

ARTICLE

Diagonal p -permutation functors in characteristic p

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Received: 22 March 2025 UTC; **Revised:** 23 March 2026 UTC; **Accepted:** 8 May 2026 UTC

2020 Mathematics Subject Classification: Primary – 16S50, 18B99, 20C20, 20J15

Keywords: diagonal p -permutation functor; simple functor; essential algebra; functorial equivalence

Abstract

Let p be a prime number. We consider diagonal p -permutation functors over a (commutative, unital) ring R in which all prime numbers different from p are invertible. We first determine the finite groups G for which the associated essential algebra $\mathcal{E}_R(G)$ is non-zero: These are groups of the form $G = L\langle u \rangle$, where (L, u) is a D^Δ -pair. When R is an algebraically closed field $\mathbb{F}$ of characteristic 0 or p , this yields a parameterization of the simple diagonal p -permutation functors over $\mathbb{F}$ by triples (L, u, W) , where (L, u) is a D^Δ -pair, and W is a simple $\mathbb{F}\text{Out}(L, u)$ -module. Finally, we describe the evaluations of the simple functor $S_{L, u, W}$ parameterized by the triple (L, u, W) . We show in particular that if G is a finite group and $\mathbb{F}$ has characteristic p , the dimension of $S_{L, 1, \mathbb{F}}(G)$ is equal to the number of conjugacy classes of p -regular elements of G with a defect group isomorphic to L .

1. Introduction

Let k be an algebraically closed field of positive characteristic p , and R be a commutative ring (with 1). As in [6] and [7], we consider the following category Rpp_k^Δ :

- The objects of Rpp_k^Δ are the finite groups.
- For finite groups G and H , the hom set $\text{Hom}_{Rpp_k^\Delta}(G, H)$ is defined as $RT^\Delta(H, G) = R \otimes_{\mathbb{Z}} T^\Delta(H, G)$, where $T^\Delta(H, G)$ is the Grothendieck group of diagonal p -permutation (kH, kG) -bimodules.
- The composition in Rpp_k^Δ is induced by the usual tensor product of bimodules.
- The identity morphism of the group G is the class of the (kG, kG) -bimodule kG .

The category $\mathcal{F}_{Rpp_k}^\Delta$ of *diagonal p -permutation functors over R* is the category of R -linear functors from Rpp_k^Δ to the category $R\text{-Mod}$ of all R -modules. It is an abelian category.

In [6] and [7], we mainly considered the case, where R is an algebraically closed field $\mathbb{F}$ of characteristic 0. In [7], we showed in particular that the category $\mathcal{F}_{\mathbb{F}pp_k}^\Delta$ is semisimple, and we classified and described its simple objects. For an arbitrary commutative ring R , we also introduced a new equivalence for blocks of groups algebras, called *functorial equivalence* over R , using diagonal p -permutation functors over R naturally attached to pairs (G, b) of a finite group G and a block idempotent b of kG . This led us in particular to prove the following finiteness theorem in the spirit of Donovan's and Puig's finiteness conjectures: For a given finite p -group D , there is only a finite number of pairs (G, b) of a finite group G and a block idempotent b of kG , with defect groups isomorphic

to D , up to functorial equivalence over $\mathbb{F}$. We also showed that important invariants of blocks (like the number of simple modules or the number of ordinary irreducible characters) are preserved by functorial equivalence over $\mathbb{F}$, and we gave a characterization of nilpotent blocks in terms of functorial equivalence.

A natural question is then to see what happens when R is a field of *positive characteristic*, in particular when $R = k$. An additional motivation for considering this case is that there are diagonal p -permutation functors attached to blocks, which vanish when considered over $\mathbb{F}$, but are of a fundamental interest when defined over k : For example, the Hochschild cohomology functors, sending a pair (G, b) as above to the i th Hochschild cohomology group $HH^i(kGb, kGb)$ (or its k -dual, isomorphic to the i th Hochschild homology group $HH_i(kGb, kGb)$). Let us mention here a question raised by Linckelmann [11]: If b has a non-trivial defect group, is it true that $HH^1(kGb, kGb) \neq 0$?

This article is a first step in the study of diagonal p -permutation functors in characteristic p , and we focus on simple functors. The main result we obtain (Theorem 5.25) is a parameterization and a description of these objects. In particular, we show how to compute the evaluations of such simple functors.

A key ingredient to the parameterization and description of simple functors is *the essential algebra* $\mathcal{E}_R(G)$ of a group G , namely, the quotient of the endomorphism algebra $RT^\Delta(G, G)$ of G in the category Rpp_k^Δ by the ideal of linear combinations of endomorphisms which factor through a group of order strictly smaller than $|G|$. We first find some conditions on G (Corollary 3.4 and Theorems 3.6 and 3.7) for the (non-)vanishing of $\mathcal{E}_R(G)$. In particular, if any prime number different from p is invertible in R , we show (Corollary 3.8) that $\mathcal{E}_R(G)$ is non-zero if and only if G is a semidirect product $L \rtimes \langle u \rangle$ (which we denote by $L\langle u \rangle$), where (L, u) is a D^Δ -pair, that is, a pair of a finite p -group L and a p' -automorphism u of L (Definition 3.9). Moreover, we describe completely (Theorem 4.9) the structure of the algebra $\mathcal{E}_R(G)$ in this case.

In Section 5, we study simple diagonal p -permutation functors, so we assume that R is a field $\mathbb{F}$ of characteristic 0 or p . Applying the results of the previous sections, we know that if S is a simple diagonal p -permutation functor over $\mathbb{F}$, then a minimal group for S is of the form $L\langle u \rangle$, where (L, u) is a D^Δ -pair, and the evaluation $V = S(L\langle u \rangle)$ is a simple $\mathcal{E}_{\mathbb{F}}(L\langle u \rangle)$ -module. Conversely, to any triple (L, u, V) , where (L, u) is a D^Δ -pair and V is a simple $\mathcal{E}_{\mathbb{F}}(L\langle u \rangle)$ -module, we associate a simple functor $S_{L\langle u \rangle, V}$ with minimal group $L\langle u \rangle$, and such that $S_{L\langle u \rangle, V}(L\langle u \rangle) \cong V$. Then we compute (Theorem 5.23) the evaluation $S_{L\langle u \rangle, V}(G)$ at an arbitrary finite group G .

The precise structure of the essential algebra given by Theorem 4.9 now allows for another parameterization of the simple functors, namely, by triples (L, u, W) , where (L, u) is a D^Δ -pair, and W is a simple module for the algebra $\mathbb{F}\text{Out}(L, u)$ -module of the group $\text{Out}(L, u)$ (Notation 3.10) of outer automorphisms of (L, u) . In Theorem 5.25, we describe the evaluations of the simple functor $S_{L, u, W}$ parameterized by such a triple (L, u, W) .

Section 6 is devoted to some examples: First, the simple functor $S_{1, 1, \mathbb{F}}$, which turns out to be closely related to the Cartan map (Lemma 6.2 and Proposition 6.8). This example shows in particular that the category $\mathcal{F}_{\mathbb{F}pp_k}$ is *not* semisimple when $\mathbb{F}$ has characteristic p . Then we describe (Theorem 6.11) the evaluations of the simple functor $S_{L, 1, W}$. In particular (Corollary 6.14), we show that for a finite group G , the dimension of $S_{L, 1, k}(G)$ is equal to the number of conjugacy classes of p' -elements of G with a defect group isomorphic to L (Definition 6.13).

2. Notation and terminology¹

Throughout the article:

- k is an algebraically closed field of positive characteristic p .
- R is a commutative ring (with 1).

¹An additional list of symbols is included at the end of the article.

- For a finite group G , we denote by $\text{Proj}(kG)$ the group of projective kG -modules, and by $R_k(G)$ the Grothendieck group of the category of finite dimensional kG -modules. We set $\text{RProj}(kG) = R \otimes_{\mathbb{Z}} \text{Proj}(kG)$ and $\text{RR}_k(G) = R \otimes_{\mathbb{Z}} R_k(G)$.
- If P is a p -subgroup of a finite group G , and M is a kG -module, we denote by $M[P]$ the Brauer quotient of M at P , and by $\text{Br}_P : M^P \rightarrow M[P]$ the projection map. The module $M[P]$ is a $k\overline{N}_G(P)$ -module, where $\overline{N}_G(P) = N_G(P)/P$.
- For a finite group G , a p -permutation kG -module (see [9]) is a direct summand of a permutation kG -module, that is, of a module admitting a G -invariant k -basis. Equivalently, a kG -module M is a p -permutation module if the restriction $\text{Res}_S^G M$ of M to a Sylow p -subgroup S of G is a permutation kS -module.
- From [9], we know that the indecomposable p -permutation kG -modules (up to isomorphism) are parameterized by pairs (P, E) , where P is a p -subgroup of G , up to conjugation, and E is an indecomposable projective $k\overline{N}_G(P)$ -module, up to isomorphism. The indecomposable module $M(P, E)$ parameterized by the pair (P, E) is the only indecomposable direct summand with vertex P of $L_{P,E} = \text{Ind}_{N_G(P)}^G \text{Inf}_{\overline{N}_G(P)}^{N_G(P)} E$, the other direct indecomposable summands having vertex strictly contained in P , up to conjugation. The module $M(P, E)$ has vertex P , and $M(P, E)[P] \cong E$ as $k\overline{N}_G(P)$ -modules.
- It follows that the Grothendieck group of p -permutation kG -modules, for relations given by direct sum decomposition, has a basis consisting of the modules $L_{P,E}$, where P is a p -subgroup of G , up to conjugation, and E is an indecomposable projective $k\overline{N}_G(P)$ -module.
- When G and H are finite groups, and L is a subgroup of $H \times G$, we denote by $p_1(L)$ (resp. $p_2(L)$) the projection of L in H (resp. in G), and we set

$$k_1(L) = \{h \in H \mid (h, 1) \in L\} \quad \text{and} \quad k_2(L) = \{g \in G \mid (1, g) \in L\}.$$

We say that L is *diagonal* if $k_1(L) = k_2(L) = 1$. Equivalently,

$$L = \Delta(Y, \pi, X) = \{(\pi(x), x) \mid x \in X\},$$

where X is a subgroup of G and $\pi : X \rightarrow Y$ is a group isomorphism from X to a subgroup Y of H . If $X = Y$ and $\pi = \text{Id}$, we simply write $\Delta(X) = \Delta(X, \text{Id}, X)$. For $X \leq G$ and an embedding $\psi : X \hookrightarrow H$, we also write $\Delta_\psi(X)$ instead of $\Delta(\psi(X), \psi, X)$.

- For finite groups G and H , a p -permutation (kH, kG) -bimodule is a (kH, kG) -bimodule which is a p -permutation module when viewed as a $k(H \times G)$ -module. A p -permutation (kH, kG) -bimodule M is *diagonal* if in addition M is projective when viewed as a left kH -module and a right kG -module. Equivalently, M is a p -permutation (kH, kG) -bimodule, and all the vertices of the indecomposable summands of M are diagonal p -subgroups of $H \times G$.
- For finite groups G and H , we denote by $T^\Delta(H, G)$ the Grothendieck group of diagonal p -permutation (kH, kG) -bimodules, for relations given by direct sum decomposition. We set $\text{RT}^\Delta(H, G) = R \otimes_{\mathbb{Z}} T^\Delta(H, G)$. The group $T^\Delta(H, G)$ has a basis consisting of the bimodules of the form

$$\text{Ind}_{N_{H \times G}(P)}^{H \times G} \text{Inf}_{\overline{N}_{H \times G}(P)}^{N_{H \times G}(P)} E,$$

where P is a diagonal p -subgroup of $H \times G$ (up to conjugation), and E is an indecomposable projective $\overline{N}_{H \times G}(P)$ -module.

- When G, H , and K are finite groups, if M is a diagonal p -permutation (kG, kH) -bimodule, and N is a diagonal p -permutation (kK, kH) -bimodule, then $N \otimes_{kH} M$ is a diagonal p -permutation (kK, kG) -bimodule. This induces a well-defined bilinear map

$$T^\Delta(K, H) \times T^\Delta(H, G) \rightarrow T^\Delta(K, G),$$

still denoted $(v, u) \mapsto v \otimes_{kH} u$. This bilinear map is also the composition in the category Rpp_k^Δ of the introduction, so it will be sometimes denoted by $(v, u) \mapsto v \circ u$.

- For finite groups G and H , we say that an element $u \in \text{RT}^\Delta(G, H)$ is *right essential* (resp. *left essential*) if it cannot be factored through groups of order strictly smaller than $|H|$ (resp. of order strictly smaller than $|G|$), that is, if $u \notin \sum_{|K| < |H|} \text{RT}^\Delta(G, K) \circ \text{RT}^\Delta(K, H)$ (resp. if $u \notin \sum_{|K| < |G|} \text{RT}^\Delta(G, K) \circ \text{RT}^\Delta(K, H)$). A (kG, kH) -bimodule M is called *right essential over R*—or simply *right essential*—(resp. *left essential*) if the element M of $\text{RT}^\Delta(G, H)$ is right essential (resp. left essential).

If $|G| = |H|$, being left essential is equivalent to being right essential, so we simply say *essential*.

- In particular, for a finite group G , the endomorphism algebra of G in the category Rpp_k^Δ is $\text{RT}^\Delta(G, G)$. The *essential algebra* (over R) of G is the quotient

$$\mathcal{E}_R(G) = \text{RT}^\Delta(G, G) / \sum_{|H| < |G|} \text{RT}^\Delta(G, H) \circ \text{RT}^\Delta(H, G)$$

of $\text{RT}^\Delta(G, G)$ by the (two sided) ideal of non-essential elements. We denote by $u \mapsto \varepsilon_R(u)$ the projection map $\text{RT}^\Delta(G, G) \rightarrow \mathcal{E}_R(G)$.

- The main reason for considering the previous essential algebra is the following: By standard results (see, e.g., [5, Lemma 2.5 and Proposition 2.7]), if S is a simple diagonal p -permutation functor over R , and if G is a group such that $V := S(G) \neq 0$, then V is a simple $\text{RT}^\Delta(G, G)$ -module, and S is isomorphic to the unique simple quotient $S_{G,V}$ of the functor $L_{G,V} : H \mapsto \text{RT}^\Delta(H, G) \otimes_{\text{RT}^\Delta(G, G)} V$. Moreover, if G is a group of minimal order such that $S(G) \neq 0$, then in fact V is a simple $\mathcal{E}_R(G)$ -module, and $S \cong S_{G,V}$. So we are looking for pairs (G, V) of a finite group G and a simple $\mathcal{E}_R(G)$ -module. In particular, for such a pair, the essential algebra $\mathcal{E}_R(G)$ is non-zero.
- An elementary group (or Brauer elementary group) is a finite group of the form $Q \times C$, where Q is a q -group for some prime number q , and C is a cyclic group (that can be assumed of order prime to q). When p is a prime number, an elementary p' -group is an elementary group of order prime to p .

3. Vanishing of $\mathcal{E}_R(G)$

Let G be a finite group. We want to know when the essential algebra $\mathcal{E}_R(G)$ is non-zero. We start with some classical lemmas.

Lemma 3.1. *Let $G = P \rtimes K$, where P and K are finite groups of coprime order. Let moreover φ be an automorphism of G . Then:*

1. $\varphi(P) = P$.
2. $C_G(P) = Z(P)C_K(P)$.
3. Suppose that $C_G(P) = Z(P)$, or equivalently by Assertion 2, that K acts faithfully on P . Then the following are equivalent:
 - (a) The restriction of φ to P is the identity.
 - (b) φ is an inner automorphism i_w of G , for some $w \in Z(P)$.

Proof. 1. This is clear, since P is the set of elements of G of order dividing the order of P .

2. The inclusion $Z(P)C_K(P) \leq C_G(P)$ is clear. Conversely, if $xt \in C_G(P)$, where $x \in P$ and $t \in K$, then ${}^t y = y^x$ for any $y \in P$. It follows that ${}^{t^n} y = y^{x^n}$ for any $n \in \mathbb{N}$ and any $y \in P$. Taking $n = |x|$ gives ${}^{t^n} y = y$, hence $t \in C_K(P)$ since $(n, |t|) = 1$. Then $x \in C_P(P) = Z(P)$.

3. It is clear that (b) implies (a).² For the converse, assume that (a) holds, and that $C_G(P) = Z(P)$. Let $x, y \in P$ and $s, t \in K$. Then

²and we do not need the assumption $C_G(P) = Z(P)$ for that. . .

$$\begin{aligned}\varphi(xs \cdot yt) &= \varphi(x {}^s y \cdot st) = x {}^s y \varphi(st) \\ &= \varphi(xs) \varphi(yt) = x \varphi(s) y \varphi(t) = x {}^{\varphi(s)} y \varphi(s) \varphi(t).\end{aligned}$$

Hence, ${}^s y = {}^{\varphi(s)} y$ for any $y \in P$. In other words, $z(s) := s^{-1} \varphi(s) \in C_G(P)$, so z is a map from K to $Z(P)$.

Now $\varphi(s) = sz(s)$, so $z(st) = z(s) {}^t z(t)$ for any $s, t \in K$. In other words, the map z is a crossed morphism from K to $Z(P)$. Since K and $Z(P)$ have coprime order, it follows that there exists $w \in Z(P)$ such that $z(s) = w^s \cdot w^{-1}$, for any $s \in K$. In other words $\varphi(s) = s \cdot w^s \cdot w^{-1} = wsw^{-1} = i_w(s)$. Since $i_w(x) = x = \varphi(x)$ for any $x \in P$, it follows that $\varphi = i_w$. $\square$

Lemma 3.2. *Let G be a finite group, and P be a normal p -subgroup of G . Then:*

1. $N_{G \times G}(\Delta(P)) = \{(a, b) \in G \times G \mid ab^{-1} \in C_G(P)\}$.
2. Set $N = N_{G \times G}(\Delta(P))$ and $\bar{N} = N/\Delta(P)$. There is an isomorphism of (kG, kG) -bimodules

$$kG \cong \text{Ind}_N^{G \times G} \text{Inf}_N^N kC_G(P),$$

where the action of $\bar{N}$ on $kC_G(P)$ is given by $(a, b)\Delta(P) \cdot \gamma = a\gamma b^{-1}$.

Proof. 1. This is clear, since $(a, b) \in N$ if and only if $x^a = x^b$, that is, $x^{ab^{-1}} = x$, for all $x \in P$.

2. The group $\bar{N}$ permutes the set $C_G(P)$ transitively, and the stabilizer in $\bar{N}$ of $1 \in C_G(P)$ is the group $\{(a, a)\Delta(P) \mid a \in G\} = \Delta(G)/\Delta(P)$. So $kC_G(P) \cong \text{Ind}_{\Delta(G)/\Delta(P)}^{\bar{N}} k$, and

$$\begin{aligned}\text{Ind}_N^{G \times G} \text{Inf}_N^N kC_G(P) &\cong \text{Ind}_N^{G \times G} \text{Inf}_N^N \text{Ind}_{\Delta(G)/\Delta(P)}^{\bar{N}} k \\ &\cong \text{Ind}_N^{G \times G} \text{Ind}_{\Delta(G)}^N \text{Inf}_{\Delta(G)/\Delta(P)}^{\Delta(G)} k \\ &\cong \text{Ind}_{\Delta(G)}^{G \times G} k \cong kG\end{aligned}$$

as (kG, kG) -bimodules. $\square$

The next step is an important reduction allowed by the following stronger version of Dress induction theorem, due to Boltje and Külshammer [2, Theorem 3.3].

Theorem 3.3. *Let H be a finite group, and U be an indecomposable kH -module with vertex D and source Z . Then, in the Green ring of kH , we have*

$$[U] = \sum_{i=1}^n a_i [\text{Ind}_{H_i}^G V_i],$$

where, for $i = 1, \dots, n$:

- a_i is an integer.
- H_i is a subgroup of H such that $D_i := O_p(H_i) \leq D$ and H_i/D_i is an elementary p' -group.
- V_i is an indecomposable kH_i -module with vertex D_i and source $\text{Res}_{D_i}^{H_i} V_i$, which is a direct summand of $\text{Res}_{D_i}^D Z$.

Corollary 3.4. *Let G be a finite group. Then $\mathcal{E}_R(G) = 0$ unless $G \cong P \rtimes K$, where P is a p -group, and K is an elementary p' -group.*

Proof. We apply Theorem 3.3 to the case $H = G \times G$ and $U = kGb$, where b is a block idempotent of kG . Then U is a diagonal p -permutation (kG, kG) -bimodule with diagonal vertex $\Delta(D) = \Delta(D, \text{Id}, D)$, where $D \leq G$ is a defect group of b . We can conclude that $[U]$ is a linear combination with integer coefficients of (isomorphism classes of) induced bimodules $[\text{Ind}_{H_i}^{G \times G} V_i]$,

where H_i is a subgroup of $G \times G$ such that $D_i = O_p(H_i) \leq \Delta(D)$ and $H_i/O_p(H_i)$ is an elementary p' -group. Let G_i be the first projection of H_i on G . Then the bimodule $\text{Ind}_{H_i}^{G \times G} V_i$ factors as

$$\text{Ind}_{H_i}^{G \times G} V_i \cong \text{Ind}_{\Delta(G_i)}^{G \times G_i} \otimes_{kG_i} \text{Ind}_{H_i}^{G_i \times G} V_i,$$

where H_i on the right-hand side is viewed as a subgroup of $G_i \times G$. Now if $G_i < G$, then the image of kGb in $\mathcal{E}_R(G)$ is equal to 0. And if $G_i = G$, then G is a quotient of H_i , so $G/O_p(G)$ is an elementary p' -group. In other words, $G \cong P \rtimes K$, where P is a p -group, and K is an elementary p' -group.

Now $\mathcal{E}_R(G)$ is non-zero if and only if its identity element is non-zero, that is, if the image of the bimodule kG in $\mathcal{E}_R(G)$ is non-zero. Since kG is the direct sum of the bimodules kGb , when b runs through block idempotents of kG , there is at least one such idempotent b such that the image of kGb in $\mathcal{E}_R(G)$ is non-zero. Hence, $G \cong P \rtimes K$, where P is a p -group and K is an elementary p' -group. $\square$

Lemma 3.5. *Let $G = P \rtimes K$, where P is a finite p -group, and K is an elementary p' -group. Let H be a finite group, and U be a right essential indecomposable diagonal p -permutation (kG, kH) -bimodule. Then:*

1. *The essential algebra $\mathcal{E}_R(H)$ is non-zero. In particular, $H = Q \rtimes L$, where Q is a p -group and L is an elementary p' -group.*
2. *There exist an injective group homomorphism $\pi : Q \hookrightarrow P$, a subgroup T of $N_{K \times L}(\Delta_\pi(Q))$ with $p_2(T) = L$, and a simple kT -module W such that*

$$U \cong \text{Ind}_{\Delta_\pi(Q) \cdot T}^{G \times H} \text{Inf}_T^{\Delta_\pi(Q) \cdot T} W$$

as (kG, kH) -bimodules.

