Representations of finite sets and correspondences

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joint work with

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- LICMA'17 -

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Let E be a finite set. Then $\mathcal{R}_E := k\mathcal{C}(E,E) \cong End_{\mathcal{F}_k}(\mathsf{Y}_{E,k})$ is a symmetric algebra (for an explicit symmetrizing form).

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3. *If L has bounded type*

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Let $M \in \mathcal{F}_k$ and E be a finite set. Define

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Let k be a noetherian ring, let $M \subseteq L$ in $\mathcal{F}_k$, and let E and F be finite sets.

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Evaluation - Adjunction

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Theorem

1. *The assignment* $(E, U) \mapsto \mathcal{L}_{E,U}$ *is a fully faithful k-linear functor* $\mathcal{G}_k \to \mathcal{F}_k^b$.

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Theorem

1. *The assignment $(E, U) \mapsto \mathcal{L}_{E,U}$ is a fully faithful k-linear functor $\mathcal{G}_k \to \mathcal{F}_k^b$.*
2. *When k is noetherian, it is an equivalence of categories*

An equivalence of categories

Definition

Let $\mathcal{G}_k$ be the following category:

- the objects are pairs (E, U), where E is a finite set, and U is an $\mathcal{R}_E$-module.
- a morphism $(E, U) \to (F, V)$ is a morphism of $\mathcal{R}_E$-modules $U \to k\mathcal{C}(E, F) \otimes_{\mathcal{R}_F} V$.
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- There is an associated fundamental functor $\mathbb{S}_{E,R} = S_{E,\mathcal{P}_E f_R}$.

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Simple correspondence functors over k

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Functors and lattices

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For $n \in \mathbb{N}$, set $S(\underline{n}) = F_{\underline{n}}/H_{\underline{n}}$. Then:

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2. *If X is a finite set, then $S(\underline{n})(X)$ is a free k-module of rank* $\displaystyle\sum_{i=0}^{n}(-1)^i\binom{n}{i}(n+1-i)^{|X|}$.
3. $\displaystyle F_{\underline{n}} \cong \bigoplus_{A\subseteq[n]} S\big(\underline{|A|}\big) \cong \bigoplus_{j=0}^{n} S(\underline{j})^{\oplus\binom{n}{j}}$.
4. $\displaystyle End_{k\mathcal{L}}(\underline{n}) \cong End_{\mathcal{F}_k}(F_{\underline{n}}) \cong \prod_{j=0}^{n} M_{\binom{n}{j}}(k)$.
5. *If k is a field, then $S(\underline{n})$ is simple (and projective, and injective), isomorphic to $S_{[n],tot,k}$.*

Simple functors: the general case

Theorem

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Let (E, R) be a finite poset

Simple functors: the general case

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Let (E, R) be a finite poset, with $n = |E|$.

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Simple functors: the general case

Theorem

Let (E, R) be a finite poset, with $n = |E|$.

1. *There exists a canonical poset $G(E, R)$ containing (E, R)*

Simple functors: the general case

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Let (E,R) *be a finite poset, with* $n = |E|$.

1. *There exists a canonical poset* $G(E,R)$ *containing* (E,R) *such that for any finite set* X,

Simple functors: the general case

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Let (E, R) *be a finite poset, with* $n = |E|$.

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2. *In particular the k-module $\mathbb{S}_{E,R}(X)$ is free*

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It is moreover a free right $kAut(E, R)$*-module.*

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3. *Let W be a $kAut(E, R)$-module. Then the assignment $X \mapsto \mathbb{S}_{E,R}(X) \otimes_{kAut(E,R)} W$*

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   It is moreover a free right $k\mathrm{Aut}(E, R)$*-module.*
3. *Let* W *be a* $k\mathrm{Aut}(E, R)$*-module. Then the assignment* $X \mapsto \mathbb{S}_{E,R}(X) \otimes_{k\mathrm{Aut}(E,R)} W$ *is a correspondence functor*

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3. *Let W be a $kAut(E, R)$-module. Then the assignment $X \mapsto \mathbb{S}_{E,R}(X) \otimes_{kAut(E,R)} W$ is a correspondence functor, denoted by $S(E, R, W)$.*

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4. *If k is a field and W is simple*

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4. *If* k *is a field and* W *is simple, then* $S(E, R, W) \cong S_{E,R,W}$.

Simple functors: the general case

Simple functors: the general case

Corollary

Simple functors: the general case

Corollary

Let k *be a field.*

Simple functors: the general case

Corollary

Let k be a field. Let (E, R) be a finite poset

Simple functors: the general case

Corollary

Let k be a field. Let (E, R) be a finite poset, and W be a simple $kAut(E, R)$-module.

