YOU!

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