Proof. 1. If $\mathcal{E}_R(H) = 0$, the identity (kH, kH) -bimodule kH factors through groups of order strictly smaller than $|H|$, so the same holds for U (by right composition with kH). Hence, $\mathcal{E}_R(H) \neq 0$, and Assertion 1 follows from Corollary 3.4.

2. From Assertion 1 follows in particular that the group $G \times H$ is solvable, with a normal Sylow p -subgroup $P \times Q$. Then there is a diagonal p -subgroup $\Delta_\pi(S)$ of $G \times H$, where S is a p -subgroup of H (i.e., a subgroup of Q) and $\pi : S \hookrightarrow P$ is an injective group homomorphism, such that

$$U \cong \text{Ind}_N^{G \times H} \text{Inf}_N^N E,$$

where $N = N_{G \times H}(\Delta_\pi(S))$ and $\bar{N} = N/\Delta_\pi(S)$, and E is an indecomposable projective $k\bar{N}$ -module.

Now $\bar{N}$ itself also has a normal Sylow p -subgroup X , and there is a p' -subgroup $\bar{T}$ of $\bar{N}$ such that $\bar{N} = X \rtimes \bar{T}$. Moreover, $\bar{T}$ lifts to a p' -subgroup T of N , that we can assume contained in the p' -Hall subgroup $K \times L$ of $G \times H$, up to replacing $\Delta_\pi(S)$ by a conjugate subgroup. Finally, $E \cong \text{Ind}_{\bar{T}}^{\bar{N}} \bar{W}$, where $\bar{W}$ is a simple $k\bar{T}$ -module. Let W be the simple kT -module corresponding to $\bar{W}$ via the isomorphism $\bar{T} \cong T$. It follows that

$$\begin{aligned} U &\cong \text{Ind}_N^{G \times H} \text{Inf}_N^N \text{Ind}_{\bar{T}}^{\bar{N}} \bar{W} \\ &\cong \text{Ind}_N^{G \times H} \text{Ind}_{\Delta_\pi(S) \cdot T}^N \text{Inf}_T^{\Delta_\pi(S) \cdot T} W \\ &\cong \text{Ind}_{\Delta_\pi(S) \cdot T}^{G \times H} \text{Inf}_T^{\Delta_\pi(S) \cdot T} W. \end{aligned}$$

Since U is right essential, we have $p_2(\Delta_\pi(S) \cdot T) = H = Q \cdot L$. This forces $S = Q$, and $p_2(T) = L$, proving Assertion 2. $\square$

Theorem 3.6. *Let G be a finite group of the form $G = P \rtimes K$, where P is a p -group and K is a non-cyclic elementary p' -group. Then $|K|^2 \mathcal{E}_R(G) = 0$. In particular, if $|K|$ is invertible in $\mathbb{R}$, then $\mathcal{E}_R(G) = 0$.*

Proof. Let M be an essential indecomposable diagonal p -permutation (kG, kG) -bimodule. By Lemma 3.5, applied to $H = G$ and $U = M$, we know that

$$M \cong \text{Ind}_{\Delta_\pi(P) \cdot T}^{G \times G} \text{Inf}_T^{\Delta_\pi(P) \cdot T} W,$$

for some $\pi \in \text{Aut}(P)$, some subgroup T of $N_{K \times K}(\Delta_\pi(P))$ with $p_2(T) = K$, and some simple kT -module W .

Now $T \leq K \times K$, so T is a p' -group, and $|T|$ divides $|K|^2$. Moreover, by Artin's induction theorem, in $R_k(T) \cong R_{\mathbb{C}}(T)$, we have an equality of the form

$$|T|W = \sum_{i=1}^n n_i \text{Ind}_{C_i}^T k_{\lambda_i},$$

where, for $1 \leq i \leq n$, C_i is a cyclic subgroup of T , n_i is an integer, and k_{λ_i} is a one-dimensional kC_i -module. Hence, in $RT^\Delta(G, G)$, we have that

$$\begin{aligned} |T|M &= \sum_{i=1}^n n_i \text{Ind}_{\Delta_\pi(P) \cdot T}^{G \times G} \text{Inf}_T^{\Delta_\pi(P) \cdot T} \text{Ind}_{C_i}^T k_{\lambda_i} \\ &= \sum_{i=1}^n n_i \text{Ind}_{\Delta_\pi(P) \cdot T}^{G \times G} \text{Ind}_{\Delta_\pi(P) \cdot C_i}^{\Delta_\pi(P) \cdot T} \text{Inf}_{C_i}^{\Delta_\pi(P) \cdot C_i} k_{\lambda_i} \\ &= \sum_{i=1}^n n_i \text{Ind}_{\Delta_\pi(P) \cdot C_i}^{G \times G} \text{Inf}_{C_i}^{\Delta_\pi(P) \cdot C_i} k_{\lambda_i}. \end{aligned}$$

The image of $|K|^2 M$ in $\mathcal{E}_R(G)$ is equal to the image of

$$\frac{|K|^2}{|T|} \sum_{i=1}^n n_i \text{Ind}_{\Delta_\pi(P) \cdot C_i}^{G \times G} \text{Inf}_{C_i}^{\Delta_\pi(P) \cdot C_i} k_{\lambda_i},$$

which is equal to zero unless there exists $i \in \{1, \dots, n\}$ such that

$$p_1(\Delta_\pi(P) \cdot C_i) = p_2(\Delta_\pi(P) \cdot C_i) = G = P \cdot K.$$

This implies that K is a quotient of C_i . Hence, K is cyclic, which completes the proof. $\square$

Theorem 3.7. Let $G = P \rtimes K$, where P is a p -group and K is a cyclic p' -group. If $C_K(P) \neq 1$, then $\mathcal{E}_R(G) = 0$.

*Proof.*³ Since G has a normal Sylow p -subgroup, all the blocks of G have defect group P . Moreover, if b is a block idempotent of G , then b is a linear combination of p -regular elements of $C_G(P) = Z(P) \times C_K(P)$, so $b \in kC_K(P)$.

Since $C_K(P) \leq Z(G)$, it means that the block idempotents of kG are exactly the primitive idempotents of the (split semisimple commutative) algebra $kC_K(P)$. Let e be one of them, and $k_\lambda = kC_K(P)e$ be the corresponding (one-dimensional) simple $kC_K(P)$ -module, where $\lambda : C_K(P) \rightarrow k^\times$ is the associated group homomorphism.

Let $\varpi : G \rightarrow K$ denote the projection map. Set $N = N_{G \times G}(\Delta(P))$ and $\overline{N} = N/\Delta(P)$. There is a short exact sequence

$$1 \longrightarrow Z(P) \xrightarrow{i} \overline{N} \xrightarrow{s} \widetilde{K} \longrightarrow 1,$$

³This proof is a slightly simplified and generalized version of the proof given by Ducellier in Proposition 4.1.2 of his thesis [10] in the case $R = \mathbb{C}$.

where

- $\widetilde{K} = \{(a, b) \in K \times K \mid a^{-1}b \in C_K(P)\}$.
- i is the map sending $z \in Z(P)$ to $(z, 1)\Delta(P) \in \overline{N}$.
- s is the map sending $(a, b)\Delta(P)$ to $(\varpi(a), \varpi(b))$.

So $\overline{N} \cong Z(P) \rtimes \widetilde{K}$, with the explicit embedding $\widetilde{K} \rightarrow \overline{N}$ sending $(a, b) \in \widetilde{K}$ to $(a, b)\Delta(P) \in \overline{N}$. We consider $\widetilde{K}$ as a subgroup of $\overline{N}$ via this embedding.

The Brauer quotient of the (kG, kG) -bimodule kG at $\Delta(P)$ is isomorphic to $kC_G(P)$, so $kGe[\Delta(P)] \cong kC_G(P)\text{Br}_P(e) = kC_G(P)e = kZ(P) \otimes_k k_\lambda$, since $\text{Br}_P(e) = e$ as $e \in kC_K(P)$. It follows that

$$kGe \cong \text{Ind}_N^{G \times G} \text{Inf}_N^N kC_G(P)e \cong \text{Ind}_N^{G \times G} \text{Inf}_N^N (kZ(P) \otimes_k k_\lambda),$$

where $\overline{N} \cong Z(P) \rtimes \widetilde{K}$ acts on $kZ(P) \otimes_k k_\lambda$ by

$$(a, b)\Delta(P) \cdot (z \otimes 1) = (ab^{-1})_p z \otimes (ab^{-1})_{p'} \cdot 1 = \lambda((ab^{-1})_{p'}) (ab^{-1})_p z \otimes 1$$

for $(a, b) \in N$ and $z \in Z(P)$, where $(ab^{-1})_p \in Z(P)$ and $(ab^{-1})_{p'} \in C_K(P)$ are the p -part and p' -part of $ab^{-1} \in C_G(P) = Z(P) \times C_K(P)$, respectively. Then $kZ(P) \otimes_k k_\lambda$ is isomorphic to $\text{Ind}_{\widetilde{K}}^{\overline{N}} k_{\widetilde{\lambda}}$, where $\widetilde{\lambda} : \widetilde{K} \rightarrow k^\times$ sends $(a, b) \in \widetilde{K}$ to $\lambda(ab^{-1}) \in k^\times$. So

$$\begin{aligned} kGe &\cong \text{Ind}_N^{G \times G} \text{Inf}_N^N \text{Ind}_{\widetilde{K}}^{\overline{N}} k_{\widetilde{\lambda}} \\ &\cong \text{Ind}_N^{G \times G} \text{Ind}_{\Delta(P)\widetilde{K}}^N \text{Inf}_{\widetilde{K}}^{\Delta(P)\widetilde{K}} k_{\widetilde{\lambda}} \\ &\cong \text{Ind}_{\Delta(P)\widetilde{K}}^{G \times G} \text{Inf}_{\widetilde{K}}^{\Delta(P)\widetilde{K}} k_{\widetilde{\lambda}}. \end{aligned} \tag{3.7.1}$$

Now we set $\overline{G} = G/C_K(P)$. We denote by $g \mapsto \overline{g}$ the projection map, and by $\delta : G \rightarrow G \times \overline{G}$ the map $g \mapsto (g, \overline{g})$. We will show that the (kG, kG) -bimodule kGe factors through the group $\overline{G}$, that is, there is a diagonal p -permutation $(kG, k\overline{G})$ -bimodule U and a diagonal p -permutation $(k\overline{G}, kG)$ -bimodule V such that $kGe \cong U \otimes_{k\overline{G}} V$.

The group P embeds in $G \times \overline{G}$ via δ . Its image $\delta(P)$ is a diagonal subgroup of $G \times \overline{G}$, and its normalizer is

$$N_\delta := N_{G \times \overline{G}}(\delta(P)) = \{(a, \overline{b}) \in G \times \overline{G} \mid \overline{x^a} = \overline{x^b}, \forall x \in P\}.$$

In other words, $(a, \overline{b}) \in N_\delta$ if and only if $\overline{x^{ab^{-1}}} = \overline{x}$ for all $x \in P$, or equivalently, if the commutator $[x, ab^{-1}]$ is in $C_K(P)$. But since $P \trianglelefteq G$, we have that $[P, ab^{-1}] \subseteq P$. Hence, $(a, \overline{b})$ normalizes $\delta(P)$ if and only if $[P, ab^{-1}] \subseteq P \cap C_K(P) = 1$, that is, $ab^{-1} \in C_G(P)$. Thus,

$$N_\delta = \{(a, \overline{b}) \in G \times \overline{G} \mid ab^{-1} \in Z(P) \times C_K(P)\}.$$

Recall that $\varpi : G \rightarrow K$ denotes the projection map. We have a surjective group homomorphism $\sigma : N_\delta \rightarrow K$ sending $(a, \overline{b})$ to $\varpi(a)$. It induces a surjective group homomorphism

$$\overline{\sigma} : \overline{N}_\delta := N_\delta / \delta(P) \rightarrow K$$

sending $(a, \overline{b})\delta(P)$ to $\varpi(a)$. The kernel of this morphism consists of the elements $(a, \overline{b})\delta(P)$ such that $a \in P$ and $ab^{-1} \in Z(P) \times C_K(P)$. Since

$$(a, \overline{b})\delta(P) = (1, \overline{ba^{-1}})(a, \overline{a})\delta(P) = (1, \overline{ba^{-1}})\delta(P),$$

and since $\overline{ba^{-1}} \in C_G(P)/C_K(P) = Z(P)$, we get a short exact sequence

$$1 \longrightarrow Z(P) \xrightarrow{\iota} \overline{N}_\delta \xrightarrow{\overline{\sigma}} K \longrightarrow 1,$$

where $\iota(z) = (1, \overline{z})\delta(P)$ for $z \in Z(P)$. This sequence is split, via the morphism $a \in K \mapsto (a, \overline{a})\delta(P)$, for $a \in K$, so $\overline{N}_\delta \cong Z(P) \rtimes K$.

Now since K is cyclic, we can extend $\lambda : C_K(P) \rightarrow k^\times$ to a group homomorphism $\beta : K \rightarrow k^\times$. This gives a one-dimensional kK -module k_β , that we can induce to $\overline{N}_\delta = Z(P) \rtimes K$. We get a projective $k\overline{N}_\delta$ -module, and we set

$$U := \text{Ind}_{N_\delta}^{G \times \overline{G}} \text{Inf}_{N_\delta}^{N_\delta} \text{Ind}_K^{\overline{N}_\delta} k_\beta.$$

This is a diagonal p -permutation $(kG, k\overline{G})$ -bimodule, and

$$U \cong \text{Ind}_{N_\delta}^{G \times \overline{G}} \text{Ind}_{\delta(P)K}^{N_\delta} \text{Inf}_K^{\delta(P)K} k_\beta \cong \text{Ind}_{\delta(P)K}^{G \times \overline{G}} \text{Inf}_K^{\delta(P)K} k_\beta,$$

where K is viewed as a subgroup of N_δ via the map $a \in K \mapsto (a, \overline{a}) \in N_\delta$. We observe that $\delta(P)K$ is equal to $\delta(G)$, so

$$U \cong \text{Ind}_{\delta(G)}^{G \times \overline{G}} \text{Inf}_K^{\delta(G)} k_\beta.$$

We define similarly a $(k\overline{G}, kG)$ -bimodule V by

$$V := \text{Ind}_{\delta^o(G)}^{\overline{G} \times G} \text{Inf}_{K^o}^{\delta^o(G)} k_{\beta^{-1}},$$

where $\delta^o : G \rightarrow \overline{G} \times G$ sends x to $(\overline{x}, x)$, and $K^o = \{(\overline{a}, a) \mid a \in K\}$.

Now we compute the tensor product $U \otimes_{k\overline{G}} V$ using Theorem 1.1 of [3]. Since $p_2(\delta(G)) = \overline{G}$, there is a single double coset $p_2(\delta(G)) \backslash \overline{G} / p_1(\delta^o(G))$. Moreover, $k_2(\delta(G)) = 1$, so we have

$$U \otimes_{k\overline{G}} V \cong \text{Ind}_{\delta(G) * \delta^o(G)}^{G \times G} (\text{Inf}_K^{\delta(G)} k_\beta \otimes_k \text{Inf}_{K^o}^{\delta^o(G)} k_{\beta^{-1}}), \quad (3.7.2)$$

where

$$\begin{aligned} \delta(G) * \delta^o(G) &= \{(a, b) \in G \times G \mid \exists \overline{c} \in \overline{G}, (a, \overline{c}) \in \delta(G) \text{ and } (\overline{c}, b) \in \delta^o(G)\} \\ &= \{(a, b) \in G \times G \mid \overline{a} = \overline{b}\} \\ &= \{(a, b) \in G \times G \mid ab^{-1} \in C_K(P)\}. \end{aligned}$$

Now if $a = xu$ and $b = yv$, with $x, y \in P$ and $u, v \in K$, we have

$$ab^{-1} = xuv^{-1}y^{-1} = x \cdot uv^{-1}(y^{-1}) \cdot uv^{-1},$$

so $ab^{-1} \in C_K(P)$ if and only if $uv^{-1} \in C_K(P)$ and $x = y$, that is, $(u, v) \in \widetilde{K}$ and $(x, y) \in \Delta(P)$. It follows that $\delta(G) * \delta^o(G) = \Delta(P)\widetilde{K}$.

The action of $(a, b) \in \delta(G) * \delta^o(G)$ on the tensor product

$$T := \text{Inf}_K^{\delta(G)} k_\beta \otimes_k \text{Inf}_{K^o}^{\delta^o(G)} k_{\beta^{-1}}$$

is obtained as follows: Let $\bar{c} \in \bar{G}$ such that $(a, \bar{c}) \in \delta(G)$ and $(\bar{c}, b) \in \delta^o(G)$, that is, $\bar{c} = \bar{a} = \bar{b}$. Then for $v \in \text{Inf}_K^{\delta(G)} k_\beta$ and $w \in \text{Inf}_{K^o}^{\delta^o(G)} k_{\beta^{-1}}$, we have $(a, b) \cdot (v \otimes w) = (a, \bar{c}) \cdot v \otimes (\bar{c}, b) \cdot w$. Here, T is one-dimensional, with basis $1 \otimes 1$, and

$$\begin{aligned} (a, b) \cdot (1 \otimes 1) &= \beta(\varpi(a))\beta(\varpi(b))^{-1}(1 \otimes 1) \\ &= \beta(\varpi(ab^{-1}))(1 \otimes 1) \\ &= \lambda(\varpi(ab^{-1}))(1 \otimes 1), \end{aligned}$$

since the restriction of β to $C_K(P)$ is equal to λ .

Now for $(a, b) \in \delta(G) * \delta^o(G) = \Delta(P)\tilde{K}$, we have $\lambda(\varpi(ab^{-1})) = \tilde{\lambda}(\rho(a, b))$, where $\rho : \Delta(P)\tilde{K} \rightarrow \tilde{K}$ is the projection map. Then $T \cong \text{Inf}_{\tilde{K}}^{\Delta(P)\tilde{K}} k_{\tilde{\lambda}}$, and by (3.7.1) and (3.7.2), we get that

$$U \otimes_{k\bar{G}} V \cong \text{Ind}_{\delta(G)*\delta^o(G)}^{G \times G} T \cong \text{Ind}_{\Delta(P)\tilde{K}}^{G \times G} \text{Inf}_{\tilde{K}}^{\Delta(P)\tilde{K}} k_{\tilde{\lambda}} \cong kGe,$$

as was to be shown.

So if $C_K(P) \neq 1$, all the bimodules kGe are mapped to 0 in $\mathcal{E}_R(G)$, and the image of their direct sum kG is also 0. Now as its identity element is equal to 0, the algebra $\mathcal{E}_R(G)$ itself is equal to 0. $\square$

Corollary 3.8. *Let $G = P \rtimes K$, where K is an elementary p' -group of order invertible in R . If $\mathcal{E}_R(G) \neq 0$, then K is cyclic and acts faithfully on P .*

Proof. Indeed if $\mathcal{E}_R(G) \neq 0$, we know by Theorem 3.6 that K is cyclic, and by Theorem 3.7, that K acts faithfully on P . $\square$

In other words, $G = P \rtimes \langle u \rangle$, where (P, u) is a D^Δ -pair, as defined hereafter.⁴

Definition 3.9. A D^Δ -pair is a pair (P, u) of a finite p -group P and a p' -automorphism u of P .

We recall the following notation [7, Notation 6.8].

Notation 3.10. For a D^Δ -pair (L, u) , we denote by $\text{Aut}(L, u)$ the group of automorphisms of the semidirect product $L\langle u \rangle = L \rtimes \langle u \rangle$ which send u to a conjugate of u , and by $\text{Out}(L, u)$ the quotient $\text{Aut}(L, u)/\text{Inn}(L\langle u \rangle)$ of this group by the group of inner automorphisms of $L\langle u \rangle$.

4. Generators and relations for $\mathcal{E}_R(G)$

Notation 4.1. For a finite group G , denote by $G^\natural$ the group $\text{Hom}(G, k^\times)$. For $\lambda \in G^\natural$, let k_λ denote the corresponding one-dimensional kG -module. For $\gamma \in \text{Aut}(G)$, denote⁵ by $kG_{\gamma, \lambda}$ the (kG, kG) -bimodule equal to kG as a vector space, with action given by

$$\forall (x, y, z) \in G^3, \quad x \cdot z \cdot y = \lambda(y)^{-1} xz\gamma(y).$$

Lemma 4.2. *Let G be a finite group.*

1. *Let $\gamma \in \text{Aut}(G)$ and $\lambda \in G^\natural$. Then $kG_{\gamma, \lambda}$ is a diagonal p -permutation bimodule, and there is an isomorphism of (kG, kG) -bimodules*

⁴ D^Δ -pairs were first introduced in the slightly different Definition 4.4 of [6]. The subsequent Lemma 4.5 there showed that the two definitions are equivalent.

⁵This generalizes Definition 4.1.3 of [10], up to replacing λ with λ^{-1} , which is more convenient in our setting.

$$kG_{\gamma,\lambda} \cong \text{Ind}_{\Delta_{\gamma}(G)}^{G \times G} k_{\lambda},$$

where k_{λ} is viewed as a $k\Delta_{\gamma}(G)$ -module via

$$(\gamma(g), g) \cdot m = \lambda(g)m \quad \text{for all } g \in G, m \in k_{\lambda}.$$

2. ⁶ Let $\delta \in \text{Aut}(G)$ and $\mu \in G^{\natural}$. Then there is an isomorphism of (kG, kG) -bimodules

$$kG_{\gamma,\lambda} \otimes_{kG} kG_{\delta,\mu} \cong kG_{\gamma \circ \delta, (\lambda \circ \delta) \times \mu}.$$

3. If kG has only one block, then $kG_{\gamma,\lambda}$ is an indecomposable (kG, kG) -bimodule, for any $\gamma \in \text{Aut}(G)$ and any $\lambda \in G^{\natural}$.

Proof. 1. Let S be a Sylow p -subgroup of G . Then $\lambda(x) = 1$ for any $x \in S$, so the restriction of $kG_{\gamma,\lambda}$ to the Sylow p -subgroup $S \times S$ of $G \times G$ is the permutation bimodule kG , with action $x \cdot z \cdot y = xz\gamma(y)$, for $x, y \in S$ and $z \in G$. Moreover, this action is free on both sides, so $kG_{\gamma,\lambda}$ is a diagonal p -permutation bimodule. Finally, the map $g \in G \mapsto (g, 1)\Delta_{\gamma}(G)$ is a bijection from G to $(G \times G)/\Delta_{\gamma}(G)$, and using this bijection, it is easy to check that $kG_{\gamma,\lambda} \cong \text{Ind}_{\Delta_{\gamma}(G)}^{G \times G} k_{\lambda}$.

2. The map $(g \otimes g') \mapsto \lambda(g')^{-1}g\gamma(g')$, for $g, g' \in G$, from $kG_{\gamma,\lambda} \otimes_{kG} kG_{\delta,\mu}$ to $kG_{\gamma \circ \delta, (\lambda \circ \delta) \times \mu}$ induces a well-defined isomorphism of (kG, kG) -bimodules.