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Simple functors: the general case

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Simple functors: the general case

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$$\dim_k S_{E,R,W}(X) = \frac{\dim_k W}{|Aut(E,R)|}\sum_{i=0}^{|E|}(-1)^i\binom{|E|}{i}(g_{E,R}-i)^{|X|} \quad .$$

The simple $\mathcal{R}_X$-modules

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Theorem

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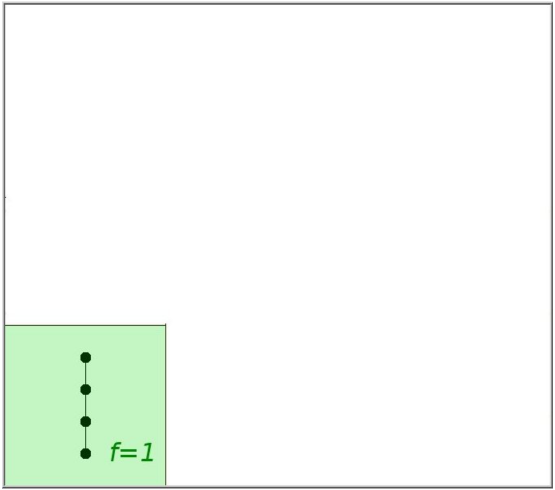
2. *The simple module parametrized by (E, R, W) is $S_{E,R,W}(X)$.*

Posets of cardinality 4

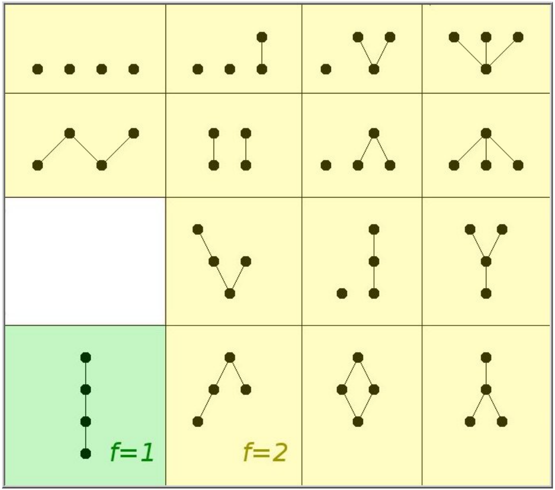
Posets of cardinality 4 ($f = g_{E,R} - 4$)

Posets of cardinality 4

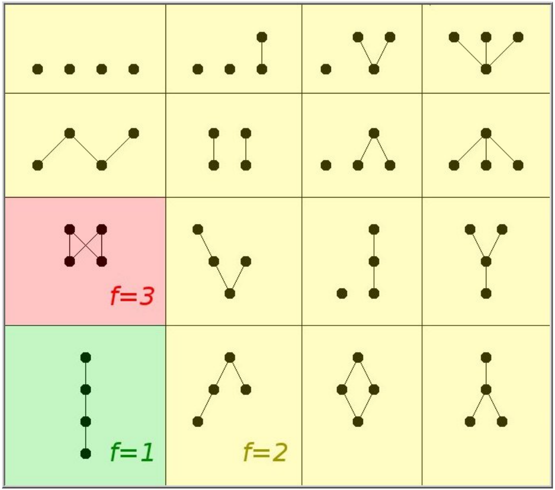
Posets of cardinality 4



Posets of cardinality 4



Posets of cardinality 4



Splitting the diamond

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The diamond is the following lattice D

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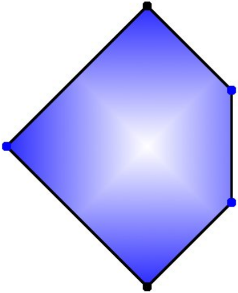

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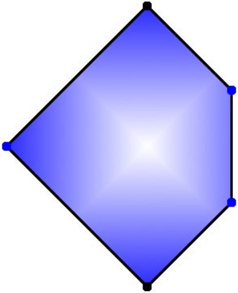

Over a field of characteristic different from 2, the functor F_D is semisimple:

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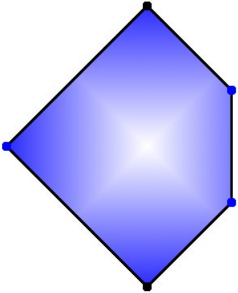

Over a field of characteristic different from 2, the functor F_D is semisimple:

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Splitting the diamond

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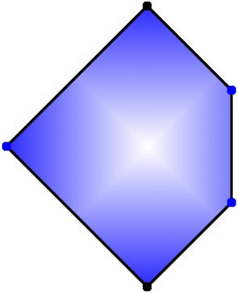

Over a field of characteristic different from 2, the functor F_D is semisimple:

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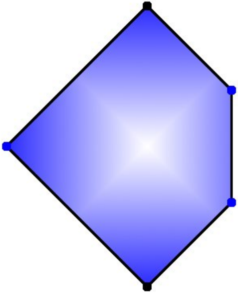

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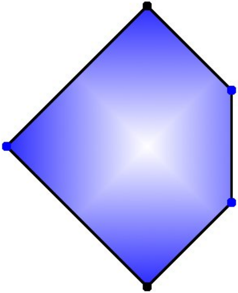

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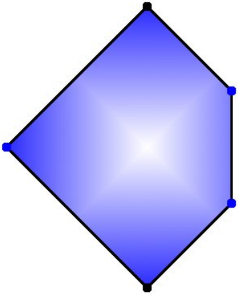

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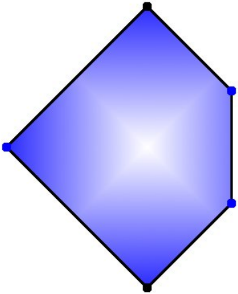

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