3. It is clear that for any (kG, kG) -bimodule M , any $\delta \in \text{Aut}(G)$, and any $\mu \in G^{\natural}$, the k -vector space $M \otimes_{kG} kG_{\delta,\mu}$ is isomorphic to M . So if $kG_{\gamma,\lambda}$ splits as a direct sum of non-zero (kG, kG) -bimodules M and M' , the tensor product $kG_{\gamma,\lambda} \otimes_{kG} kG_{\delta,\mu}$ splits as the direct sum of (kG, kG) -bimodules $M \otimes_{kG} kG_{\delta,\mu}$ and $M' \otimes_{kG} kG_{\delta,\mu}$, none of which is equal to zero. Then $kG_{\gamma \circ \delta, (\lambda \circ \delta) \times \mu}$ splits as a direct sum of non-zero bimodules. Taking $\delta = \gamma^{-1}$ and $\mu = (\lambda \circ \gamma^{-1})^{-1}$, we get that the (kG, kG) -bimodule $kG_{\text{Id},1} \cong kG$ splits non-trivially. So kG is not indecomposable, that is, kG has more than one block. $\square$

In the rest of this section, in view of Corollary 3.8, we assume the following.

Hypothesis 4.3. The group G is of the form $P \rtimes K$, where P is a p -group and K is a cyclic p' -group of order invertible in R , acting faithfully on P .

We want to find the structure of the algebra $\mathcal{E}_R(G)$. First, we look for generators of $\mathcal{E}_R(G)$ as an R -module.

Lemma 4.4. Assume that 4.3 holds. Then:

1. kG has only one block. So for $\gamma \in \text{Aut}(G)$ and $\lambda \in G^{\natural}$, the bimodule $kG_{\gamma,\lambda}$ is indecomposable, with vertex $\Delta_{\gamma}(P)$.
2. Let $\gamma, \delta \in \text{Aut}(G)$ and $\lambda, \mu \in G^{\natural}$. Then the bimodules $kG_{\gamma,\lambda}$ and $kG_{\delta,\mu}$ are isomorphic if and only if $\lambda = \mu$ and $\delta \circ \gamma^{-1}$ is an inner automorphism of G .
3. Conversely, if M is an indecomposable diagonal p -permutation bimodule with vertex $\Delta_{\gamma}(P)$, for $\gamma \in \text{Aut}(G)$, then there exists $\lambda \in G^{\natural}$ such that $M \cong kG_{\gamma,\lambda}$.
4. In particular, if M is an essential indecomposable diagonal p -permutation (kG, kG) -bimodule, there exist $\gamma \in \text{Aut}(G)$ and $\lambda \in G^{\natural}$ such that $M \cong kG_{\gamma,\lambda}$ as (kG, kG) -bimodule.

Proof. 1. The p -subgroup P of G is normal, and $C_G(P) = Z(P) \times C_K(P) = Z(P) \leq P$ since $C_K(P) = 1$ as K acts faithfully on P . So kG has only one block [1, Proposition 6.2.2]. Then all the bimodules $kG_{\gamma,\lambda}$ are indecomposable by Lemma 4.2.

Let $\gamma \in \text{Aut}(G)$. Set $N := N_{G \times G}(\Delta_{\gamma}(P))$ and $\overline{N} = N/\Delta_{\gamma}(P)$. Then

$$N = \{(a, b) \in G \times G \mid a^{-1}\gamma(b) \in C_G(P) = Z(P)\},$$

⁶Up to the previous change of notation, this is Proposition 4.1.4 in [10].

so the second projection p_2 induces a short exact sequence

$$1 \longrightarrow Z(P) \longrightarrow \overline{N} \longrightarrow K \longrightarrow 1,$$

which is split (by the map $x \in K \mapsto (\gamma(x), x)\Delta_\gamma(P)$). Now the indecomposable projective $k\overline{N}$ -modules are the modules $\text{Ind}_K^{\overline{N}} k_\lambda$, for $\lambda \in K^\natural$. Moreover,

$$\begin{aligned} \text{Ind}_N^{G \times G} \text{Inf}_N^N \text{Ind}_K^{\overline{N}} k_\lambda &\cong \text{Ind}_N^{G \times G} \text{Ind}_{\Delta_\gamma(P) \cdot \Delta_\gamma(K)}^N k_\lambda \\ &\cong \text{Ind}_{\Delta_\gamma(G)}^{G \times G} k_\lambda \cong kG_{\gamma, \lambda} \end{aligned}$$

by Lemma 4.2. This gives another proof that $kG_{\gamma, \lambda}$ is indecomposable and also shows that it has vertex $\Delta_\gamma(P)$ and Brauer quotient

$$kG_{\gamma, \lambda}[\Delta_\gamma(P)] \cong \text{Ind}_K^{\overline{N}} k_\lambda \cong kZ(P) \otimes_k k_\lambda. \quad (4.4.1)$$

2. First, if $\lambda = \mu$ and $\delta = i_x \circ \gamma$, where i_x is conjugation by $x \in G$, then one checks easily that the map $g \in G \mapsto gx \in G$ induces a bimodule isomorphism $kG_{\gamma, \lambda} \cong kG_{\delta, \mu}$.

For the converse, let $\gamma' = \gamma^{-1}$ and $\lambda' = (\lambda \circ \gamma')^{-1}$. Then by Lemma 4.2, we have an isomorphism of (kG, kG) -bimodules

$$kG_{\gamma, \lambda} \otimes_{kG} kG_{\gamma', \lambda'} \cong kG_{\gamma \circ \gamma', (\lambda \circ \gamma') \times \lambda'} = kG_{\text{Id}, 1} = kG.$$

So if $kG_{\delta, \mu} \cong kG_{\gamma, \lambda}$, we have an isomorphism of (kG, kG) -bimodules

$$kG_{\delta, \mu} \otimes_{kG} kG_{\gamma', \lambda'} \cong kG,$$

that is,

$$kG_{\delta \circ \gamma', (\mu \circ \gamma') \times \lambda'} \cong kG_{\text{Id}, 1}.$$

So if we know that an isomorphism of bimodules $kG_{\theta, \rho} \cong kG_{\text{Id}, 1}$, where $\theta \in \text{Aut}(G)$ and $\rho \in G^\natural$ implies that θ is inner and $\rho = 1$, we are done, since we can conclude that $\delta \circ \gamma' = \delta \circ \gamma^{-1}$ is inner, and that $(\mu \circ \gamma') \times \lambda' = 1$, that is, $\mu = \lambda$. In other words, we can assume $\gamma = \text{Id}$ and $\lambda = 1$, and that $kG_{\delta, \mu} \cong kG$.

Now if $kG_{\delta, \mu}$ is isomorphic to kG , then its vertex $\Delta_\delta(P)$ is contained in—hence equal to— $\Delta(P)$, up to conjugation in $G \times G$. It means that the restriction of δ to P is equal to the conjugation by some element of G . Up to composing δ with some inner automorphism of G , we can assume that this restriction is equal to the identity, and then δ is equal to the conjugation by some element of $Z(P)$, by Lemma 3.1. This shows that δ is inner.

Then we have a bimodule isomorphism $kG_{\delta, \lambda} \cong kG_{\text{Id}, \lambda}$ by the first remark in the proof of Assertion 2. In other words, we can assume $\delta = \text{Id}$ and $kG_{\text{Id}, \lambda} \cong kG_{\text{Id}, 1}$. Then the Brauer quotients at $\Delta(P)$ of these bimodules are isomorphic. Hence, $kZ(P) \otimes k_\lambda \cong kZ(P) \otimes k \cong kZ(P)$. Now the fixed points of $Z(P)$ on $kZ(P) \otimes k_\lambda$ form a kK -module isomorphic to k_λ , so $k_\lambda \cong k$ as K -module, hence $\lambda = 1$, as was to be shown.

3. Suppose conversely that M is an indecomposable diagonal p -permutation (kG, kG) -bimodule with vertex $\Delta_\gamma(P)$, where $\gamma \in \text{Aut}(G)$. Then $M[\Delta_\gamma(P)]$ is an indecomposable projective $k\overline{N}$ -module of the form $\text{Ind}_K^{\overline{N}} k_\lambda$ for some $\lambda \in K^\natural$, and then

$$M \cong \text{Ind}_N^{G \times G} \text{Inf}_N^N \text{Ind}_K^{\overline{N}} k_\lambda \cong kG_{\gamma, \lambda}.$$

4. As in the proof of Theorem 3.6, we apply Lemma 3.5, in the case $H = G$ and $U = M$. We know that

$$M \cong \text{Ind}_{\Delta_\pi(P) \cdot T}^{G \times G} \text{Inf}_T^{\Delta_\varphi(P) \cdot T} W,$$

for some $\pi \in \text{Aut}(P)$, some subgroup T of $N_{K \times K}(\Delta_\pi(P))$ with $p_2(T) = K$, and some simple kT -module W .

Moreover, $k_1(T) \leq k_1(N_{G \times G}(\Delta_\pi(P))) = C_G(P) = Z(P)$. So $k_1(T) = 1$ since T is a p' -group. Then $p_2 : T \rightarrow K$ is an isomorphism, with inverse θ , and $T = \Delta_\theta(K)$. Then T is cyclic, and $W \cong k_\lambda$ for some $\lambda \in T^\natural$.

Now T normalizes $\Delta_\pi(P)$ if and only if $\theta^{(x)}\pi(y) = \pi(x)y$ for any $x \in K$ and any $y \in P$. Then the map $\gamma : y \cdot x \mapsto \pi(y) \cdot \theta(x)$, where $y \in P$ and $x \in K$, is an automorphism of G , such that $\gamma(K) = K$ and $\Delta_\pi(P) \cdot L = \Delta_\gamma(G)$. Then

$$M \cong \text{Ind}_{\Delta_\gamma(G)}^{G \times G} k_\mu,$$

where $\mu = \lambda \circ \varpi \in G^\natural \cong \Delta_\gamma(G)^\natural$. Now $M \cong kG_{\gamma, \mu}$ by Lemma 4.2. $\square$

It follows that $\mathcal{E}_R(G)$ is linearly generated by the images of the (kG, kG) -bimodules $kG_{\gamma, \lambda}$, for $\gamma \in \text{Aut}(G)$ and $\lambda \in G^\natural$. By Lemma 4.4, we can take γ in a set of representatives of elements of $\text{Out}(G)$. We want to describe the linear relations between these generators. In other words, we want to find equalities of the form

$$\sum_{\substack{\gamma \in \text{Out}(G) \\ \lambda \in G^\natural}} r_{\gamma, \lambda} kG_{\gamma, \lambda} = \sum_{i=1}^n s_i U_i \otimes_{kH_i} V_i, \quad (4.4.2)$$

in $RT^\Delta(G, G)$, where $r_{\gamma, \lambda} \in R$, $n \in \mathbb{N}$, and for $1 \leq i \leq n$, H_i is a finite group of order smaller than the order of G , U_i is a diagonal p -permutation (kG, kH_i) -bimodule, V_i is a diagonal p -permutation (kH_i, kG) -bimodule, and $s_i \in R$. We can assume, moreover, that for $1 \leq i \leq n$, the essential algebra $\mathcal{E}_R(H_i)$ is non-zero: Indeed otherwise, the identity bimodule $kH_i \in RT^\Delta(H_i, H_i)$ is a linear combination with coefficients in R of elements of $RT^\Delta(H_i, X) \otimes_{kX} RT^\Delta(X, H_i)$, for $|X| < |H_i|$, and we can replace H_i by smaller groups in (4.4.2).

Hence, by Corollary 3.4, we can assume that for $1 \leq i \leq n$, we have $H_i = Q_i \rtimes L_i$, where Q_i is a p -group, and L_i is an elementary p' -group. We can also assume that U_i and V_i are indecomposable, and that U_i is right essential and V_i is left essential.

Lemma 4.5. *Assume that Hypothesis 4.3 holds. Let H be a finite group, and U be a right essential indecomposable diagonal p -permutation (kG, kH) -bimodule. Then the vertices of U have order at most $|P|$. If U has vertex of order $|P|$, then there exists an injective group homomorphism $\sigma : H \hookrightarrow G$ such that $P \leq \sigma(H)$ and $\lambda \in H^\natural = \Delta_\sigma(H)^\natural$ such that*

$$U \cong \text{Ind}_{\Delta_\sigma(H)}^{G \times H} k_\lambda.$$

Proof. We know from Lemma 3.5 that $H = Q \rtimes L$, where Q is a p -group with an embedding $\pi : Q \hookrightarrow P$, and L is an elementary p' -group. Moreover, there is a subgroup T of $N_{K \times L}(\Delta_\pi(Q))$ with $p_2(T) = L$, and a simple kT -module W such that

$$U \cong \text{Ind}_{\Delta_\pi(Q) \cdot T}^{G \times H} \text{Inf}_T^{\Delta_\pi(Q) \cdot T} W.$$

Then a vertex of U is contained in $\Delta_\pi(Q) \cdot T$ up to conjugation, so it has order at most $|Q| \leq |P|$. And if it has order $|P|$, the embedding $\pi : Q \hookrightarrow P$ is an isomorphism. Moreover, $k_1(T) \leq k_1(N_{G \times H}(\Delta_\pi(P))) = C_G(P)$, so $k_1(T) \leq C_K(P) = 1$ since K acts faithfully on P . Then the projection map $p_2 : T \rightarrow L$ is an

isomorphism, and then $T = \Delta_\tau(L)$, for some injective group homomorphism $\tau : L \hookrightarrow K$. In particular, L and T are cyclic.

Moreover, since $T = \Delta_\tau(L) \leq N_{G \times H}(\Delta_\pi(P))$, we have ${}^{\tau(l)}\pi(x) = \pi({}^l x)$ for any $l \in L$ and $x \in Q$. Then the map $\sigma : H = Q \cdot L \rightarrow G$ sending $x \cdot l$, for $x \in Q$ and $l \in L$, to $\pi(x) \cdot \tau(l)$, is an injective group homomorphism, such that $P \leq \gamma(H) \leq G$. Moreover, $\Delta_\pi(Q) \cdot T = \Delta_\sigma(H)$. Finally, T is cyclic, so there is $\lambda \in T^\natural = L^\natural \cong H^\natural$ such that $W = k_\lambda$, and $U \cong \text{Ind}_{\Delta_\sigma(H)}^{G \times H} k_\lambda$, completing the proof. $\square$

Theorem 4.6. *Assume that Hypothesis 4.3 holds. Let H be a finite group, let U (resp. V) be a right (resp. left) essential indecomposable diagonal p -permutation (kG, kH) -bimodule (resp. (kH, kG) -bimodule).*

1. *If $U \otimes_{kH} V$ has an indecomposable direct summand with vertex of order $|P|$, then there is a subgroup $I \cong H$ of G , containing P , an automorphism ψ of I , and $\zeta \in I^\natural = \Delta_\psi(I)^\natural$ such that*

$$U \otimes_{kH} V \cong \text{Ind}_{\Delta_\psi(I)}^{G \times G} k_\zeta.$$

2. *If $U \otimes_{kH} V$ admits an essential indecomposable summand, then this summand has vertex $\Delta_\gamma(P)$ for some $\gamma \in \text{Aut}(G)$, and there exists $J \leq K$ and $\theta \in J^\natural$ such that $P \cdot J \cong H$ and*

$$U \otimes_{kH} V \cong \bigoplus_{\alpha \in \mathcal{I}_\theta} kG_{\gamma, \alpha}$$

as (kG, kG) -bimodules, where $\mathcal{I}_\theta = \{\alpha \in G^\natural = K^\natural \mid \text{Res}_J^K \alpha = \theta\}$.

Proof. 1. Let X be a vertex of U . Then X is a diagonal subgroup of $P \times H$, so $|X| \leq |P|$, and U is a direct summand of $\text{Ind}_X^{G \times H} k$. Similarly, if Y is a vertex of V , then Y is a diagonal subgroup of $H \times P$, hence $|Y| \leq |P|$, and V is a direct summand of $\text{Ind}_Y^{H \times G} k$. It follows that $U \otimes_{kH} V$ is a direct summand of

$$\bigoplus_{h \in p_2(X) \backslash H / p_1(Y)} \text{Ind}_{X * {}^{(h,1)}Y}^{G \times G} k.$$

So the vertices of the indecomposable summands of $U \otimes_{kH} V$ are contained (up to conjugation) in some group $X * {}^{(h,1)}Y$, which has order at most $\min(|X|, |Y|)$. If one of them has order $|P|$, then $|P| \leq |X| \leq |P|$, hence $|X| = |P|$, and $|P| \leq |Y| \leq |P|$, so $|Y| = |P|$.

By Lemma 4.5, it follows that there is an embedding $\sigma : H \hookrightarrow G$ with $P \leq \sigma(H) \leq G$, and $\lambda \in H^\natural$, such that $U \cong \text{Ind}_{\Delta_\sigma(H)}^{G \times H} k_\lambda$. Similarly, swapping G and H , there is an embedding $\tau : H \rightarrow G$ with $P \leq \tau(H) \leq G$, and $\mu \in H^\natural$, such that $V \cong \text{Ind}_{\Delta_\tau(H)}^{H \times G} k_\mu$, where $\Delta_\tau^\circ(H) = \{(h, \tau(h)) \mid h \in H\}$. Then

$$U \otimes_{kH} V \cong \text{Ind}_{\Delta_\sigma(H) * \Delta_\tau^\circ(H)}^{G \times G} (k_\lambda \otimes_k k_\mu).$$

Now $\sigma(H)/P$ and $\tau(H)/P$ are subgroups of the same order of the cyclic group K , so $\sigma(H) = \tau(H)$. Set $I := \tau(H)$. There is a unique automorphism ψ of I such that $\psi(\tau(h)) = \sigma(h)$ for all $h \in H$. Then

$$\Delta_\sigma(H) * \Delta_\tau^\circ(H) = \{(\sigma(h), \tau(h)) \mid h \in H\} = \{(\psi(x), x) \mid x \in I\} = \Delta_\psi(I).$$

Moreover, $k_\lambda \otimes k_\mu$ is one-dimensional, so there is a unique $\zeta \in I^\natural$ such that $k_\lambda \otimes k_\mu \cong k_\zeta$ as kI -modules, defined by $\zeta(x) = (\lambda\mu)(\tau^{-1}(x))$ for $x \in I$. This completes the proof of Assertion 1.

2. By Lemma 4.4, an essential diagonal p -permutation bimodule M is isomorphic to $kG_{\gamma, \lambda}$ for some $\gamma \in \text{Aut}(G)$ and $\lambda \in G^\natural$. Then M has vertex $\Delta_\gamma(P)$, of order $|P|$, so the conclusion of Assertion 1 holds. Hence, there is a subgroup I of G , containing P , an automorphism ψ of I , and $\zeta \in I^\natural$, such that

$$U \otimes_{kH} V \cong \text{Ind}_{\Delta_\psi(I)}^{G \times G} k_\zeta.$$

In particular, $\Delta_\gamma(P)$ is contained in $\Delta_\psi(P)$, up to conjugation, and we can assume that $\Delta_\gamma(P) = \Delta_\psi(P)$, that is, that ψ is the restriction of γ to P . We have $\Delta_\psi(P) \leq \Delta_\psi(I) \leq N := N_{G \times G}(\Delta_\psi(P)) = N_{G \times G}(\Delta_\gamma(P))$. So N fits in a short exact sequence of groups

$$1 \longrightarrow Z(P) \longrightarrow N \xrightarrow{p_2} G \longrightarrow 1.$$

Similarly, the group $\bar{N} := N/\Delta_\psi(P)$ fits in the sequence

$$1 \longrightarrow Z(P) \longrightarrow \bar{N} \longrightarrow K \longrightarrow 1.$$

This sequence splits via $x \in K \mapsto (\gamma(x), x)\Delta_\psi(P)$, and we view K as a subgroup of $\bar{N}$ via this map.

The group $\bar{\Delta}_\psi(I) = \Delta_\psi(I)/\Delta_\psi(P)$ is a subgroup of $\bar{N}$ and intersects $Z(P)$ trivially. So $\bar{\Delta}_\psi(I)$ is isomorphic to a subgroup J of K . Let $\theta : J \rightarrow k^\times$ be the image of $\bar{\zeta}$ under this isomorphism.

Since $\Delta_\psi(P)$ acts trivially on k_ζ , we have $k_\zeta = \text{Inf}_{\Delta_\psi(I)}^{\Delta_\psi(I)} k_{\bar{\zeta}}$, where $\bar{\zeta}$ is the homomorphism $\bar{\Delta}_\psi(I) \rightarrow k^\times$ corresponding to ζ . Hence, we have

$$\begin{aligned} U \otimes_{kH} V &\cong \text{Ind}_{\Delta_\psi(I)}^{G \times G} k_\zeta \\ &= \text{Ind}_N^{G \times G} \text{Ind}_{\Delta_\psi(I)}^N \text{Inf}_{\Delta_\psi(I)}^{\Delta_\psi(I)} k_{\bar{\zeta}} \\ &\cong \text{Ind}_N^{G \times G} \text{Inf}_N^N \text{Ind}_{\Delta_\psi(I)}^{\bar{N}} k_{\bar{\zeta}} \\ &\cong \text{Ind}_N^{G \times G} \text{Inf}_N^N \text{Ind}_{\Delta_\psi(I)}^{\bar{N}} \text{Iso}_J^{\bar{\Delta}_\psi(I)} k_\theta \\ &\cong \text{Ind}_N^{G \times G} \text{Inf}_N^N \text{Ind}_K^{\bar{N}} \text{Ind}_J^K k_\theta. \end{aligned}$$

Now K is cyclic, so $\text{Ind}_J^K k_\theta = \sum_{\substack{\alpha \in K^\natural \\ \text{Res}_J^K \alpha = \theta}} k_\alpha$, and if $\alpha \in K^\natural$, then $\text{Ind}_K^{\bar{N}} k_\alpha$ is an indecomposable projective $k\bar{N}$ -module. Then $\text{Ind}_N^{G \times G} \text{Inf}_N^N \text{Ind}_K^{\bar{N}} k_\alpha \cong kG_{\gamma, \alpha}$. Now

$$U \otimes_{kH} V \cong \bigoplus_{\beta \in \mathcal{I}_\theta} kG_{\gamma, \beta},$$

as was to be shown. □

Notation 4.7. Assume that Hypothesis 4.3 holds.

1. We abuse notation identifying $\lambda \in K^\natural$ with $k_\lambda \in R_k(K)$.
2. We set $\bar{R}_k(K) = R_k(K)/\sum_{L < K} \text{Ind}_L^K R_k(L)$, and we let $\alpha \mapsto \bar{\alpha}$ denote the projection map.
3. Let $\gamma \in \text{Aut}(G)$. Then $\bar{N}_{G \times G}(\Delta_\gamma(P)) \cong Z(P) \rtimes K$, so taking coinvariants by $Z(P)$ yields an isomorphism

$$v \in \text{Proj}\left(k\bar{N}_{G \times G}(\Delta_\gamma(P))\right) \mapsto v_{Z(P)} \in R_k(K).$$

For $u \in T^\Delta(G, G)$, let $r_\gamma(u)$ denote $\overline{u[\Delta_\gamma(P)]_{Z(P)}} \in \bar{R}_k(K)$.

We note that $\sum_{L < K} \text{Ind}_L^K R_k(L)$ is an ideal of the ring $R_k(K)$, so $\bar{R}_k(K)$ has a natural quotient ring structure. Moreover, the group $\text{Aut}(G)$ acts on $G^\natural = K^\natural$, and $\text{Inn}(G)$ acts trivially on $G^\natural$, since $[G, G] \leq P$. So $\text{Out}(G)$ acts on $R_k(K)$ and $\bar{R}_k(K)$ by ring automorphisms.

Notation 4.8. We denote by $\text{Out}(G) \ltimes \overline{R}_k(K)$ the semidirect product of $\text{Out}(G)$ with $\overline{R}_k(K)$, that is,

$$\text{Out}(G) \ltimes \overline{R}_k(K) = \bigoplus_{\gamma \in \text{Out}(G)} \gamma \ltimes \overline{R}_k(K),$$

where $\gamma \ltimes \overline{R}_k(K)$ denotes a copy of $\overline{R}_k(K)$ indexed by γ .

Then $\text{Out}(G) \ltimes \overline{R}_k(K)$ is a ring for the product defined by

$$\forall \gamma, \delta \in \text{Out}(G), \forall \lambda, \mu \in R_k(K), (\gamma \ltimes \overline{\lambda}) \cdot (\delta \ltimes \overline{\mu}) := (\gamma \circ \delta) \ltimes \overline{(\lambda \circ \delta) \times \mu}.$$

We set

$$\text{Out}(G) \ltimes \text{R}\overline{R}_k(K) := \text{R} \otimes_{\mathbb{Z}} (\text{Out}(G) \ltimes \overline{R}_k(K)) \cong \bigoplus_{\gamma \in \text{Out}(G)} \gamma \ltimes \text{R}\overline{R}_k(K).$$

In the next statement, we recall explicitly Hypothesis 4.3 for the reader's convenience.

Theorem 4.9. *Let G be a group of the form $P \rtimes K$, where P is a p -group and K is a cyclic p' -group of order invertible in R , acting faithfully on P . Then:*

1. *The map*

$$\gamma \ltimes \overline{\alpha} \in \text{Out}(G) \ltimes \text{R}\overline{R}_k(K) \mapsto \varepsilon_{\text{R}}(kG_{\gamma, \alpha}) \in \mathcal{E}_{\text{R}}(G),$$

where $\alpha \in K^{\natural}$ and $\gamma \in \text{Out}(G)$, extends to a well-defined algebra homomorphism $\mathbf{T}$.

2. *The map*

$$\varepsilon_{\text{R}}(u) \in \mathcal{E}_{\text{R}}(G) \mapsto \sum_{\gamma \in \text{Out}(G)} \gamma \ltimes r_{\gamma}(u) \in \text{Out}(G) \ltimes \text{R}\overline{R}_k(K),$$

where $u \in \text{RT}^{\Delta}(G, G)$, is a well-defined algebra homomorphism $\mathbf{S}$.

3. *The maps*

$$\mathcal{E}_{\text{R}}(G) \xrightleftharpoons[\mathbf{T}]{\mathbf{S}} \text{Out}(G) \ltimes \text{R}\overline{R}_k(K)$$

are isomorphisms of algebras, inverse to each other.

Proof. Proving that the map $\mathbf{T}$ is well-defined amounts to proving that if $u \in K^{\natural}$ is induced from a proper subgroup J of K , and if $\gamma \in \text{Out}(G)$, then $\mathbf{T}(\gamma \ltimes \overline{u}) = 0$. Let $J < K$ and $\theta \in J^{\natural}$. Then $u = \text{Ind}_J^K \theta = \sum_{\alpha \in \mathcal{I}_{\theta}} k_{\alpha}$, so

$$\mathbf{T}(\gamma \ltimes u) = \varepsilon_{\text{R}}\left(\bigoplus_{\alpha \in \mathcal{I}_{\theta}} kG_{\gamma, \alpha}\right).$$

But setting $N := N_{G \times G}(\Delta_{\gamma}(P))$ and $\overline{N} := N/\Delta_{\gamma}(P)$, we have

$$\begin{aligned} \bigoplus_{\alpha \in \mathcal{I}_{\theta}} kG_{\gamma, \alpha} &\cong \bigoplus_{\alpha \in \mathcal{I}_{\theta}} \text{Ind}_N^{G \times G} \text{Inf}_N^N \text{Ind}_{\overline{N}}^{\overline{N}} k_{\alpha} \\ &\cong \text{Ind}_N^{G \times G} \text{Inf}_N^N \text{Ind}_{\overline{N}}^{\overline{N}} \left(\bigoplus_{\alpha \in \mathcal{I}_{\theta}} k_{\alpha} \right) \\ &\cong \text{Ind}_N^{G \times G} \text{Inf}_N^N \text{Ind}_{\overline{N}}^{\overline{N}} \text{Ind}_J^K k_{\theta} \\ &\cong \text{Ind}_N^{G \times G} \text{Ind}_{\Delta_{\gamma}(P) \cdot \Delta_{\gamma}(K)}^N \text{Inf}_K^{\Delta_{\gamma}(P) \cdot \Delta_{\gamma}(K)} \text{Ind}_J^K k_{\theta} \end{aligned}$$

$$\begin{aligned}
&\cong \operatorname{Ind}_{\Delta_\gamma(G)}^{G \times G} \operatorname{Inf}_K^{\Delta_\gamma(G)} \operatorname{Ind}_J^K k_\theta \\
&\cong \operatorname{Ind}_{\Delta_\gamma(G)}^{G \times G} \operatorname{Ind}_{\Delta_\gamma(P) \cdot \Delta_\gamma(J)}^{\Delta_\gamma(G)} \operatorname{Inf}_J^{\Delta_\gamma(P) \cdot \Delta_\gamma(J)} k_\theta \\
&\cong \operatorname{Ind}_{\Delta_\gamma(P \cdot J)}^{G \times G} \operatorname{Inf}_J^{\Delta_\gamma(P \cdot J)} k_\theta.
\end{aligned}$$

But $p_2(\Delta_\gamma(P \cdot J)) = P \cdot J < G$ since $J < K$. Hence, $\mathcal{E}_R\left(\bigoplus_{\alpha \in \mathcal{I}_\theta} kG_{\gamma, \alpha}\right) = 0$, as was to be shown, so the map $\mathbf{T}$ is well defined.

Now comparing the products in Lemma 4.2 and Notation 4.8, we get that $\mathbf{T}$ is a homomorphism of R -algebras. Moreover, the identity element of $\operatorname{Out}(G) \ltimes \overline{R\bar{R}}_k(K)$, which is $\operatorname{Id} \ltimes \bar{1}$, is mapped by $\mathbf{T}$ to $\mathcal{E}_R(kG_{\operatorname{Id}, 1}) = \mathcal{E}_R(kG)$, which is the identity element of $\mathcal{E}_R(G)$.

2. Proving that the map $\mathbf{S}$ is well-defined amounts to proving that if H is a finite group with $|H| < |G|$, if U (resp. V) is a right (resp. left) essential diagonal p -permutation (kG, kH) -bimodule (resp. (kH, kG) -bimodule), then $\mathbf{S}(U \otimes_{kH} V) = 0$. So let $\gamma \in \operatorname{Aut}(G)$ such that $r_\gamma(U \otimes_{kH} V) \neq 0$. Then, in particular, $(U \otimes_{kH} V)[\Delta_\gamma(P)] \neq 0$, so $U \otimes_{kH} V$ admits an indecomposable summand with vertex $\Delta_\gamma(P)$. By Lemma 4.4, this summand is isomorphic to $kG_{\gamma, \lambda}$, for some $\lambda \in G^\natural$. By Theorem 4.6, there is a subgroup J of K with $P \cdot J \cong H$, and $\theta \in J^\natural$ such that

$$U \otimes_{kH} V \cong \bigoplus_{\alpha \in \mathcal{I}_\theta} kG_{\gamma, \alpha}.$$

Now by (4.4.1)

$$(U \otimes_{kH} V)[\Delta_\gamma(P)] \cong \bigoplus_{\alpha \in \mathcal{I}_\theta} \operatorname{Ind}_K^{\bar{N}} k_\alpha \cong \operatorname{Ind}_K^{\bar{N}} \left(\bigoplus_{\alpha \in \mathcal{I}_\theta} k_\alpha \right) \cong kZ(P) \otimes_k \operatorname{Ind}_J^K k_\theta.$$

It follows that in $\overline{R\bar{R}}_k(K)$, we have $r_\gamma(U \otimes_{kH} V) = \overline{\operatorname{Ind}_J^K k_\theta} = 0$ since $J < K$ as $P \cdot J \cong H$ and $|H| < |G| = |P||K|$. This contradiction shows that $\mathbf{S}$ is well defined.

We postpone the proof that $\mathbf{S}$ is an algebra homomorphism at the end of the proof of Assertion 3.

3. Let $\gamma \in \operatorname{Aut}(G)$ and $\alpha \in K^\natural$. Then $kG_{\gamma, \alpha}$ has vertex $\Delta_\gamma(P)$, and $kG_{\gamma, \alpha}[\Delta_\gamma(P)] \cong kZ(P) \otimes k_\alpha$ by (4.4.1). We also get from Lemma 4.4 that $kG_{\gamma, \alpha}[\Delta_\delta(P)] = 0$ if $\delta \in \operatorname{Aut}(G)$ and $\delta \neq \gamma$ in $\operatorname{Out}(G)$: Indeed, if $\Delta_\gamma(P)$ and $\Delta_\delta(P)$ are conjugate in $G \times G$, then the restriction of δ to P is equal to the restriction of γ to P , up to an automorphism given by conjugation by some element of G , which we may assume to be trivial. Then $\delta^{-1}\gamma$ is inner, by Lemma 3.1.

It follows that $\mathbf{S}(\mathcal{E}_R(kG_{\gamma, \alpha})) = \gamma \ltimes \bar{\alpha}$. Since $\mathbf{T}(\gamma \ltimes \bar{\alpha}) = \mathcal{E}_R(kG_{\gamma, \alpha})$, the maps $\mathbf{S}$ and $\mathbf{T}$ are inverse to each other. In particular, they are bijections, so $\mathbf{S}$ is a map of R -algebras, as $\mathbf{T}$ is. This completes the proof. $\square$

5. The simple functors

5.1.

We want to consider the simple diagonal p -permutation functors, so by general arguments, we can assume that our ring R of coefficients is a field $\mathbb{F}$. Moreover, in order to apply the results of the previous sections, we want that p' -groups have order invertible in $\mathbb{F}$. So we are left with the cases, where $\mathbb{F}$ has characteristic 0 or p .

If S is a simple diagonal p -permutation functor over $\mathbb{F}$, and H is a finite group of minimal order such that $S(H) \neq 0$, then $V = S(H)$ is a simple module for the essential algebra $\mathcal{E}_{\mathbb{F}}(H)$. In particular, $\mathcal{E}_{\mathbb{F}}(H) \neq 0$, so $H = L\langle u \rangle$ for some D^Δ -pair (L, u) . Moreover, we have an isomorphism of algebras

$$\mathcal{E}_{\mathbb{F}}(L\langle u \rangle) \cong \operatorname{Out}(L\langle u \rangle) \ltimes \overline{R\bar{R}}_k(L\langle u \rangle).$$

5.2.

Conversely, if (L, u) is a D^Δ -pair, and V is a simple $\mathcal{E}_{\mathbb{F}}(L\langle u \rangle)$ -module, then we denote by $S_{L\langle u \rangle, V}$ the unique simple diagonal p -permutation functor with minimal group $L\langle u \rangle$ and such that $S_{L\langle u \rangle, V}(L\langle u \rangle) \cong V$ as an $\mathcal{E}_{\mathbb{F}}(L\langle u \rangle)$ -module. The evaluation of $S_{L\langle u \rangle, V}$ at a finite group G is isomorphic to

$$S_{L\langle u \rangle, V}(G) \cong \left(\mathbb{F}T^\Delta(G, L\langle u \rangle) \otimes_{\mathbb{F}T^\Delta(L\langle u \rangle, L\langle u \rangle)} V \right) / \mathcal{R},$$

where $\mathcal{R}$ is the subspace generated by all finite sums $\sum_{i \in I} f_i \otimes v_i$, with $f_i \in \mathbb{F}T^\Delta(G, L\langle u \rangle)$ and $v_i \in V$ for $i \in I$, such that $\sum_{i \in I} \pi(\varphi \circ f_i) \cdot v_i = 0$ for all $\varphi \in \mathbb{F}T^\Delta(L\langle u \rangle, G)$, where $\pi : \mathbb{F}T^\Delta(L\langle u \rangle, L\langle u \rangle) \rightarrow \mathcal{E}_{\mathbb{F}}(L\langle u \rangle)$ is the projection map.

5.3.

In particular, let $f \in \mathbb{F}T^\Delta(G, L\langle u \rangle)$ be of the form $f = \text{Ind}_N^{G \times L\langle u \rangle} \text{Inf}_N^N E$, where

- $N = N_{G \times L\langle u \rangle}(\Delta(P, \gamma, R))$, for some $R \leq L$ and $\gamma : R \xrightarrow{\cong} P \leq G$;
- $\overline{N} = N/\Delta(P, \gamma, R)$;
- E is a projective $k\overline{N}$ -module.

If there exists some $v \in V$ such that $f \otimes v \notin \mathcal{R}$, then there exists $\varphi \in \mathbb{F}T^\Delta(L\langle u \rangle, G)$ such that $\pi(\varphi \circ f) \neq 0$. In particular, the Brauer quotient $(\varphi \circ f)[\Delta(L, \theta, L)]$ has to be non-zero for some $\theta \in \text{Aut}(L)$, which forces $R = L$.

5.4.

Moreover, if $f \in \mathbb{F}T^\Delta(G, L\langle u \rangle)$ can be factorized through a group of order strictly smaller than the order of $L\langle u \rangle$, then $f \otimes v \in \mathcal{R}$ for any $v \in V$. So if there exists $v \in V$ with $f \otimes v \notin \mathcal{R}$, then in particular $p_2(N) = L\langle u \rangle$.

Remark 5.5. This condition $p_2(N) = L\langle u \rangle$ means that for any $x \in L\langle u \rangle$, there exists $g \in N_G(P)$ such that ${}^g\gamma(l) = \gamma(xl)$ for all $l \in L$. Equivalently, there exists $s \in N_G(P)$ such that ${}^s\gamma(l) = \gamma(u l)$ for all $l \in L$: Suppose indeed that such an element s exists. Any element $x \in L\langle u \rangle$ is equal to $l_0 u^\alpha$ for some $l_0 \in L$ and $\alpha \in \mathbb{N}$. Then

$$\gamma \circ i_x = \gamma \circ i_{l_0} \circ i_{u^\alpha} = i_{\gamma(l_0)} \circ \gamma \circ (i_u)^\alpha = i_{\gamma(l_0)} \circ (i_s)^\alpha \circ \gamma = i_{\gamma(l_0)s^\alpha} \circ \gamma,$$

that is, $i_g \circ \gamma = \gamma \circ i_x$, for $g = \gamma(l_0)s^\alpha$.

In other words, saying that $p_2(N) = L\langle u \rangle$ amounts to saying that (P, γ) belongs to the set

$$\mathcal{P}(G, L, u) := \{(P, \gamma) \mid \gamma : L \xrightarrow{\cong} P \leq G, \exists s \in N_G(P), i_s \circ \gamma = \gamma \circ i_u\}.$$

For $\varphi \in \text{Aut}(L\langle u \rangle)$, we have that

$$\begin{aligned} N_{G \times L\langle u \rangle}(\Delta(P, \gamma\varphi, L)) &= \{(a, b) \mid (a, \varphi(b)) \in N_{G \times L\langle u \rangle}(\Delta(P, \gamma, L))\} \\ &= (1 \times \varphi^{-1})N_{G \times L\langle u \rangle}(\Delta(P, \gamma, L)) \end{aligned}$$

so the set $\mathcal{P}(G, L, u)$ is a $(G, \text{Aut}(L\langle u \rangle))$ -biset by

$$\forall g \in G, \forall \varphi \in \text{Aut}(L\langle u \rangle), g \cdot (P, \gamma) \cdot \varphi = ({}^gP, i_g\gamma\varphi).$$

5.6.

Let $\mathcal{T}(G, L, u)$ denote the set of triples (P, γ, E) , where $(P, \gamma) \in \mathcal{P}(G, L, u)$ and E is an indecomposable projective $kN(\Delta(P, \gamma, L))/\Delta(P, \gamma, L)$ -module. The set $\mathcal{T}(G, L, u)$ is also a $(G, \text{Aut}(L\langle u \rangle))$ -biset by

$$\forall g \in G, \forall \varphi \in \text{Aut}(L\langle u \rangle), g \cdot (P, \gamma, E) \cdot \varphi = ({}^g P, i_g \gamma \varphi, {}^g E^\varphi),$$

where ${}^g E^\varphi$ is the $kN(\Delta({}^g P, i_g \gamma \varphi, L))/\Delta({}^g P, i_g \gamma \varphi, L)$ -module obtained from E via the group isomorphism

$$(a, b)\Delta({}^g P, i_g \gamma \varphi, L) \mapsto (a^g, \varphi(b))\Delta(P, \gamma, L)$$

from $kN(\Delta({}^g P, i_g \gamma \varphi, L))/\Delta({}^g P, i_g \gamma \varphi, L)$ to $kN(\Delta(P, \gamma, L))/\Delta(P, \gamma, L)$.

For $(P, \gamma, E) \in \mathcal{T}(G, L, u)$, set

$$T(P, \gamma, E) = \text{Ind}_N^{G \times L\langle u \rangle} \text{Inf}_N^N E,$$

where $N = N_{G \times L\langle u \rangle}(\Delta(P, \gamma, L))$ and $\overline{N} = N/\Delta(P, \gamma, L)$. Then $T(P, \gamma, E)$ is a diagonal p -permutation $(kG, kL\langle u \rangle)$ -bimodule, and we abuse notation writing $T(P, \gamma, E) \in \mathbb{F}T^\Delta(G, L\langle u \rangle)$. We observe that for $(g, \varphi) \in G \times \text{Aut}(L\langle u \rangle)$, we have

$$T({}^g P, i_g \gamma \varphi, {}^g E^\varphi) \cong T(P, \gamma, E) \otimes_{kL\langle u \rangle} k(L\langle u \rangle)_\varphi,$$

where $k(L\langle u \rangle)_\varphi$ is the bimodule $k(L\langle u \rangle)$ twisted by φ on the right, that is, $x \cdot m \cdot y$ (in $k(L\langle u \rangle)_\varphi$) = $xm\varphi(y)$ (in $k(L\langle u \rangle)$).

5.7.

Now a lemma:

Lemma 5.8. *Let J be a finite group, let $K \trianglelefteq J$ such that J/K is a cyclic p' -group. Let E be an indecomposable projective kJ -module, let V be an indecomposable summand of $\text{Res}_K^J E$, and H be the inertia group of V in J . Then V extends to an indecomposable projective kH -module F , and there exists a group homomorphism $\lambda : H/K \rightarrow k^\times$ such that*

$$E \cong \text{Ind}_H^J (F \otimes_k \text{Inf}_{H/K}^H k_\lambda).$$

Proof. Use Theorem 3.13.2 in [1] and Theorem 4.1 of [7]. □

5.9.

Let $(P, \gamma, E) \in \mathcal{T}(G, L, u)$. Set $N_{P, \gamma} = N_{G \times L\langle u \rangle}(\Delta(P, \gamma, L))$ and $\overline{N}_{P, \gamma} = N_{P, \gamma}/\Delta(P, \gamma, L)$, so E is an indecomposable projective $k\overline{N}_{P, \gamma}$ -module. There is an exact sequence of groups

$$1 \rightarrow C_G(P) \rightarrow \overline{N}_{P, \gamma} \rightarrow \langle u \rangle \rightarrow 1, \quad (5.9.1)$$

so by the previous lemma, if V is an indecomposable summand of $\text{Res}_{C_G(P)}^{\overline{N}_{P, \gamma}} E$, and H its stabilizer in $\overline{N}_{P, \gamma}$, there exists an indecomposable projective kH -module F such that

$$E \cong \text{Ind}_H^{\overline{N}_{P, \gamma}} F.$$

From this follows that

$$\begin{aligned} T(P, \gamma, E) &= \text{Ind}_{N_{P,\gamma}}^{G \times L\langle u \rangle} \text{Inf}_{\overline{N}_{P,\gamma}}^{N_{P,\gamma}} E \\ &\cong \text{Ind}_{N_{P,\gamma}}^{G \times L\langle u \rangle} \text{Inf}_{\overline{N}_{P,\gamma}}^{N_{P,\gamma}} \text{Ind}_H^{\overline{N}_{P,\gamma}} F \\ &\cong \text{Ind}_{N_{P,\gamma}}^{G \times L\langle u \rangle} \text{Ind}_H^{N_{P,\gamma}} \text{Inf}_H^{\hat{H}} F \cong \text{Ind}_{\hat{H}}^{G \times L\langle u \rangle} \text{Inf}_H^{\hat{H}} F, \end{aligned}$$

where $\hat{H}$ is the inverse image in $N_{P,\gamma}$ of $H \leq \overline{N}_{P,\gamma}$ by the projection map $N_{P,\gamma} \rightarrow \overline{N}_{P,\gamma}$. Now $T(P, \gamma, E)$ factors through the second projection of $\hat{H} \leq G \times L\langle u \rangle$, so we can assume that this projection is the whole of $L\langle u \rangle$, that is, equivalently that $H = \overline{N}_{P,\gamma}$. In other words, by Theorem 4.1 of [7] already quoted above, we can assume that $\text{Res}_{G(P)}^{\overline{N}_{P,\gamma}} E$ is indecomposable. We denote by $\text{Pim}^\#(k\overline{N}_{P,\gamma})$ the set of isomorphism classes of such indecomposable projective $k\overline{N}_{P,\gamma}$ -modules, and by $\mathcal{T}^\#(G, L, u)$ the subset of $\mathcal{T}(G, L, u)$ consisting of triples (P, γ, E) such that $E \in \text{Pim}^\#(k\overline{N}_{P,\gamma})$.

5.10.

It follows from the above remarks that $S_{L\langle u \rangle, V}(G)$ is generated by the images of the elements $T(P, \gamma, E) \otimes v$, where

- (P, γ) runs through a set $[\mathcal{P}(G, L, u)]$ of representatives of orbits of $(G \times \text{Aut}(L\langle u \rangle))$ on $\mathcal{P}(G, L, u)$;
- $E \in \text{Pim}^\#(k\overline{N}_{P,\gamma})$;
- $v \in V$.

In other words, we have a surjective map

$$\bigoplus_{\substack{(P,\gamma) \in [\mathcal{P}(G,L,u)] \\ E \in \text{Pim}^\#(k\overline{N}_{P,\gamma})}} T(P, \gamma, E) \otimes V \longrightarrow S_{L\langle u \rangle, V}(G)$$

sending $T(P, \gamma, E) \otimes v \in \mathbb{F}T^\Delta(G, L\langle u \rangle) \otimes V$ to its image in $S_{L\langle u \rangle, V}(G)$. The kernel of this map is the set of sums

$$\sum_{\substack{(P,\gamma) \in [\mathcal{P}(G,L,u)] \\ E \in \text{Pim}^\#(k\overline{N}_{P,\gamma})}} T(P, \gamma, E) \otimes v_{P,\gamma,E},$$

where $v_{P,\gamma,E} \in V$, such that for any (Q, δ) in $[\mathcal{P}(G, L, u)]$ and any indecomposable projective $k\overline{N}_{Q,\delta}$ -module F , or equivalently, for any F in $\text{Pim}^\#(k\overline{N}_{Q,\delta})$,

$$\sum_{\substack{(P,\gamma) \in [\mathcal{P}(G,L,u)] \\ E \in \text{Pim}^\#(k\overline{N}_{P,\gamma})}} \pi(T^o(Q, \delta, F) \otimes_{kG} T(P, \gamma, E)) \cdot v_{P,\gamma,E} = 0, \quad (5.10.1)$$

where $T^o(Q, \delta, F)$ is the $(kL\langle u \rangle, kG)$ -bimodule “opposite” of the $(kG, kL\langle u \rangle)$ -bimodule $T(Q, \delta, F)$. In other words,

$$T^o(Q, \delta, F) = \text{Ind}_{N_{Q,\delta}^o}^{L\langle u \rangle \times G} \text{Inf}_{\overline{N}_{Q,\delta}^o}^{N_{Q,\delta}^o} F^o,$$

where

- $N_{Q,\delta}^o = N_{L\langle u \rangle \times G}(\Delta(L, \delta^{-1}, Q))$;
- $\overline{N}_{Q,\delta}^o = N_{Q,\delta}^o / \Delta(L, \delta^{-1}, Q)$;
- F^o is the opposite module of F .

Now if $\pi(T^o(Q, \delta, F) \otimes_{kG} T(P, \gamma, E)) \neq 0$, there exists an automorphism θ of $L\langle u \rangle$ such that

$$(T^o(Q, \delta, F) \otimes_{kG} T(P, \gamma, E))[\Delta(L, \theta, L)] \neq 0.$$

Hence, there exist $V \leq G$, $\alpha : V \xrightarrow{\cong} L$, and $\beta : L \xrightarrow{\cong} V$ such that $\theta = \alpha\beta$ and

$$T^o(Q, \delta, F)[\Delta(L, \alpha, V)] \neq 0 \text{ and } T(P, \gamma, E)[\Delta(V, \beta, L)] \neq 0.$$

So $\Delta(Q, \delta, L)$ is conjugate to $\Delta(V, \alpha^{-1}, L)$ in $G \times L\langle u \rangle$, and $\Delta(P, \gamma, L)$ is conjugate to $\Delta(V, \beta, L)$ in $G \times L\langle u \rangle$. By Remark 5.5, this amounts to saying that there exist $g, h \in G$ such that $(V, \alpha^{-1}) = ({}^g Q, i_g \delta)$ and $(V, \beta) = ({}^h P, i_h \gamma)$. Hence, $P = V^h = h^{-1}gQ$ and $\theta = \delta^{-1}i_{g^{-1}h}\gamma$.

In other words, setting $z = h^{-1}g$, we have $P = {}^z Q$ and $\gamma = i_z \delta \theta$, that is, $(P, \gamma) = z \cdot (Q, \delta) \cdot \theta$. As (P, γ) and (Q, δ) are both in our set $[\mathcal{P}(G, L, u)]$ of representatives of $G \times \text{Aut}(L\langle u \rangle)$ -orbits on $\mathcal{P}(G, L, u)$, this forces $P = Q$ and $\gamma = \delta$.

Now Equation (5.10.1) reduces to the fact that for every (Q, δ) in $[\mathcal{P}(G, L, u)]$ and any $F \in \text{Pim}^\#(k\overline{N}_{Q, \delta})$

$$\sum_{E \in \text{Pim}^\#(k\overline{N}_{Q, \delta})} \pi(T^o(Q, \delta, F) \otimes_{kG} T(Q, \delta, E)) \cdot v_{Q, \delta, E} = 0. \quad (5.10.2)$$

It follows that

$$S_{L\langle u \rangle, V}(G) \cong \bigoplus_{(Q, \delta) \in [\mathcal{P}(G, L, u)]} \left(\bigoplus_{E \in \text{Pim}^\#(k\overline{N}_{Q, \delta})} T(Q, \delta, E) \otimes V \right) / \mathcal{R}_{Q, \delta}, \quad (5.10.3)$$

where the relations $\mathcal{R}_{Q, \delta}$ are given by (5.10.2) for every $F \in \text{Pim}(k\overline{N}_{Q, \delta})$.

Now we set

$$G_{Q, \delta} = \{g \in G \mid \exists x \in L\langle u \rangle, (g, x) \in N_{G \times L\langle u \rangle}(\Delta(Q, \delta, L))\}.$$

With this notation and using the main theorem in [3], the $(kL\langle u \rangle, kL\langle u \rangle)$ -bimodule $M = T^o(Q, \delta, F) \otimes_{kG} T(Q, \delta, E)$ is isomorphic to

$$\begin{aligned} M &\cong \bigoplus_{g \in G_{Q, \delta} \setminus G/G_{Q, \delta}} \text{Ind}_{N_{Q, \delta}^o * {}^{(g, 1)}N_{Q, \delta}}^{L\langle u \rangle \times L\langle u \rangle} \left(\text{Inf}_{N_{Q, \delta}^o}^{N_{Q, \delta}^o} F^o \otimes_{kC_G(Q, {}^g Q)} {}^{(g, 1)} \text{Inf}_{N_{Q, \delta}}^{N_{Q, \delta}} E \right) \\ &= \bigoplus_{g \in G_{Q, \delta} \setminus G/G_{Q, \delta}} \text{Ind}_{N_{Q, \delta}^o * N_{Q, \delta}^o \cdot i_g \delta}^{L\langle u \rangle \times L\langle u \rangle} \left(\text{Inf}_{N_{Q, \delta}^o}^{N_{Q, \delta}^o} F^o \otimes_{kC_G(Q, {}^g Q)} \text{Inf}_{N_{Q, \delta}^o \cdot i_g \delta}^{N_{Q, \delta}^o \cdot i_g \delta} {}^{(g, 1)} E \right). \end{aligned} \quad (5.10.4)$$

5.II.

Now another lemma:

Lemma 5.12. *Let G, H, K be finite groups, let $Z \leq X \leq G \times H$ and $T \leq Y \leq H \times K$ be subgroups. Set $D = k_2(X) \cap k_1(Y)$. Then $X/Z \times_D Y/T$ is an $X * Y$ -set, and for $(u, v)Z \in X/Z$ and $(w, r)T \in Y/T$, the stabilizer of $(u, v)Z \times_D (w, r)T$ in $X * Y$ is equal to ${}^{(u, v)}Z * {}^{(w, r)}T$.*

Proof. This is straightforward. □

5.13.

Let M_g denote the term of the direct sum (5.10.4) indexed by $g \in G$, that is,

$$M_g = \text{Ind}_{N_{Q,\delta}^o * N_{gQ,i_g\delta}}^{L\langle u \rangle \times L\langle u \rangle} \left(\text{Inf}_{N_{Q,\delta}^o}^{N_{Q,\delta}^o} F^o \otimes_{kC_G(Q, {}^gQ)} \text{Inf}_{N_{gQ,i_g\delta}}^{N_{gQ,i_g\delta}({}^{g,1})} E \right).$$

To apply the previous lemma, we set

$$X = N_{Q,\delta}^o, \quad Y = N_{gQ,i_g\delta}, \quad Z = \Delta^o(Q, \delta, L), \quad T = \Delta({}^gQ, i_g\delta, L).$$

Then $k_2(X) = C_G(Q)$ and $k_1(Y) = C_G({}^gQ)$, thus $D = C_G(Q, {}^gQ)$. Moreover, $Z \trianglelefteq X$ and $T \trianglelefteq Y$.

Now since E is a direct summand of $k\bar{N}_{Q,\delta}$, it follows that $\text{Inf}_{N_{gQ,i_g\delta}}^{N_{gQ,i_g\delta}({}^{g,1})} E$ is a direct summand of kY/T . Similarly, $\text{Inf}_{N_{Q,\delta}^o}^{N_{Q,\delta}^o} F^o$ is a direct summand of kX/Z . Hence, $\text{Inf}_{N_{Q,\delta}^o}^{N_{Q,\delta}^o} F^o \otimes_{kC_G(Q, {}^gQ)} \text{Inf}_{N_{gQ,i_g\delta}}^{N_{gQ,i_g\delta}({}^{g,1})} E$ is a direct summand of

$$kX/Z \otimes_{kC_G(Q, {}^gQ)} kY/T \cong k(X/Z) \times_D (Y/T).$$

By the previous lemma, for $(u, v)Z \in X/Z$ and $(w, r)T \in Y/T$, the stabilizer of $(u, v)Z \times_D (w, r)T$ in $X * Y$ is equal to $(u, v)Z * (w, r)T = Z * T$. Hence, the vertices of the indecomposable direct summands of $kX/Z \otimes_{kC_G(Q, {}^gQ)} kY/T$ are subgroups of $Z * T$.

It follows that if $\pi(M_g) \neq 0$, then there exists $\theta \in \text{Aut}(L\langle u \rangle)$ such that $\Delta(L, \theta, L)$ is conjugate in $L\langle u \rangle \times L\langle u \rangle$ to a subgroup of $Z * T$. Up to changing θ by some inner automorphism of $L\langle u \rangle$, we can assume that $\Delta(L, \theta, L) \leq Z * T$. But

$$\begin{aligned} Z * T &= \Delta^o(Q, \delta, L) * \Delta({}^gQ, i_g\delta, L) \\ &= \left\{ (a, b) \in L\langle u \rangle \times L\langle u \rangle \mid \exists c \in G, \begin{cases} (c, a) \in \Delta(Q, \delta, L) \\ (c, b) \in \Delta({}^gQ, i_g\delta, L) \end{cases} \right\} \\ &= \{ (\delta^{-1}(c), \delta^{-1}(c^g)) \mid c \in Q \cap {}^gQ \}. \end{aligned}$$

Then $Z * T$ contains $\Delta(L, \theta, L)$ if and only if $Q = {}^gQ$ and $\theta(l) = \delta^{-1}({}^g\delta(l))$ for any $l \in L$. In other words, $g \in N_G(Q)$ and $\delta^{-1}i_g\delta$ is the restriction of θ to L .

For $g \in N_G(Q)$, we set $i_g^\delta = \delta^{-1}i_g\delta \in \text{Aut}(L)$ and

$$\widehat{G}_{Q,\delta} = \{ g \in N_G(Q) \mid \exists \theta \in \text{Aut}(L\langle u \rangle), i_g^\delta = \theta|_L \}.$$

With this notation, we see that $\pi(M_g) = 0$ unless $g \in \widehat{G}_{Q,\delta}$.

We also observe that $G_{Q,\delta}$ is a normal subgroup of $\widehat{G}_{Q,\delta}$, and we set

$$\overline{G}_{Q,\delta} = \widehat{G}_{Q,\delta} / G_{Q,\delta}.$$

We observe moreover that for $g \in \widehat{G}_{Q,\delta}$, there is a unique $\theta \in \text{Aut}(L\langle u \rangle)$ such that $\theta|_L = i_g^\delta$, up to inner automorphism i_w of $L\langle u \rangle$ for some $w \in Z(L)$, thanks to Lemma 3.1.

5.14.

For $g \in \widehat{G}_{Q,\delta}$, we will denote by $\theta_g \in \text{Out}(L\langle u \rangle)$ the unique outer automorphism such that $(\theta_g)|_L = i_g^\delta$, and by $\hat{\theta}_g$ a representative of θ_g in $\text{Aut}(L\langle u \rangle)$ satisfying $(\hat{\theta}_g)|_L = i_g^\delta$. The map $g \in \widehat{G}_{Q,\delta} \mapsto \theta_g \in$

$\text{Out}(L\langle u \rangle)$ is a group homomorphism, with kernel $G_{Q,\delta}$. In other words, $\overline{G}_{Q,\delta} = \widehat{G}_{Q,\delta}/G_{Q,\delta}$ embeds in $\text{Out}(L\langle u \rangle)$.

Now let $(a, b) \in N_{Q,\delta}$, that is, $(a, b) \in G \times L\langle u \rangle$, and ${}^a\delta(l) = \delta({}^b l)$ for all $l \in L$. If $g \in \widehat{G}_{Q,\delta}$, we claim that $({}^g a, \hat{\theta}_g(b))$ also lies in $N_{Q,\delta}$. Indeed for $l \in L$, setting $l' = \hat{\theta}_g^{-1}(l)$, we have $\delta(l) = i_g \delta(l')$, so

$$\begin{aligned} ({}^g a) \delta(l) &= ({}^g a) i_g \delta(l') = g a \delta(l') a^{-1} g^{-1} \\ &= g \delta({}^b l') g^{-1} = i_g \delta({}^b l'), \end{aligned}$$

whereas

$$\begin{aligned} \delta(\hat{\theta}_g({}^b l)) &= \delta(\hat{\theta}_g({}^b) \hat{\theta}_g(l')) \\ &= \delta(\hat{\theta}_g({}^b l')) \\ &= \delta \delta^{-1} i_g \delta({}^b l') = i_g \delta({}^b l'), \end{aligned}$$

proving the claim.

In other words, the map $\Phi_g : (a, b) \mapsto ({}^g a, \hat{\theta}_g(b))$ is an automorphism of $N_{Q,\delta}$. Moreover, it sends $\Delta(Q, \delta, L)$ to itself. Indeed, for $l \in L$, we have

$$({}^g \delta(l), \hat{\theta}_g(l)) = (i_g \delta(l), \delta^{-1} i_g \delta(l)) = (\delta(\theta_g(l)), \theta_g(l)).$$

It follows that Φ_g induces an automorphism $\bar{\Phi}_g$ of $\overline{N}_{Q,\delta}$. But there is a little more: Let $\overline{N}_{Q,\delta}^b$ denote the quotient $\overline{N}_{Q,\delta}/Z(Q)$. Then by the above lemma, for $g, h \in \widehat{G}_{Q,\delta}$, the composition $\Phi_g \Phi_h$ is equal to Φ_{gh} modulo an inner automorphism induced by an element of $Z(L)$. It follows that Φ_g induces an automorphism Φ_g^b of $\overline{N}_{Q,\delta}^b$, and that $\Phi_g^b \Phi_h^b = \Phi_{gh}^b$. In other words, $\widehat{G}_{Q,\delta}$ acts on $\overline{N}_{Q,\delta}^b$.

5.15.

Assuming now that $g \in \widehat{G}_{Q,\delta}$, we have

$$M_g = \text{Ind}_{N_{Q,\delta}^o * N_{Q,i_g \delta}}^{L\langle u \rangle \times L\langle u \rangle} \left(\text{Inf}_{N_{Q,\delta}^o}^{N_{Q,\delta}^o} F^o \otimes_{kC_G(Q)} \text{Inf}_{N_{Q,i_g \delta}}^{N_{Q,i_g \delta}} (g, 1) E \right).$$

Moreover,

$$\begin{aligned} N_{Q,\delta}^o * N_{Q,i_g \delta} &= \{(a, b) \in L\langle u \rangle \times L\langle u \rangle \mid \exists c \in G, (c, a) \in N_{Q,\delta}, (c, b) \in N_{Q,i_g \delta}\} \\ &= \left\{ (a, b) \in L\langle u \rangle \times L\langle u \rangle \mid \exists c \in G, \forall l \in L, \begin{cases} {}^c \delta(l) = \delta({}^a l) \\ {}^{cg} \delta(l) = {}^g \delta({}^b l) \end{cases} \right\} \\ &\leq \{(a, b) \in L\langle u \rangle \times L\langle u \rangle \mid \forall l \in L, {}^a (i_g \delta(l)) = i_g \delta({}^b l)\}. \end{aligned}$$

The last group is the normalizer of $\Delta(L, i_g \delta, L)$ in $L\langle u \rangle \times L\langle u \rangle$. Conversely, if $(a, b) \in N_{L\langle u \rangle \times L\langle u \rangle}(\Delta(L, i_g \delta, L))$, since $(Q, \delta) \in \mathcal{P}(G, L, u)$, there exists $c \in N_G(Q)$ such that $(c, a) \in N_{Q,\delta}$. In other words, we have

$$\forall l \in L, {}^a (i_g \delta(l)) = i_g \delta({}^b l) \text{ and } {}^c \delta(l) = \delta({}^a l).$$

It follows that

$$\forall l \in L, i_g \delta({}^b l) = \delta({}^a i_g \delta(l)) = {}^c \delta i_g \delta(l) = {}^c i_g \delta(l),$$

that is, $(c, b) \in N_{Q, i_g \delta}$. Thus, $(a, b) \in N_{Q, \delta}^o * N_{Q, i_g \delta}$, and this gives

$$N_{Q, \delta}^o * N_{Q, i_g \delta} = N_{L\langle u \rangle \times L\langle u \rangle}(\Delta(L, i_g \delta, L)).$$

To simplify the notation, we denote this group by $N_{L, u}(i_g \delta)$.

We also observe that $\Delta(L, i_g \delta, L) = \Delta^o(Q, \delta, L) * \Delta(Q, i_g \delta, L)$. We denote this group by $\Delta_{L, u}(i_g \delta)$, and we set

$$\overline{N}_{L, u}(i_g \delta) = N_{L, u}(i_g \delta) / \Delta_{L, u}(i_g \delta).$$

Since (L, u) is a D^Δ -pair, this group fits into a short exact sequence of groups

$$1 \rightarrow Z(L) \rightarrow \overline{N}_{L, u}(i_g \delta) \rightarrow \langle u \rangle \rightarrow 1. \quad (5.15.1)$$

This sequence splits by the map sending $v \in \langle u \rangle$ to the image of $(\hat{\theta}_g(v), v)$ in $\overline{N}_{L, u}(i_g \delta)$.

5.16.

From the above discussion follows that

$$\pi(M) = \sum_{g \in \overline{G}_{Q, \delta}} \pi \left(\text{Ind}_{N_{L, u}(i_g \delta)}^{L\langle u \rangle \times L\langle u \rangle} \left(\text{Inf}_{N_{Q, \delta}^o}^{N_{Q, \delta}^o} F^o \otimes_{kC_G(Q)} \text{Inf}_{N_{Q, i_g \delta}}^{N_{Q, i_g \delta}(g, 1)} E \right) \right).$$

The subgroup $\Delta_{L, u}(i_g \delta)$ of $N_{L, u}(i_g \delta)$ acts trivially on the module

$$T(F, g, E) = \text{Inf}_{N_{Q, \delta}^o}^{N_{Q, \delta}^o} F^o \otimes_{kC_G(Q)} \text{Inf}_{N_{Q, i_g \delta}}^{N_{Q, i_g \delta}(g, 1)} E,$$

so $T(F, g, E)$ is inflated from a $k\overline{N}_{L, u}(i_g \delta)$ -module

$$T(F, g, E) = \text{Inf}_{\overline{N}_{L, u}(i_g \delta)}^{N_{L, u}(i_g \delta)} \overline{T}(F, g, E),$$

where $\overline{T}(F, g, E) = F^o \otimes_{kC_G(Q)} (g, 1)E$.

The discussion in Section 5.13 above shows that the vertices of the indecomposable summands of $T(F, g, E)$ are equal to $\Delta_{L, u}(i_g \delta)$. In other words, $\overline{T}(F, g, E)$ is a projective $\overline{N}_{L, u}(i_g \delta)$ -module. In view of the split exact sequence (5.15.1), we have that $\overline{N}_{L, u}(i_g \delta) \cong Z(L) \rtimes \langle u \rangle$, and

$$\overline{T}(F, g, E) \cong \bigoplus_{\lambda \in \langle u \rangle^\natural} m_\lambda(F, g, E) \text{Ind}_{\langle u \rangle}^{Z(L) \rtimes \langle u \rangle} k_\lambda,$$

for some multiplicities $m_\lambda(F, g, E) \in \mathbb{N}$, where $\langle u \rangle^\natural$ is the set of group homomorphisms $\lambda : \langle u \rangle \rightarrow k^\times$. It is easy to check that

$$\begin{aligned} m_\lambda(F, g, E) &= \dim_k \text{Hom}_{Z(L) \rtimes \langle u \rangle} \left(\text{Inf}_{\langle u \rangle}^{Z(L) \rtimes \langle u \rangle} k_\lambda, \overline{T}(F, g, E) \right) \\ &= \dim_k \text{Hom}_{\langle u \rangle} \left(k_\lambda, (\overline{T}(F, g, E))^{Z(L)} \right). \end{aligned} \quad (5.16.1)$$

Now the image of the module $\text{Ind}_{N_{L, u}(i_g \delta)}^{L\langle u \rangle \times L\langle u \rangle} \text{Inf}_{\overline{N}_{L, u}(i_g \delta)}^{N_{L, u}(i_g \delta)} \text{Ind}_{\langle u \rangle}^{Z(L) \rtimes \langle u \rangle} k_\lambda$ in the essential algebra $\mathcal{E}_{\mathbb{F}}(L\langle u \rangle)$ is equal to $kL\langle u \rangle_{\hat{\theta}_g, \lambda^{-1}}$, where $\theta_g \in \text{Out}(L\langle u \rangle)$ and $\hat{\theta}_g \in \text{Aut}(L\langle u \rangle)$ are defined in Section 5.14.⁷ So

⁷Note that for $g \in \widehat{G}_{Q, \delta}$, the outer automorphism θ_g only depends on $gG_{Q, \delta} \in \overline{G}_{Q, \delta}$.

our relations $\mathcal{R}_{Q,\delta}$ of (5.10.3) become

$$\forall F \in \text{Pim}(k\overline{N}_{Q,\delta}), \quad \sum_{\substack{g \in \overline{G}_{Q,\delta} \\ \lambda \in \langle u \rangle^{\natural} \\ E \in \text{Pim}^{\sharp}(k\overline{N}_{Q,\delta})}} m_{\lambda}(F, g, E) kG_{\theta_g, \lambda} \cdot v_E = 0. \quad (5.16.2)$$

5.17.

In order to understand these relations, we are going to change them a little bit, by replacing F by its dual and the tensor product $- \otimes_{kC_G(Q)} -$ appearing in $T(F, g, E)$ by $\text{Hom}_{kC_G(Q)}(-, -)$, in other words, by setting now

$$\overline{T}(F, g, E) = \text{Hom}_{kC_G(Q)}(F, {}^{(g,1)}E). \quad (5.17.1)$$

In particular, the action of $\overline{N}_{L,u}(i_g^{\delta})$ on $\overline{T}(F, g, E)$ is given as follows: for $(a, b) \in N_{L,u}(i_g^{\delta})$, choose $s \in G$ such that $(s, a) \in N_{Q,\delta}$. Then $(s^g, b) \in N_{Q,\delta}$, and for $\varphi \in \text{Hom}_{kC_G(Q)}(F, {}^{(g,1)}E)$, we have

$$\forall f \in F, \quad ((a, b)\varphi)(f) = (s^g, b)\varphi((s, a)^{-1}f). \quad (5.17.2)$$

The action of $Z(L)$ on $\overline{T}(F, g, E)$ is simply given by multiplication

$$\forall z \in Z(L), \forall \varphi \in \text{Hom}_{kC_G(Q)}(F, {}^{(g,1)}E), \forall f \in F, \quad (z\varphi)(f) = \delta(z)^g \cdot \varphi(f),$$

where $\delta(z) \in Z(Q)$. It follows that

$$\begin{aligned} \overline{T}(F, g, E)^{Z(L)} &= \text{Hom}_{kC_G(Q)}(F, {}^{(g,1)}E^{Z(Q)}) \\ &\cong \text{Hom}_{kC_G(Q)/Z(Q)}(F^{Z(Q)}, {}^{(g,1)}E^{Z(Q)}), \end{aligned}$$

where the last isomorphism comes from the fact that since F is projective, the module of co-invariants $F_{Z(Q)}$ is isomorphic to the module of invariants $F^{Z(Q)}$.

Now in view of (5.16.1), we have to describe the action of u , that is, the element $(\hat{\theta}_g(u), u)$ of $N_{L,u}(i_g^{\delta})$, on $\overline{T}(F, g, E)$. For this, we use (5.17.2), and we choose $s \in G$ such that $(s, \hat{\theta}_g(u)) \in N_{Q,\delta}$. Now for $\varphi \in \text{Hom}_{kC_G(Q)}(F, {}^{(g,1)}E)$, we have

$$\forall f \in F, \quad (u\varphi)(f) = (s^g, u)\varphi((s, \hat{\theta}_g(u))^{-1}f).$$

For each $\lambda \in \langle u \rangle^{\natural}$, an element of $\text{Hom}_{\langle u \rangle}(k_{\lambda}, \text{Hom}_{kC_G(Q)}(F^{Z(Q)}, {}^{(g,1)}E^{Z(Q)}))$ is now determined by an element $\varphi \in \text{Hom}_{kC_G(Q)}(F, {}^{(g,1)}E)$ such that

$$\forall f \in F, \quad \lambda(u)\varphi(f) = (s^g, u)\varphi((s, \hat{\theta}_g(u))^{-1}f).$$

In other words,

$$\forall f \in F, \quad \varphi((s, \hat{\theta}_g(u))f) = \lambda(u)^{-1}(s^g, u)\varphi(f).$$

Since $(s^g, u) = \Phi_g^{-1}((s, \hat{\theta}_g(u)))$, we get that

$$\forall f \in F, \quad \varphi((s, \hat{\theta}_g(u))f) = \lambda(u)^{-1}\Phi_g^{-1}((s, \hat{\theta}_g(u)))\varphi(f).$$

Now let $c \in C_G(Q)$. Then $(c, 1) \in N_{Q,\delta}$, and

$$\forall f \in F, \varphi((c, 1)f) = (c^g, 1)\varphi(f) = \lambda(1)^{-1}\Phi_g^{-1}((c, 1))\varphi(f).$$

Let $\overline{(a, b)}$ denote the image in $\overline{N}_{Q,\delta}$ of $(a, b) \in N_{Q,\delta}$. Now the element $\overline{(s, \hat{\theta}_g(u))}$, together with the elements $\overline{(c, 1)}$, for $c \in C_G(Q)$, generate the whole of $\overline{N}_{Q,\delta}$. It follows that for any $\overline{(a, b)} \in \overline{N}_{Q,\delta}$,

$$\varphi(\overline{(a, b)}f) = \lambda(\hat{\theta}_g^{-1}(b))^{-1}\bar{\Phi}_g^{-1}(\overline{(a, b)})\varphi(f).$$

In other words, φ is a morphism of $k\overline{N}_{Q,\delta}$ -modules from F to the module $^{[g,\lambda]}E$, equal to E as a k -vector space, but with action of $\overline{(a, b)} \in \overline{N}_{Q,\delta}$ given by

$$\forall e \in E, \overline{(a, b)} \cdot e \text{ (in } ^{[g,\lambda]}E) := \lambda(\hat{\theta}_g^{-1}(b))^{-1}\bar{\Phi}_g^{-1}(\overline{(a, b)}) \cdot e \text{ (in } E). \quad (5.17.3)$$

It follows that

$$m_\lambda(F, g, E) = \dim_k \operatorname{Hom}_{\overline{N}_{Q,\delta}}(F^{Z(Q)}, ^{[g,\lambda]}E^{Z(Q)}),$$

so our relations (5.16.2) become

$$\forall F \in \operatorname{Pim}(k\overline{N}_{Q,\delta}), \sum_{\substack{g \in \overline{G}_{Q,\delta} \\ \lambda \in \langle u \rangle^{\natural} \\ E \in \operatorname{Pim}^{\#}(k\overline{N}_{Q,\delta})}} \dim_k \operatorname{Hom}_{\overline{N}_{Q,\delta}}(F^{Z(Q)}, ^{[g,\lambda]}E^{Z(Q)}) kG_{\theta_g, \lambda} \cdot \nu_E = 0. \quad (5.17.4)$$

Remark 5.18. It should be noted that in (5.17.3), the coefficient $\lambda(\hat{\theta}_g^{-1}(b))^{-1}$ does not depend on the choice of $\hat{\theta}_g \in \operatorname{Aut}(L\langle u \rangle)$ in the class $\theta_g \in \operatorname{Out}(L\langle u \rangle)$, as two different choices differ by an inner automorphism, so the corresponding values $\hat{\theta}_g(b)$ are conjugate. Hence, we could write $\lambda(\theta_g^{-1}(b))^{-1}$ instead of $\lambda(\hat{\theta}_g^{-1}(b))^{-1}$.

5.19.

Suppose now that $g, h \in \widehat{G}_{Q,\delta}$, and $\lambda, \mu \in \langle u \rangle^{\natural}$. If E is a projective $k\overline{N}_{Q,\delta}$ -module, we claim that the $k\overline{N}_{Q,\delta}$ -modules $^{[h,\mu]}(^{[g,\lambda]}E)$ and $^{[hg, (\mu \circ \hat{\theta}_g) \cdot \lambda]}E$ are isomorphic. Indeed, there exists an element $w \in Z(L)$ such that $\hat{\theta}_{hg} = \hat{\theta}_h \hat{\theta}_g i_w$. Now $(1, w) \in N_{Q,\delta}$, and we can define $f : E \rightarrow E$ by $f(e) = \overline{(1, w^{-1})} \cdot e$. Then f is clearly an automorphism of the k -vector space E .

Claim. The map f is an isomorphism of $k\overline{N}_{Q,\delta}$ -modules from $^{[h,\mu]}(^{[g,\lambda]}E)$ to $^{[hg, (\mu \circ \hat{\theta}_g) \cdot \lambda]}E$.

Proof. Indeed, for $e \in E$, let $^{[g,\lambda]}e$ denote the element e of $^{[g,\lambda]}E$. Then for $(a, b) \in N_{Q,\delta}$,

$$\begin{aligned} \overline{(a, b)} \cdot ^{[h,\mu]}(^{[g,\lambda]}e) &= \mu(\hat{\theta}_h^{-1}(b))\bar{\Phi}_h^{-1}(\overline{(a, b)}) \cdot ^{[g,\lambda]}e \\ &= \mu(\hat{\theta}_h^{-1}(b))\overline{(a^h, \hat{\theta}_h^{-1}(b))} \cdot ^{[g,\lambda]}e \\ &= \mu(\hat{\theta}_h^{-1}(b))\lambda(\hat{\theta}_g^{-1}\hat{\theta}_h^{-1}(b))\overline{(a^{hg}, \hat{\theta}_g^{-1}\hat{\theta}_h^{-1}(b))} \cdot e. \end{aligned}$$

It follows that

$$f(\overline{(a, b)} \cdot ^{[h,\mu]}(^{[g,\lambda]}e)) = \mu(\hat{\theta}_h^{-1}(b))\lambda(\hat{\theta}_g^{-1}\hat{\theta}_h^{-1}(b))\overline{(a^{hg}, w^{-1}\hat{\theta}_g^{-1}\hat{\theta}_h^{-1}(b))} \cdot e. \quad (5.19.1)$$

On the other hand,

$$\begin{aligned} \overline{(a, b)} \cdot [hg, (\mu \circ \hat{\theta}_g) \cdot \lambda] f(e) &= ((\mu \circ \hat{\theta}_g) \cdot \lambda)(\hat{\theta}_{hg}^{-1}(b)) \overline{(a^{hg}, \hat{\theta}_{hg}^{-1}(b))} \cdot f(e) \\ &= \mu(\hat{\theta}_g \hat{\theta}_{hg}^{-1}(b)) \lambda(\hat{\theta}_{hg}^{-1}(b)) \overline{(a^{hg}, \hat{\theta}_{hg}^{-1}(b)w^{-1})} \cdot e. \end{aligned} \quad (5.19.2)$$

Since $\hat{\theta}_{hg} = \hat{\theta}_h \hat{\theta}_g i_w$, we have $i_w^{-1} \hat{\theta}_g^{-1} \hat{\theta}_h^{-1} = \hat{\theta}_{hg}^{-1}$, so $\hat{\theta}_{hg}^{-1}(b)w^{-1} = w^{-1} \hat{\theta}_g^{-1} \hat{\theta}_h^{-1}(b)$. Moreover,

$$\lambda(\hat{\theta}_{hg}^{-1}(b)) = \lambda(w^{-1} \hat{\theta}_g^{-1} \hat{\theta}_h^{-1}(b)w) = \lambda(\hat{\theta}_g^{-1} \hat{\theta}_h^{-1}(b)),$$

and

$$\begin{aligned} \mu(\hat{\theta}_g \hat{\theta}_{hg}^{-1}(b)) &= \mu(\hat{\theta}_g(w^{-1} \hat{\theta}_g^{-1} \hat{\theta}_h^{-1}(b)w)) \\ &= \mu(\hat{\theta}_g(w^{-1}) \hat{\theta}_h^{-1}(b) \theta_g(w)) = \mu(\hat{\theta}_h^{-1}(b)). \end{aligned}$$

Now it follows from (5.19.1) and (5.19.2) that

$$f\left(\overline{(a, b)} \cdot [h, \mu]([g, \lambda]e)\right) = \overline{(a, b)} \cdot [hg, (\mu \circ \hat{\theta}_g) \cdot \lambda] f(e),$$

proving the claim. $\square$

In addition to the above claim, we observe that if $g \in \widehat{G}_{Q, \delta}$ is actually in $G_{Q, \delta}$, then the $k\overline{N}_{Q, \delta}$ -modules E and $^{[g, 1]}E$ are isomorphic: Indeed, saying that $g \in G_{Q, \delta}$ amounts to saying that $\hat{\theta}_g$ is an inner automorphism i_x of $L\langle u \rangle$, for some $x \in L\langle u \rangle$ with $(g, x) \in N_{Q, \delta}$. Then (5.17.3) becomes

$$\begin{aligned} \forall e \in E, \quad \overline{(a, b)} \cdot e \text{ (in } ^{[g, 1]}E) &= \bar{\Phi}_g^{-1}(\overline{(a, b)}) \cdot e \text{ (in } E) \\ &= \overline{(a^g, b^x)} \cdot e. \end{aligned}$$

Then the map $e \in E \mapsto \overline{(g^{-1}, x^{-1})} \cdot e \in ^{[g, 1]}E$ is an isomorphism of $k\overline{N}_{Q, \delta}$ -modules.

We can now introduce the semidirect product

$$\overline{G}_{Q, \delta} \ltimes \langle u \rangle^{\natural} := (\widehat{G}_{Q, \delta} / G_{Q, \delta}) \ltimes \langle u \rangle^{\natural}.$$

As a set $\overline{G}_{Q, \delta} \ltimes \langle u \rangle^{\natural}$ is the Cartesian product $\overline{G}_{Q, \delta} \times \langle u \rangle^{\natural}$. For $g \in G_{Q, \delta}$ and $\lambda \in \langle u \rangle^{\natural}$, let $[g, \lambda]$ denote the pair $(gG_{Q, \delta}, \lambda)$ in $\overline{G}_{Q, \delta} \ltimes \langle u \rangle^{\natural}$. The product in $\overline{G}_{Q, \delta} \ltimes \langle u \rangle^{\natural}$ is given as follows: For $g, h \in G_{Q, \delta}$ and $\lambda, \mu \in \langle u \rangle^{\natural}$, we have

$$[h, \mu][g, \lambda] = [hg, (\mu \circ \theta_g) \cdot \lambda].$$

The above discussion now shows that there is an action of $\overline{G}_{Q, \delta} \ltimes \langle u \rangle^{\natural}$ on the group $\text{Proj}(k\overline{N}_{Q, \delta})$. The group $\overline{G}_{Q, \delta} \ltimes \langle u \rangle^{\natural}$ also acts on the set $\text{Pim}^{\sharp}(k\overline{N}_{Q, \delta})$ introduced at Section 5.9. Indeed the restriction of $^{[g, \lambda]}E$ to $C_G(Q)$ is isomorphic to the restriction of $^{[g, 1]}E$, and $\widehat{G}_{Q, \delta}$ permutes the indecomposable $kC_G(Q)$ -modules.

5.20.

Yet another (well-known) lemma⁸:

Lemma 5.21. *Let H be a finite group, and R be a normal p -subgroup of H .*

⁸We will not use Assertion 2 of this lemma here, but we state it for completeness.

1. The assignment $E \mapsto E^R$ induces a bijection between the set of isomorphism classes of indecomposable projective kH -modules and the set of isomorphism classes of indecomposable projective $k(H/R)$ -modules.
2. Moreover, if R is central in H , then for any projective kH -modules E and F ,

$$\dim_k \operatorname{Hom}_{kH}(E, F) = |R| \dim_{k(H/R)}(E^R, F^R).$$

Proof. 1. Let E be a projective kH -module, and M be any $k(H/R)$ -module. Then

$$\operatorname{Hom}_{kH}(E, \operatorname{Inf}_{H/R}^H M) \cong \operatorname{Hom}_{k(H/R)}(E_R, M) \cong \operatorname{Hom}_{k(H/R)}(E^R, M)$$

as $E_R \cong E^R$ if E is projective. Now the functor $M \mapsto \operatorname{Hom}_{kH}(E, \operatorname{Inf}_{H/R}^H M)$ is exact, since inflation is exact, so E^R is a projective $k(H/R)$ -module. Moreover, the simple kH -modules are inflated from H/R , and the previous isomorphism shows that if E is indecomposable, then E^R has a unique simple $k(H/R)$ -quotient, thus it is indecomposable.

2. If now R is central in H , then R acts on $\operatorname{Hom}_{kH}(E, F)$ by left multiplication, that is, $(r\varphi)(e) = r \cdot \varphi(e)$ for $r \in R$, $e \in E$, and $\varphi \in \operatorname{Hom}_{kH}(E, F)$. Moreover, this action is free, since if $E = F = kH$, then $\operatorname{Hom}_{kH}(E, F) \cong kH$ is free as a kR -module. Thus,

$$\begin{aligned} \dim_k \operatorname{Hom}_{kH}(E, F) &= |R| \dim_k (\operatorname{Hom}_{kH}(E, F))^R = |R| \dim_k \operatorname{Hom}_{kH}(E, F^R) \\ &= |R| \dim_k \operatorname{Hom}_{k(H/R)}(E_R, F^R) \\ &= |R| \dim_k \operatorname{Hom}_{k(H/R)}(E^R, F^R), \end{aligned}$$

as was to be shown. $\square$

5.22.

Let $\mathbb{F}R_k(\overline{N}_{Q,\delta}^b) = \mathbb{F} \otimes_{\mathbb{Z}} R_k(\overline{N}_{Q,\delta}^b)$ denote the Grothendieck group of finite-dimensional $k\overline{N}_{Q,\delta}^b$ -modules, extended by $\mathbb{F}$. For a projective $k\overline{N}_{Q,\delta}^b$ -module X , let $[X]$ denote its image in $\mathbb{F}R_k(\overline{N}_{Q,\delta}^b)$. The group $\overline{G}_{Q,\delta} \ltimes \langle u \rangle^{\natural}$ also acts on $R_k(\overline{N}_{Q,\delta}^b)$ by permutation of its base consisting of the isomorphism classes of simple modules.

We denote by $\Gamma_{Q,\delta}$ the image of the linear map

$$\gamma_{Q,\delta}^{L,u} : \mathbb{F}\operatorname{Pim}^{\#}(k\overline{N}_{Q,\delta}^b) \rightarrow \mathbb{F}R_k(\overline{N}_{Q,\delta}^b)$$

induced by $E \mapsto [E^{Z(Q)}]$. The map $\gamma_{Q,\delta}^{L,u}$ is a map of $\mathbb{F}(\overline{G}_{Q,\delta} \ltimes \langle u \rangle^{\natural})$ -modules, so its image $\Gamma_{Q,\delta}$ is also an $\mathbb{F}(\overline{G}_{Q,\delta} \ltimes \langle u \rangle^{\natural})$ -module.

The $\mathcal{E}_{\mathbb{F}}(L\langle u \rangle)$ -module V is also an $\mathbb{F}(\overline{G}_{Q,\delta} \ltimes \langle u \rangle^{\natural})$ -module, thanks to the (surjective) homomorphism of algebras sending $[g, \lambda] \in \mathbb{F}(\overline{G}_{Q,\delta} \ltimes \langle u \rangle^{\natural})$ to the image of $kG_{\theta_g, \lambda}$ in $\mathcal{E}_{\mathbb{F}}(L\langle u \rangle)$. Tensoring with V gives a surjective map

$$\gamma_{Q,\delta}^{L,u} \otimes \operatorname{Id}_V : \mathbb{F}\operatorname{Pim}^{\#}(k\overline{N}_{Q,\delta}^b) \otimes_{\mathbb{F}} V \rightarrow \Gamma_{Q,\delta} \otimes_{\mathbb{F}} V,$$

of $\mathbb{F}(\overline{G}_{Q,\delta} \ltimes \langle u \rangle^{\natural})$ -modules, where the action of $\overline{G}_{Q,\delta} \ltimes \langle u \rangle^{\natural}$ on both tensor products is diagonal.

Now rephrasing (5.10.3), we get a surjective map

$$\sigma = \bigoplus \sigma_{Q,\delta} : \bigoplus_{(Q,\delta) \in [\mathcal{P}(G,L,u)]} \mathbb{F}\operatorname{Pim}^{\#}(k\overline{N}_{Q,\delta}^b) \otimes_{\mathbb{F}} V \rightarrow S_{L\langle u \rangle, V}(G),$$

where $\sigma_{Q,\delta}$ sends $E \otimes v \in \mathbb{F}\operatorname{Pim}^{\#}(k\overline{N}_{Q,\delta}^b) \otimes_{\mathbb{F}} V$ to the image of $T(Q, \delta, E) \otimes v$ in $S_{L\langle u \rangle, V}(G)$.

The kernel of this map is the direct sum for $(Q, \delta) \in [\mathcal{P}(G, L, u)]$ of the kernels of its components $\sigma_{Q, \delta}$. By (5.17.4), the sum $\sum_{E \in \text{Pim}^\#(k\overline{N}_{Q, \delta})} E \otimes v_E$ is in the kernel of $\sigma_{Q, \delta}$ if and only if

$$\forall F \in \text{Pim}(k\overline{N}_{Q, \delta}), \sum_{\substack{[g, \lambda] \in \overline{G}_{Q, \delta} \ltimes \langle u \rangle^{\mathfrak{h}} \\ E \in \text{Pim}^\#(k\overline{N}_{Q, \delta})}} \dim_k \text{Hom}_{\overline{N}_{Q, \delta}^b} (F^{Z(Q)}, [g, \lambda] E^{Z(Q)}) [g, \lambda] \cdot v_E = 0. \quad (5.22.1)$$

Now $R_k(\overline{N}_{Q, \delta}^b)$ has a basis consisting of the $[X_F]$ for $F \in \text{Pim}(k\overline{N}_{Q, \delta})$, where X_F is the unique simple quotient of the $k\overline{N}_{Q, \delta}^b$ -module $F^{Z(Q)}$. So (5.22.1) is equivalent to

$$\sum_{F \in \text{Pim}(k\overline{N}_{Q, \delta})} [X_F] \otimes \sum_{\substack{[g, \lambda] \in \overline{G}_{Q, \delta} \ltimes \langle u \rangle^{\mathfrak{h}} \\ E \in \text{Pim}^\#(k\overline{N}_{Q, \delta})}} \dim_k \text{Hom}_{\overline{N}_{Q, \delta}^b} (F^{Z(Q)}, [g, \lambda] E^{Z(Q)}) [g, \lambda] \cdot v_E = 0$$

in $\mathbb{F}R_k(\overline{N}_{Q, \delta}) \otimes V$. This in turn is equivalent to

$$\sum_{F \in \text{Pim}(k\overline{N}_{Q, \delta})} \sum_{\substack{[g, \lambda] \in \overline{G}_{Q, \delta} \ltimes \langle u \rangle^{\mathfrak{h}} \\ E \in \text{Pim}^\#(k\overline{N}_{Q, \delta})}} \dim_k \text{Hom}_{\overline{N}_{Q, \delta}^b} (F^{Z(Q)}, [g, \lambda] E^{Z(Q)}) [X_F] \otimes [g, \lambda] \cdot v_E = 0.$$

Now for any $[g, \lambda] \in \overline{G}_{Q, \delta} \ltimes \langle u \rangle^{\mathfrak{h}}$ and any $E \in \text{Pim}^\#(k\overline{N}_{Q, \delta})$,

$$\sum_{F \in \text{Pim}(k\overline{N}_{Q, \delta})} \dim_k \text{Hom}_{\overline{N}_{Q, \delta}^b} (F^{Z(Q)}, [g, \lambda] E^{Z(Q)}) [X_F] = [[g, \lambda] E^{Z(Q)}] = [g, \lambda] \gamma_{Q, \delta}^{L, u}(E).$$

It follows that $\sum_{E \in \text{Pim}^\#(k\overline{N}_{Q, \delta})} E \otimes v_E$ is in the kernel of $\sigma_{Q, \delta}$ if and only if

$$\sum_{E \in \text{Pim}^\#(k\overline{N}_{Q, \delta})} \sum_{[g, \lambda] \in \overline{G}_{Q, \delta} \ltimes \langle u \rangle^{\mathfrak{h}}} [g, \lambda] \gamma_{Q, \delta}^{L, u}(E) \otimes [g, \lambda] \cdot v_E = 0$$

in $\mathbb{F}R_k(\overline{N}_{Q, \delta}) \otimes V$. In other words, $\sigma_{Q, \delta}$ has the same kernel as the map

$$\sum_{E \in \text{Pim}^\#(k\overline{N}_{Q, \delta})} E \otimes v_E \mapsto \sum_{[g, \lambda] \in \overline{G}_{Q, \delta} \ltimes \langle u \rangle^{\mathfrak{h}}} [g, \lambda] (\gamma_{Q, \delta}^{L, u} \otimes \text{Id}) \left(\sum_{E \in \text{Pim}^\#(k\overline{N}_{Q, \delta})} E \otimes v_E \right).$$

It follows that the image of $\sigma_{Q, \delta}$ is isomorphic to the image of the previous map, that is,

$$\text{Im}(\sigma_{Q, \delta}) \cong \text{Tr}_1^{\overline{G}_{Q, \delta} \ltimes \langle u \rangle^{\mathfrak{h}}} (\Gamma_{Q, \delta} \otimes_{\mathbb{F}} V).$$

We finally get the following theorem.

Theorem 5.23. *There is an isomorphism of k -vector spaces*

$$S_{L\langle u \rangle, V}(G) \cong \bigoplus_{(Q, \delta) \in [\mathcal{P}(G, L, u)]} \text{Tr}_1^{\overline{G}_{Q, \delta} \ltimes \langle u \rangle^{\mathfrak{h}}} (\Gamma_{Q, \delta} \otimes_{\mathbb{F}} V).$$

5.24. The simple diagonal p -permutation functors.

Now we use Theorem 5.23 to describe the simple functors. We assume that $\mathbb{F}$ is algebraically closed, of characteristic 0 or p . Recall that we have an isomorphism of algebras

$$\mathcal{E}_{\mathbb{F}}(L\langle u \rangle) \cong \text{Out}(L\langle u \rangle) \ltimes \mathbb{F}\overline{R}_k(\langle u \rangle).$$

In order to describe the simple $\mathcal{E}_{\mathbb{F}}(L\langle u \rangle)$ -modules, we set $A = \mathbb{F}\overline{R}_k(\langle u \rangle)$ and $\Omega = \text{Out}(L\langle u \rangle)$, and we use the results of [12] in our specific situation, as in Section 4.2 of [10]. First, the simple A -modules are one-dimensional, of the form $\mathbb{F}_x$, where x is a generator of $\langle u \rangle$. For such an x , we get an algebra homomorphism

$$e_x : A = \mathbb{F}\overline{R}_k(\langle u \rangle) \rightarrow \mathbb{F}$$

defined by

$$\forall \lambda \in \langle u \rangle^{\natural}, \quad e_x(\lambda) = \widetilde{\lambda}(x),$$

where $\widetilde{\lambda} : \langle u \rangle \rightarrow \mathbb{F}^{\times}$ is the character obtained from $\lambda : \langle u \rangle \rightarrow k^{\times}$ by composition with a fixed isomorphism from the group of roots of unity in k to the group of p' -roots of unity in $\mathbb{F}$. This makes sense because if λ is induced from a proper subgroup of $\langle u \rangle$, then $\widetilde{\lambda}(x) = 0$, as x cannot be contained in a proper subgroup of $\langle u \rangle$. So e_x extends to an algebra homomorphism $A = \mathbb{F}\overline{R}_k(\langle u \rangle) \rightarrow \mathbb{F}$, which in turn yields a one-dimensional A -module $\mathbb{F}_x$. We get all the simple A -modules in this way.

We also abusively denote by $\widetilde{\lambda}$ the composition $L\langle u \rangle \longrightarrow \langle u \rangle \xrightarrow{\widetilde{\lambda}} \mathbb{F}^{\times}$.

Now the stabilizer Ω_x of $\mathbb{F}_x$ in $\Omega = \text{Out}(L\langle u \rangle)$ is the set of classes of $\gamma \in \text{Aut}(L\langle u \rangle)$ such that $\gamma(x) = x$. This does not depend on the generator x of $\langle u \rangle$, and it is equal to $\text{Out}(L, u)$. We note that Ω_x acts trivially on A , since it acts trivially on $\langle u \rangle$, so $\Omega_x \ltimes A = \Omega_x \times A$.

So any simple $\mathcal{E}_{\mathbb{F}}(L\langle u \rangle)$ -module V is of the form

$$V = \text{Ind}_{\Omega_x \times A}^{\Omega \ltimes A}(W \otimes \mathbb{F}_x),$$

for some generator x of $\langle u \rangle$ and some simple $\mathbb{F}\Omega_x$ -module W . The action of $\Omega_x \times A$ on $W \otimes \mathbb{F}_x$ is given by

$$\forall \gamma \in \Omega_x, \forall a \in A, \forall w \in W, \quad (\gamma, a) \cdot (w \otimes 1) = e_x(a)((\gamma \cdot w) \otimes 1).$$

Moreover, the isomorphism type of the simple $\mathcal{E}_{\mathbb{F}}(L\langle u \rangle)$ -module V is determined by the isomorphism type of the simple $\mathbb{F}\text{Out}(L, u)$ -module W , and the choice of the generator x of $\langle u \rangle$, up to the action of Ω , that is, up to the action of the subgroup $\text{Aut}(L\langle u \rangle)^{\sharp}$ of $\text{Aut}(L\langle u \rangle)$ consisting of automorphisms which stabilize $\langle u \rangle$.

Then (L, x) is a D^{Δ} -pair, such that $\text{Out}(L, u) = \text{Out}(L, x)$, and choosing x up to the action of $\text{Aut}(L\langle u \rangle)^{\sharp}$ amounts to choosing (L, x) in a set of D^{Δ} -pairs such that $L\langle x \rangle = L\langle u \rangle$, up to isomorphism of D^{Δ} -pairs. So up to changing (L, u) to (L, x) , we can parameterize the simple functor $S_{L\langle u \rangle, V}$ by the triple (L, x, W) instead, that is, we can suppose $x = u$ in the previous calculations, and set $S_{L, u, W} = S_{L\langle u \rangle, V}$, where

$$V = \text{Ind}_{\text{Out}(L, u) \times \mathbb{F}\overline{R}_k(\langle u \rangle)}^{\text{Out}(L\langle u \rangle) \ltimes \mathbb{F}\overline{R}_k(\langle u \rangle)}(W \otimes \mathbb{F}_u).$$

By Theorem 5.23, we have that

$$S_{L, u, W}(G) \cong \bigoplus_{(Q, \delta) \in [\mathcal{P}(G, L, u)]} \text{Tr}_1^{\overline{G}_{Q, \delta \ltimes \langle u \rangle^{\natural}}}(\Gamma_{Q, \delta} \otimes_{\mathbb{F}} V),$$

so we must describe the action of $\overline{G}_{Q,\delta} \ltimes \langle u \rangle^{\natural}$ on V . We recall from 5.14 that the group homomorphism

$$g \in \widehat{G}_{Q,\delta} \rightarrow \theta_g \in \text{Out}(L\langle u \rangle)$$

induces an embedding $\overline{G}_{Q,\delta} \hookrightarrow \text{Out}(L\langle u \rangle)$, and we identify $\overline{G}_{Q,\delta}$ with its image via this embedding.

Now the semidirect product $\text{Out}(L\langle u \rangle) \ltimes \langle u \rangle^{\natural}$ embeds in $\text{Out}(L\langle u \rangle) \ltimes \mathbb{F}\overline{R}_k(\langle u \rangle)$ via $[\theta, \lambda] \mapsto \theta \ltimes \overline{\lambda}$, for $\theta \in \text{Out}(L\langle u \rangle)$ and $\lambda \in \langle u \rangle^{\natural}$. In particular, it acts on V . We observe moreover that there is an isomorphism of $\mathbb{F}$ -vector spaces

$$V \cong \bigoplus_{\psi \in \text{Out}(L\langle u \rangle)/\text{Out}(L,u)} (\psi \otimes W). \quad (*)$$

With this decomposition, for $\theta \in \text{Aut}(L\langle u \rangle)$, $\lambda \in \langle u \rangle^{\natural}$, $\psi \in \text{Out}(L\langle u \rangle)/\text{Out}(L,u)$, and $w \in W$, we have

$$\begin{aligned} [\theta, \lambda] \cdot (\psi \otimes w) &= (\theta \ltimes \overline{\lambda})(\psi \ltimes \overline{1}) \cdot (\text{Id} \otimes w) \\ &= ((\theta \circ \psi) \ltimes (\overline{\lambda} \circ \overline{\psi})) \cdot (\text{Id} \otimes w) \\ &= ((\theta \circ \psi) \ltimes \overline{1})(\text{Id} \ltimes (\overline{\lambda} \circ \overline{\psi})) \cdot (\text{Id} \otimes w) \\ &= \widetilde{\lambda}(\psi(u))((\theta \circ \psi) \ltimes \overline{1}) \cdot (\text{Id} \otimes w) \\ &= \widetilde{\lambda}(\psi(u))(\theta \ltimes \overline{1})(\psi \ltimes \overline{1})(\text{Id} \otimes w) \\ &= \widetilde{\lambda}(\psi(u))(\theta \ltimes \overline{1}) \cdot (\psi \otimes w) \\ &= \widetilde{\lambda}(\psi(u)) \theta \cdot (\psi \otimes w), \end{aligned}$$

where $\psi \otimes w \mapsto \theta \cdot (\psi \otimes w)$ denotes the action of θ on $\psi \otimes w \in \text{Ind}_{\text{Out}(L,u)}^{\text{Out}(L\langle u \rangle)} W$.

So the group $H := \text{Out}(L\langle u \rangle) \ltimes \langle u \rangle^{\natural}$ permutes the components $\psi \otimes W$ of the direct sum $(*)$, and it permutes them transitively. The stabilizer of the component $\text{Id} \otimes W$ is equal to $H_1 := \text{Out}(L,u) \ltimes \langle u \rangle^{\natural}$, and $H_1 = \text{Out}(L,u) \times \langle u \rangle^{\natural}$ as $\text{Out}(L,u)$ acts trivially on $\langle u \rangle^{\natural}$. The group H_1 acts on $\text{Id} \otimes W$ by

$$\forall (\theta, \lambda) \in H_1, (\theta, \lambda) \cdot (\text{Id} \otimes w) = \widetilde{\lambda}(u)(\text{Id} \otimes \theta \cdot w).$$

It follows that there is an isomorphism of $\mathbb{F}H$ -modules

$$V \cong \text{Ind}_{H_1}^H W,$$

where the action of $(\theta, \lambda) \in H_1$ on W is given by

$$(\theta, \lambda) \cdot w = \widetilde{\lambda}(u) \theta \cdot w.$$

Now we consider the restriction of V to $K_{Q,\delta} := \overline{G}_{Q,\delta} \ltimes \langle u \rangle^{\natural}$ and use the Mackey formula. The map

$$\psi \in \text{Out}(L\langle u \rangle) \mapsto K_{Q,\delta}(\psi \ltimes 1)H_1$$

induces a bijection from $\overline{G}_{Q,\delta} \backslash \text{Out}(L\langle u \rangle) / \text{Out}(L,u)$ to $K_{Q,\delta} \backslash H / H_1$. This gives

$$\text{Res}_{K_{Q,\delta}}^H V \cong \bigoplus_{\psi \in [\overline{G}_{Q,\delta} \backslash \text{Out}(L\langle u \rangle) / \text{Out}(L,u)]} \text{Ind}_{O_\psi}^{K_{Q,\delta}(\psi \ltimes 1)} \text{Res}_{O_\psi}^{H_1} W, \quad (*)$$

where

$$\begin{aligned} O_\psi &= K_{Q,\delta}^{\psi \ltimes 1} \cap H_1 \\ O_\psi &= K_{Q,\delta} \cap \psi \ltimes 1 H_1 = \psi \ltimes 1 (O_\psi). \end{aligned}$$

Now $K_{Q,\delta}^{\psi \times 1} = (\overline{G}_{Q,\delta})^\psi \ltimes \langle u \rangle^{\natural}$. Moreover,

$$\begin{aligned} (\overline{G}_{Q,\delta})^\psi &= \{ \theta^\psi \mid \theta \in \text{Out}(L\langle u \rangle), \exists g \in N_G(Q), \theta|_L = (i_g)^\delta \} \\ &= \{ \theta \in \text{Out}(L\langle u \rangle) \mid \exists g \in N_G(Q), (\psi\theta)|_L = (i_g)^\delta \} \\ &= \{ \theta \in \text{Out}(L\langle u \rangle) \mid \exists g \in N_G(Q), \theta|_L = (i_g)^{\delta\psi} \} \\ &= \overline{G}_{Q,\delta\psi}. \end{aligned}$$

Hence,

$$\psi O = (\overline{G}_{\delta\psi} \ltimes \langle u \rangle^{\natural}) \cap (\text{Out}(L, u) \times \langle u \rangle^{\natural}) = \overline{G}_{Q,\delta\psi,u} \times \langle u \rangle^{\natural},$$

where we have set

$$\overline{G}_{Q,\delta\psi,u} = \{ \theta \in \text{Out}(L, u) \mid \exists g \in N_G(Q), \theta|_L = (i_g)^{\delta\psi} \}.$$

From (*), and omitting the appropriate restriction symbols before $\Gamma_{Q,\delta}$, we now get

$$\begin{aligned} \text{Tr}_1^{K_{Q,\delta}}(\Gamma_{Q,\delta} \otimes \text{Res}_{K_{Q,\delta}}^H V) &\cong \bigoplus_{\psi \in [\overline{G}_{Q,\delta} \backslash \text{Out}(L\langle u \rangle) / \text{Out}(L, u)]} \text{Tr}_1^{\psi \times 1(\psi O)}(\Gamma_{Q,\delta} \otimes {}^{(\psi \times 1)}\text{Res}_{\psi O}^{H_1} W) \\ &\cong \bigoplus_{\psi \in [\overline{G}_{Q,\delta} \backslash \text{Out}(L\langle u \rangle) / \text{Out}(L, u)]} \text{Tr}_1^{\psi O}((\Gamma_{Q,\delta})^{\psi \times 1} \otimes \text{Res}_{\psi O}^{H_1} W) \\ &\cong \bigoplus_{\psi \in [\overline{G}_{Q,\delta} \backslash \text{Out}(L\langle u \rangle) / \text{Out}(L, u)]} \text{Tr}_1^{\psi O}(\Gamma_{Q,\delta\psi} \otimes \text{Res}_{\psi O}^{H_1} W). \end{aligned}$$

This finally gives

$$\begin{aligned} S_{L,u,W}(G) &\cong \bigoplus_{\substack{(Q,\delta) \in [G \backslash \mathcal{P}(G, L, u) / \text{Out}(L\langle u \rangle)] \\ \psi \in [\overline{G}_{Q,\delta} \backslash \text{Out}(L\langle u \rangle) / \text{Out}(L, u)]}} \text{Tr}_1^{\overline{G}_{Q,\delta\psi,u} \times \langle u \rangle^{\natural}}(\Gamma_{Q,\delta\psi} \otimes \text{Res}_{\overline{G}_{Q,\delta\psi,u} \times \langle u \rangle^{\natural}}^{\text{Out}(L, u) \times \langle u \rangle^{\natural}} W) \\ &\cong \bigoplus_{(Q,\delta) \in [G \backslash \mathcal{P}(G, L, u) / \text{Out}(L, u)]} \text{Tr}_1^{\overline{G}_{Q,\delta,u} \times \langle u \rangle^{\natural}}(\Gamma_{Q,\delta} \otimes \text{Res}_{\overline{G}_{Q,\delta,u} \times \langle u \rangle^{\natural}}^{\text{Out}(L, u) \times \langle u \rangle^{\natural}} W) \\ &= \bigoplus_{(Q,\delta) \in [G \backslash \mathcal{P}(G, L, u) / \text{Out}(L, u)]} \text{Tr}_1^{\overline{G}_{Q,\delta,u} \times \langle u \rangle^{\natural}}(\Gamma_{Q,\delta} \otimes W), \end{aligned}$$

where we have omitted the restriction symbol $\text{Res}_{\overline{G}_{Q,\delta,u} \times \langle u \rangle^{\natural}}^{\text{Out}(L, u) \times \langle u \rangle^{\natural}}$ in the last line.

Moreover, $\text{Tr}_1^{\overline{G}_{Q,\delta,u} \times \langle u \rangle^{\natural}} = \text{Tr}_1^{\overline{G}_{Q,\delta}} \circ \text{Tr}_1^{\langle u \rangle^{\natural}}$, so we first compute $\text{Tr}_1^{\langle u \rangle^{\natural}}(\Gamma_{Q,\delta} \otimes W)$. Let $\gamma \in \Gamma_{Q,\delta}$ and $w \in W$. Then

$$\begin{aligned} \text{Tr}_1^{\langle u \rangle^{\natural}}(\gamma \otimes w) &= \sum_{\lambda \in \langle u \rangle^{\natural}} [1, \lambda] \cdot (\gamma \otimes w) \\ &= \sum_{\lambda \in \langle u \rangle^{\natural}} ([1, \lambda] \cdot \gamma) \otimes ([1, \lambda] \cdot w) \\ &= \left(\sum_{\lambda \in \langle u \rangle^{\natural}} \tilde{\lambda}(u) [1, \lambda] \cdot \gamma \right) \otimes w. \end{aligned} \tag{**}$$

Let

$$\Xi_{Q,\delta} := \{m \in \Gamma_{Q,\delta} \mid \forall \lambda \in \langle u \rangle^{\natural}, [1, \lambda] \cdot m = \tilde{\lambda}(u)^{-1} m\}.$$

Then one checks easily that the map

$$\gamma \in \Gamma_{Q,\delta} \mapsto \frac{1}{|u|} \sum_{\lambda \in \langle u \rangle^{\natural}} \tilde{\lambda}(u) [1, \lambda] \cdot \gamma$$

is an idempotent endomorphism of $\Gamma_{Q,\delta}$, with image $\Xi_{Q,\delta}$. It follows from (**) that

$$\mathrm{Tr}_1^{\langle u \rangle^{\natural}}(\Gamma_{Q,\delta} \otimes W) = \Xi_{Q,\delta} \otimes W.$$

Hence,

$$S_{L,u,W}(G) \cong \bigoplus_{(Q,\delta) \in [G \backslash \mathcal{P}(G,L,u)/\mathrm{Out}(L,u)]} \mathrm{Tr}_1^{\overline{G}_{Q,\delta}}(\Xi_{Q,\delta} \otimes_{\mathbb{F}} W).$$

Now the space $\Xi_{Q,\delta}$ is the image by the map $\gamma_{Q,\delta}^{L,u}$ of the corresponding subspace

$$\mathbb{F}\mathrm{Pim}^{\#}(k\overline{N}_{Q,\delta})^0 = \{n \in \mathbb{F}\mathrm{Pim}^{\#}(k\overline{N}_{Q,\delta}) \mid \forall \lambda \in \langle u \rangle^{\natural}, n_{\lambda} = \tilde{\lambda}(u)^{-1} n\}$$

of $\mathbb{F}\mathrm{Pim}^{\#}(k\overline{N}_{Q,\delta})$. This space has a basis consisting of the sums $\sum_{\lambda \in \langle u \rangle^{\natural}} \lambda(u) E_{\lambda}$, where E runs through a set Σ of representatives of orbits of $\mathrm{Pim}^{\#}(k\overline{N}_{Q,\delta})$ under the action of $\langle u \rangle^{\natural}$. Since $\mathrm{Pim}^{\#}(k\overline{N}_{Q,\delta})$ is a $(\overline{G}_{Q,\delta,u}, \langle u \rangle^{\natural})$ -biset, this set of representatives can moreover be chosen invariant by the action of $\overline{G}_{Q,\delta,u}$.

Now the restriction map $\mathrm{Res}_{C_G(Q)}^{\overline{N}_{Q,\delta}}$ induces a bijection from Σ to the set $\mathrm{Pim}(kC_G(Q), u)$ of isomorphism classes of u -invariant indecomposable projective $kC_G(Q)$ -modules, and this bijection is $\overline{G}_{Q,\delta,u}$ -equivariant. Moreover, taking fixed points by $Z(Q)$ gives a bijection from $\mathrm{Pim}(kC_G(Q), u)$ to the set $\mathrm{Pim}(kC_G(Q)/Z(Q), u)$ of isomorphism classes of u -invariant indecomposable projective $kC_G(Q)/Z(Q)$ -modules.

Finally, we have proved the following theorem.

Theorem 5.25. *Let $\mathbb{F}$ be a field of characteristic 0 or p .*

1. *The simple diagonal p -permutation functors over $\mathbb{F}$ are parameterized by triples (L, u, W) , where (L, u) is a D^{Δ} -pair and W is a simple $\mathbb{F}\mathrm{Out}(L, u)$ -module.*
2. *The evaluation at a finite group G of the simple functor $S_{L,u,W}$ parameterized by the triple (L, u, W) is*

$$S_{L,u,W}(G) \cong \bigoplus_{(Q,\delta) \in [G \backslash \mathcal{P}(G,L,u)/\mathrm{Out}(L,u)]} \mathrm{Tr}_1^{\overline{G}_{Q,\delta,u}}(\mathbb{F}\mathrm{Cart}(kC_G(Q)/Z(Q), u) \otimes_{\mathbb{F}} W),$$

where $\mathbb{F}\mathrm{Cart}(kC_G(Q)/Z(Q), u)$ is the image of the map

$$\mathbb{F}\mathrm{Pim}(kC_G(Q)/Z(Q), u) \rightarrow \mathbb{F}R_k(C_G(Q)/Z(Q))$$

induced by the Cartan map.

6. Examples

6.1. The functor $S_{1,1,\mathbb{F}}$.

We apply Theorem 5.25 to the case, where $L = \mathbf{1}$, so $u = 1$, and $W = \mathbb{F}$. For a finite group G , we get that $(Q, \delta) \in \mathcal{P}(G, \mathbf{1}, 1)$ if and only if $Q = \mathbf{1}$ and $\delta : L \rightarrow Q$ is the identity. Moreover, $\overline{G}_{Q,\delta,1} = 1$, so by Theorem 5.25, we get that

$$S_{1,1,\mathbb{F}}(G) \cong \mathbb{F}\text{Cart}(G),$$

where $\mathbb{F}\text{Cart}(G)$ is the image of the map $\mathbb{F}\text{Proj}(kG) \rightarrow \mathbb{F}R_k(G)$ induced by the Cartan map $c^G : \text{Proj}(kG) \rightarrow R_k(G)$. We now show that the previous isomorphism is quite natural.

For this, we observe that the assignments $G \mapsto \text{Proj}(kG)$ and $G \mapsto R_k(G)$ are diagonal p -permutation functors: If M is a diagonal p -permutation (kH, kG) -bimodule, then M is left and right projective, so if Λ is a projective kG -module, then $M \otimes_{kG} \Lambda$ is a projective kH -module. Similarly, the functor $M \otimes_{kG} -$ changes a short exact sequence of kG -modules into a short exact sequence of kH -modules.

Moreover, the Cartan maps c^G form a morphism of diagonal p -permutation functors

$$c : \text{Proj}(k-) \rightarrow R_k(-).$$

In particular, the assignment $\mathbb{F}\text{Cart}(-) : G \mapsto \mathbb{F}\text{Cart}(G)$ is a subfunctor of $\mathbb{F}R_k(-)$.

Lemma 6.2. *The functor $\mathbb{F}\text{Cart}(-)$ is the unique minimal subfunctor of $\mathbb{F}R_k(-)$. It is isomorphic to the simple functor $S_{1,1,\mathbb{F}}$.*

Proof. Let F be a subfunctor of $\mathbb{F}R_k(-)$. Then $F(\mathbf{1}) \leq \mathbb{F}R_k(\mathbf{1}) = \mathbb{F}$, so $F(\mathbf{1})$ is either 0 or $\mathbb{F}$. Suppose first that $F(\mathbf{1}) = 0$. Let G be a finite group, and $u \in F(G)$. Then $u = \sum_{S \in \text{Irr}_k(G)} \lambda_S S$, where $\lambda_S \in \mathbb{F}$. Let $T \in \text{Irr}_k(G)$, and P_T be its projective cover. Then $P_T \in \mathbb{F}T^\Delta(\mathbf{1}, G)$, so $P_T \otimes_{kG} u \in F(\mathbf{1}) = 0$. But for $S \in \text{Irr}_k(G)$, we have $P_T \otimes_{kG} S = 0$ unless S is isomorphic to the dual $T^\natural$ of T . It follows that $\lambda_{T^\natural} = 0$ for any $T \in \text{Irr}_k(G)$, so $u = 0$. Hence, $F = 0$ if $F(\mathbf{1}) = 0$.

Suppose now that $F(\mathbf{1}) = \mathbb{F}$, that is, $F(\mathbf{1}) \ni k$. If $T \in \text{Irr}_k(G)$, then $P_T \in \mathbb{F}T^\Delta(G, \mathbf{1})$, so $P_T \otimes_k k \in F(G)$, for any $T \in \text{Irr}_k(G)$. But $P_T \otimes_k k \cong P_T$ is the image of P_T by the Cartan map. It follows that $F(G)$ contains $\mathbb{F}\text{Cart}(G)$, so $F \geq \mathbb{F}\text{Cart}(-)$. Hence, $\mathbb{F}\text{Cart}(-)$ is the unique (non-zero) minimal subfunctor of $\mathbb{F}R_k$. Since $\mathbb{F}\text{Cart}(\mathbf{1}) \cong \mathbb{F}$, it follows that $\mathbb{F}\text{Cart}(-) \cong S_{1,1,\mathbb{F}}$, completing the proof. $\square$

Remark 6.3. When $\mathbb{F}$ has characteristic p , the functor $\mathbb{F}\text{Cart}(-)$ is a proper subfunctor of $\mathbb{F}R_k(-)$, so Lemma 6.2 shows in particular that the category $\mathcal{F}_{\mathbb{F}p p_k}^\Delta$ is not semisimple.

6.4.

In the case, $\mathbb{F}$ has characteristic 0, the Cartan matrix has non-zero determinant in $\mathbb{F}$, so the Cartan map $\mathbb{F}c^G : \mathbb{F}\text{Proj}(kG) \rightarrow \mathbb{F}R_k(G)$ is invertible. So we have isomorphisms of functors

$$\mathbb{F}\text{Proj}(k-) \cong \mathbb{F}R_k(-) \cong S_{1,1,\mathbb{F}}$$

in this case. This is Theorem 5.20 in [6].

6.5.

The other case we can consider is when $\mathbb{F}$ is a field of characteristic p , and we assume that $\mathbb{F} = k$. We choose a p -modular system $(K, \mathcal{O}, k)$, and we assume that K is big enough for the group G . If S is a simple kG -module, we denote by $\Phi_S : G_{p'} \rightarrow \mathcal{O}$ the modular character of P_S , where $G_{p'}$ is the set of p -regular elements of G . If $v = \sum_{S \in \text{Irr}_k(G)} \omega_S P_S$, where $\omega_S \in \mathcal{O}$, is an element of $\mathcal{O}\text{Proj}(kG)$, we

denote by $\Phi_v : \mathcal{O}\text{Proj}(kG) \rightarrow \mathcal{O}$ the map $\sum_{S \in \text{Irr}_k(G)} \omega_S \Phi_S : G_{p'} \rightarrow \mathcal{O}$, and we call Φ_v the modular character of v .

Then for a simple kG -module T , the multiplicity of S as a composition factor of P_T is equal to the Cartan coefficient

$$c_{T,S}^G = \dim_k \text{Hom}_{kG}(P_T, P_S) = \frac{1}{|G|} \sum_{x \in G_{p'}} \Phi_T(x) \Phi_S(x^{-1}).$$

In order to describe the image of the Cartan map kc^G , we want to evaluate the image of this integer under the projection map $\rho : \mathcal{O} \rightarrow k$. For this, we denote by $[G_{p'}]$ a set of representatives of conjugacy classes of $G_{p'}$, and we observe that in the field K , we have

$$\begin{aligned} c_{S,T}^G &= \frac{1}{|G|} \sum_{x \in [G_{p'}]} \frac{|G|}{|C_G(x)|} \Phi_T(x) \Phi_S(x^{-1}) \\ &= \sum_{x \in [G_{p'}]} \frac{1}{|C_G(x)|} \frac{\Phi_T(x)}{|C_G(x)|_p} \frac{\Phi_S(x^{-1})}{|C_G(x)|_p} |C_G(x)|_p^2 \\ &= \sum_{x \in [G_{p'}]} \frac{(\Phi_T(x)/|C_G(x)|_p)(\Phi_S(x^{-1})/|C_G(x)|_p)}{|C_G(x)_{p'}|} |C_G(x)|_p. \end{aligned} \quad (6.5.1)$$

But since Φ_S and Φ_T are characters of projective kG -modules (and since $C_G(x) = C_G(x^{-1})$), the quotients $\Phi_T(x)/|C_G(x)|_p$ and $\Phi_S(x^{-1})/|C_G(x)|_p$ are in $\mathcal{O}$, so

$$\forall x \in [G_{p'}], \frac{(\Phi_T(x)/|C_G(x)|_p)(\Phi_S(x^{-1})/|C_G(x)|_p)}{|C_G(x)_{p'}|} \in \mathcal{O}.$$

Now it follows from (6.5.1) that

$$\rho(c_{T,S}^G) = \sum_{x \in [G_0]} \rho \left(\frac{\Phi_T(x) \Phi_S(x^{-1})}{|C_G(x)|} \right), \quad (6.5.2)$$

where $[G_0]$ is a set of representatives of conjugacy classes of the set G_0 of elements of *defect zero* of G , that is, the set of p' -elements x of G such that $C_G(x)$ is a p' -group.

Notation 6.6. For $x \in G_0$, we set

$$\Gamma_{G,x} = \sum_{S \in \text{Irr}(kG)} \frac{\Phi_S(x^{-1})}{|C_G(x)|} S \in \mathcal{O}R_k(G),$$

where $\text{Irr}(kG)$ is a set of representatives of isomorphism classes of simple kG -modules. We also set

$$\gamma_{G,x} = \sum_{S \in \text{Irr}(kG)} \rho \left(\frac{\Phi_S(x^{-1})}{|C_G(x)|} \right) S \in kR_k(G).$$

Remark 6.7. We note that $\Gamma_{G,x}$ and $\gamma_{G,x}$ only depend on the conjugacy class of x in G , that is, $\Gamma_{G,x} = \Gamma_{G,xg}$ and $\gamma_{G,x} = \gamma_{G,xg}$ for $g \in G$.

By Theorem 6.3.2 of [8] (see also Theorem 6.3.2 of [13]), the elementary divisors of the Cartan matrix of G are equal to $|C_G(x)|_p$, for $x \in [G_{p'}]$. It follows that the rank of the Cartan matrix modulo p is equal to the number of conjugacy classes of elements of defect 0 of G , that is, the cardinality of $[G_0]$. The following can be viewed as an explicit form of this result.

Proposition 6.8.

1. Let T be a simple kG -module. Then, in $kR_k(G)$,

$$kc^G(P_T) = \sum_{x \in [G_0]} \rho(\Phi_T(x)) \gamma_{G,x}.$$

2. The elements $\gamma_{G,x}$, for $x \in [G_0]$, form a basis of $k\text{Cart}(G) \leq kR_k(G)$.

Proof. Throughout the proof, we simply write γ_x instead of $\gamma_{G,x}$.

1. By (6.5.2), we have

$$\begin{aligned} kc^G(P_T) &= \sum_{S \in \text{Irr}(kG)} \rho(c_{T,S}^G) S = \sum_{S \in \text{Irr}(kG)} \sum_{x \in [G_0]} \rho\left(\frac{\Phi_T(x)\Phi_S(x^{-1})}{|C_G(x)|}\right) S \\ &= \sum_{x \in [G_0]} \sum_{S \in \text{Irr}(kG)} \rho\left(\frac{\Phi_T(x)\Phi_S(x^{-1})}{|C_G(x)|}\right) S \\ &= \sum_{x \in [G_0]} \rho(\Phi_T(x)) \sum_{S \in \text{Irr}(kG)} \rho\left(\frac{\Phi_S(x^{-1})}{|C_G(x)|}\right) S \\ &= \sum_{x \in [G_0]} \rho(\Phi_T(x)) \gamma_x. \end{aligned}$$

2. We first prove that γ_x lies in the image of kc^G , for any $x \in G_0$. So let $x \in G_0$, and $1_x : \langle x \rangle \rightarrow \mathcal{O}$ be the map with value 1 at x and 0 elsewhere. Then $|x|1_x = \sum_{\zeta} \zeta(x^{-1})\zeta$, where ζ runs through the simple $k\langle x \rangle$ -modules, that is, the group homomorphisms $\langle x \rangle \rightarrow \mathcal{O}^\times$, is an element of $\mathcal{O}P_k(\langle x \rangle) = \mathcal{O}R_k(\langle x \rangle)$. Let $v_x = \text{Ind}_{\langle x \rangle}^G(|x|1_x)$. Then $v_x \in \mathcal{O}P_k(G)$, and its modular character evaluated at $g \in G$ is equal to

$$\begin{aligned} \Phi_{v_x}(g) &= \frac{1}{|x|} \sum_{\substack{h \in G \\ g^h \in \langle x \rangle}} \Phi_{|x|1_x}(g^h) \\ &= \frac{1}{|x|} \sum_{\substack{h \in G \\ g^h = x}} |x| = \begin{cases} |C_G(x)| & \text{if } g =_G x \\ 0 & \text{otherwise,} \end{cases} \end{aligned} \quad (6.8.1)$$

where $g =_G x$ means that g is conjugate to x in G . Now from Assertion 1, we get that

$$kc^G(v_x) = \sum_{y \in [G_0]} \rho(\Phi_{v_x}(y)) \gamma_y = |C_G(x)| \gamma_x, \quad (6.8.2)$$

so γ_x is in the image of kc^G , since $|C_G(x)| \neq 0$ in k .

Now by Assertion 1, the elements γ_x , for $x \in [G_0]$, generate the image $\text{Cart}(G)$ of kc^G . They are, moreover, linearly independent: Suppose indeed that some linear combination $\sum_{x \in [G_0]} \lambda_x \gamma_x$, where $\lambda_x \in k$, is equal to 0. For all $x \in [G_0]$, choose $\tilde{\lambda}_x \in \mathcal{O}$ such that $\rho(\tilde{\lambda}_x) = \lambda_x$. By (6.8.2), we get an element $\sum_{x \in [G_0]} \tilde{\lambda}_x \frac{v_x}{|C_G(x)|}$ of $\mathcal{O}\text{Proj}(kG)$ whose modular character has values in the maximal ideal $J(\mathcal{O})$ of $\mathcal{O}$. But by (6.8.1), the value at $g \in G_{p'}$ of this modular character is equal to

$$\sum_{x \in [G_0]} \tilde{\lambda}_x \frac{\Phi_{v_x}(g)}{|C_G(x)|},$$

which is equal to 0 if $g \notin G_0$, and to $\tilde{\lambda}_x$ if g is conjugate to $x \in [G_0]$ in G . It follows that $\tilde{\lambda}_x \in J(\mathcal{O})$, hence $\lambda_x = \rho(\tilde{\lambda}_x) = 0$. Since $g \in G_{p'}$ was arbitrary, we get that $\lambda_x = 0$ for any $x \in [G_0]$, so the elements γ_x , for $x \in [G_0]$, are linearly independent. This completes the proof of Proposition 6.8. $\square$

6.9. The functors $S_{L,1,W}$.

In this section, we consider the case, where $u = 1$, that is, the case of simple functors $S_{L,1,W}$, where L is a p -group and W is a simple $\mathbb{F}\text{Out}(L)$ module. The case where $\mathbb{F}$ has characteristic 0 is solved by Corollary 7.4 of [7]. Then we are left with the case, where $\mathbb{F}$ has characteristic p , and we assume that $\mathbb{F} = k$. In this situation, for a finite group G , the set $\mathcal{P}(G, L, 1)$ is just the set of pairs (Q, δ) , where Q is a p -subgroup of G and $\delta : L \rightarrow Q$ is a group isomorphism. Moreover, the set $[\mathcal{P}(G, L, 1)]$ is in one-to-one correspondence with a set of representatives of conjugacy classes of subgroups Q of G which are isomorphic to L (the bijection mapping (Q, δ) to Q). Then the group $\bar{G}_{Q,\delta,1}$ is isomorphic to $N_G(Q)/QC_G(Q)$. By Theorem 5.25, we have that

$$S_{L,1,W}(G) \cong \bigoplus_{(Q,\delta) \in [\mathcal{P}(G,L,1)]} \text{Tr}_1^{N_G(Q)/QC_G(Q)} \left(k\text{Cart}(C_G(Q)/Z(Q)) \otimes_k W \right).$$

Notation 6.10. Let G be a finite group.

- For a subgroup Q of G and an element z of G , we set $N_G(Q, z) = N_G(Q) \cap C_G(z)$ and $C_G(Q, z) = C_G(Q) \cap C_G(z)$.
- For a p -subgroup Q of G , we denote by $\zeta(G, Q)$ the set of elements z in $C_G(Q)_{p'}$ for which $Z(Q)$ is a Sylow p -subgroup of $C_G(Q, z)$. The group $N_G(Q)$ acts on $\zeta(G, Q)$ by conjugation, and we denote by $[\zeta(G, Q)]$ a set of representatives of orbits under this action.
- For a finite p -group L , we denote by $\mathcal{Z}(G, L)$ the set of pairs (Q, z) , where Q is a subgroup of G isomorphic to L , and z is an element of $\zeta(G, Q)$. In other words,

$$\mathcal{Z}(G, L) = \{(Q, z) \mid L \cong Q \leq G, z \in C_G(Q)_{p'}, Z(Q) \in \text{Syl}_p(C_G(Q, z))\}.$$

The group G acts on $\mathcal{Z}(G, L)$ by conjugation, and we denote by $[\mathcal{Z}(G, L)]$ a set of representatives of orbits under this action.

Theorem 6.11. Let L be a p -group and W be a simple $k\text{Out}(L)$ -module. Let, moreover, G be a finite group.

1. Let Q be a p -subgroup of G . Then there is an isomorphism

$$k\text{Cart}(C_G(Q)/Z(Q)) \cong \bigoplus_{z \in [\zeta(G, Q)]} \text{Ind}_{N_G(Q, z)C_G(Q)/QC_G(Q)}^{N_G(Q)/QC_G(Q)} k$$

of $kN_G(Q)/QC_G(Q)$ -modules.

2. The evaluation of the simple functor $S_{L,1,W}$ at G is

$$S_{L,1,W}(G) \cong \bigoplus_{(Q,z) \in [\mathcal{Z}(G,L)]} \text{Tr}_1^{N_G(Q,z)/QC_G(Q,z)}(W).$$

Proof. 1. For a subgroup Q of G , denote by $x \mapsto \bar{x}$ the projection map $N_G(Q) \rightarrow \bar{N}_G(Q) = N_G(Q)/Q$, and set $\bar{C}_G(Q) = QC_G(Q)/Q \cong C_G(Q)/Z(Q)$.

By Proposition 6.8, the vector space $k\text{Cart}(\bar{C}_G(Q))$ has a basis consisting of the elements $\gamma_{\bar{C}_G(Q),x}$, for $x \in [\bar{C}_G(Q)_0]$, and the group $N_G(Q)$ permutes these elements. Now if $x \in \bar{C}_G(Q)_{p'}$, there is an element $z \in (QC_G(Q))_{p'}$ such that $x = \bar{z}$, and we can moreover assume that $z \in C_G(Q)_{p'}$. Then the centralizer of $\bar{z}$ in $\bar{C}_G(Q)$ is equal to $QC_{QC_G(Q)}(z)/Q = QC_G(Q, z)/Q$. So $\bar{z} \in \bar{C}_G(Q)_0$ if and only if Q is a Sylow p -subgroup of $QC_G(Q, z)$, or equivalently, if $Z(Q)$ is a Sylow p -subgroup of $C_G(Q, z)$, that is, $z \in \zeta(G, Q)$.

Moreover, an element $n \in N_G(Q)$ stabilizes $\gamma_{\bar{C}_G(Q),\bar{z}}$ if and only if $\gamma_{\bar{C}_G(Q),\bar{z}} = \gamma_{\bar{C}_G(Q),\overline{nz n^{-1}}}$, hence by Proposition 6.8, if there exists $\bar{c} \in \bar{C}_G(Q)$ such that $\overline{nz n^{-1}} = \bar{c} \bar{z} \bar{c}^{-1}$. In other words, $c^{-1}n \in$

$C_{\overline{N}_G(Q)}(\bar{z}) = N_G(Q, z)/Q$, where we set $N_G(Q, z) = N_G(Q) \cap C_G(z)$. So the stabilizer of $\gamma_{\overline{C}_G(Q), \bar{z}}$ in $N_G(Q)$ is equal to $QC_G(Q)N_G(Q, z) = N_G(Q, z)C_G(Q)$.

Hence, $k\text{Cart}(\overline{C}_G(Q))$ is isomorphic to the permutation $N_G(Q)/QC_G(Q)$ -module $k\zeta(G, Q)$. The elements $\gamma_{\overline{C}_G(Q), \bar{z}}$, for $z \in [\zeta(G, Q)]$ form a set of representatives of orbits for the action of $N_G(Q)/QC_G(Q)$, and the stabilizer of $\gamma_{\overline{C}_G(Q), \bar{z}}$ is the group $N_G(Q, z)C_G(Q)/QC_G(Q)$. This proves Assertion 1.

2. Now $N_G(Q, z)C_G(Q)/QC_G(Q) \cong N_G(Q, z)/QC_G(Q, z)$, and Assertion 2 follows from Theorem 5.25, thanks to the general following fact (see Proposition 5.6.10(ii) in [1]): If Γ' is a subgroup of a finite group Γ , if M is a finite-dimensional $k\Gamma$ -module and M' is a finite-dimensional $k\Gamma'$ -module, then

$$\text{Tr}_1^\Gamma((\text{Ind}_1^{\Gamma'} M') \otimes_k M) \cong M' \otimes_k \text{Tr}_1^{\Gamma'}(\text{Res}_1^{\Gamma'} M)$$

as k -vector spaces. □

Remark 6.12. The formula in Assertion 2 of Theorem 6.11 can be viewed as another instance of similar formulas in Proposition 8.8 of [14], Theorem 2.6 of [15], or Theorem 6.1 of [4].

The following corollary deals with the case of Theorem 6.11, where W is the trivial module k . First a definition:

Definition 6.13. Let G be a finite group, and let L be a finite p -group. An element $z \in G_{p'}$ is said to have a *defect group isomorphic to L* if L is isomorphic to a Sylow p -subgroup of $C_G(z)$.

Corollary 6.14. Let G be a finite group, and let L be a finite p -group. Then the dimension of $\mathcal{S}_{L,1,k}(G)$ is equal to the number of conjugacy classes of elements of $G_{p'}$ with a defect group isomorphic to L .

Proof. Indeed, if $(Q, z) \in \mathcal{Z}(G, L)$ then $\text{Tr}_1^{N_G(Q, z)/QC_G(Q)}(k)$ is equal to zero if p divides the order of $N_G(Q, z)/QC_G(Q)$, and one-dimensional otherwise, that is, if a Sylow p -subgroup of $N_G(Q, z)$ is contained in $QC_G(Q, z)$. But since $z \in \zeta(G, Q)$, the group Q is a Sylow p -subgroup of $QC_G(Q, z)$. So $\text{Tr}_1^{N_G(Q, z)/QC_G(Q)}(k)$ is non-zero (and then, one-dimensional) if and only if Q is a Sylow p -subgroup of $N_G(Q, z) = N_{C_G(z)}(Q)$, that is, if Q is a Sylow p -subgroup of $C_G(z)$. □

List of Symbols

c^G	33	$[\mathcal{P}(G, L, u)]$	20
$\mathcal{E}_R(G)$	4	$\text{Pim}(kC_G(Q), u)$	32
$[g, \lambda]_E$	25	$\text{Pim}^\#(k\overline{N}_{P, \gamma})$	20
$\mathbb{F}\text{Cart}(G)$	33	Φ_g	23
$\mathbb{F}\text{Proj}^\#(k\overline{N}_{Q, \delta})^0$	32	$\mathcal{S}_{L, u, W}$	30
$\mathbb{F}R_k(\overline{N}_{Q, \delta})^b$	27	$T(P, \gamma, E)$	19
$G_{Q, \delta}$	21	$T^o(Q, \delta, F)$	20
$\overline{G}_{Q, \delta}$	22	$\overline{T}(F, g, E)$	24
$\widehat{G}_{Q, \delta}$	22	$\mathcal{T}(G, L, u)$	19
$\Gamma_{Q, \delta}$	28	$\mathcal{T}^\#(G, L, u)$	20
$\gamma_{Q, \delta}^{L, u}$	28	θ_g	22
$N_{P, \gamma}$	19	$\hat{\theta}_g$	22
$\overline{N}_{P, \gamma}$	19	$\langle u \rangle^{\natural}$	24
$\overline{N}_{Q, \delta}^b$	23	$\mathcal{Z}(G, L)$	36
$\mathcal{P}(G, L, u)$	18	$\zeta(G, Q)$	36

Acknowledgements. The first author is grateful to the Mathematics Department at Bilkent University for its hospitality during a visit in March 2023, when this work was first initiated.

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