TWOSIDED TILTING COMPLEXES FOR GREEN ORDERS AND BRAUER TREE ALGEBRAS

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Abstract. We give an explicit twosided tilting complex between two Green orders having the same structural data as they were defined by K. W. Roggenkamp in [13, 14]. This yields an explicit twosided tilting complex between two Brauer tree algebras over the same field associated to trees with the same number of edges and the same exceptional multiplicity. The present work also gives a certain generalization of a result of Gabriel and Riedtmann [3].

1. Introduction

Let Λ be a ring. The (bounded) derived module category $D^b(\Lambda)$, or for short the derived category, of Λ is the category with objects being complexes of finitely generated projective modules which are bounded to the right and which have non zero homology only in finitely many degrees. Morphisms are complex morphisms up to homotopy. For details we refer to [5].

By the fundamental theorem [9] of Jeremy Rickard for any two rings Λ and Γ the derived module categories of Λ and of Γ are equivalent if and only if there is a complex T of finitely generated projective Λ modules which is bounded to the left and to the right and which has certain properties to be made more precise in Section 2. A complex satisfying these properties is called a *onesided tilting complex*, or for short, a *tilting complex*.

A more precise description can be given in case of R-projective algebras. Let R be an integral domain. An R–algebra is called an R–order if Λ is finitely generated projective as R–module and $frac(R) \otimes_R \Lambda$ is semisimple, $frac(R)$ being the field of fractions of R. Jeremy Rickard proved in [10] that if R is a Dedekind domain and if T is a tilting complex over the R–order Λ with endomorphism ring being the R–order Γ, then there is a bounded complex X of $\Lambda - \Gamma$–bimodules such that

$$X \otimes^L_\Gamma - : D^b(\Gamma) \longrightarrow D^b(\Lambda)$$

is an equivalence of triangulated categories. This complex of bimodules is called a *twosided tilting complex between* Λ *and* Γ. A more detailed description will be given in Section 2.

Given a twosided tilting complex between two Gorenstein R–orders Λ and Γ, then one can construct a Λ–Γ–bimodule M, projective if restricted to either side, and $M \otimes_\Gamma -$ induces a stable equivalence between Λ and Γ.

This paper. The aim of the present paper is to construct a twosided tilting complex explicitly in purely ring theoretical terms for derived equivalences between two Green orders as they were introduced by Roggenkamp [13]. Roughly speaking, Green orders are an analogue of Brauer tree algebras replacing a field by a complete discrete valuation domain. Given a Brauer tree algebra A over a perfect field k then there is a complete discrete valuation domain R with residue field k and a Green order Λ such that $k \otimes_R \Lambda \simeq A$. Moreover, the ring theoretical structure of a Green order is determined by some *combinatorial data* and some *ring theoretical*

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structural data. Blocks of group rings RG of finite groups G with cyclic defect group are Green orders.

We shall construct explicitly a twosided tilting complex for any two Green orders having the same structure data. In [16] an iterative process is given to prove that two such Green orders are derived equivalent no matter what the combinatorial data is. The method was purely combinatorial. The construction and the proof was simplified by Steffen König and the author in [6].

The main difficulty one has to overcome is that *there is no canonical way to produce a bimodule which induces a stable equivalence between the two Green orders.* As mentioned above, a twosided tilting complex would provide us with such a bimodule.

A main source for the construction is a careful examination of the construction of Raphaël Rouquier [15] in case of $\hat{\mathbb{Z}}_3 S_3$, the group ring of the symmetric group of degree 3 over the 3-adic integers. Here it becomes clear how a twosided two term tilting complex works. We shall construct the complex out of its homology. To first construct the homology and then the complex having this homology is used in [16] in a very special case and in this more general situation it seems to be new.

For two Brauer tree algebras over a field corresponding to Brauer trees with the same number of edges and the same multiplicity of the exceptional vertex, Gabriel and Riedtmann gave a stable equivalence in [3]. They did not give a bimodule inducing the stable equivalence; our construction gives one.

We mention that our method is entirely combinatorial.

What happened before. One may become interested in derived equivalences with a conjecture of Michel Broué [1] saying that the derived categories of a block of a group ring of a finite group with abelian defect group D and its Brauer correspondent in the group ring of the normalizer in the group of the defect group are equivalent as triangulated categories.

In [2, Remark after 4.7] Michel Broué asked for an explicit construction of a twosided tilting complex between a block of a finite group and its Brauer correspondent and stated that in December 1992 no explicit construction of a twosided tilting complex, even in case of a cyclic defect group, was known. Progress has been fast since then.

In [15] Raphaël Rouquier constructed under some conditions a twosided tilting complex for symmetric orders out of a stable equivalence given by a bimodule. The twosided tilting complex is as explicit as is the bimodule inducing the stable equivalence. For blocks with cyclic defect, the well known stable equivalence comming from Green correspondence is in fact induced by tensoring with a bimodule. The hypotheses for the construction in [15] are fulfilled. Other examples are given by Jeremy Rickard in [11] for the situation of algebraic groups.

In [13, 14] Klaus W. Roggenkamp introduced a certain class of orders, Green orders, and clarified their structure completely. Moreover, he proved that blocks of cyclic defect groups of group rings of finite groups over any finite extension of the p–adic integers are Morita equivalent to Green orders of a special shape. Furthermore, reducing modulo the radical of the coefficient domain, every Brauer tree algebra over a perfect field of finite characteristic is an image of a suitably chosen Green order. In [13, 14] the only missing part in the complete description of the blocks of cyclic defect by Green orders is the structure of the 'exceptional vertex'. For the field of fractions of the coefficient domain being a splitting field for the block, Markus Linckelmann's result [7] published in his thesis answers the remaining question of the structure of the centre of the block. For more general coefficient domains, Plesken [8] gives information for the structure of the exceptional vertex.

In Section 2 we state the Theorems of J.Rickard. The onesided tilting complexes are described in Section 3, making a summary of the relevant parts of [16] and of [13]. The complex is then constructed in Section 4.

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2. Rickard's theorems

We state, for the reader's convenience, Rickard's two main theorems. The first of which deals with onesided tilting complexes, the second deals with twosided tilting complexes.

Objects of the category $D^b(\Lambda)$ are right bounded complexes of finitely generated projective Λ-modules and with homology concentrated in only finitely many degrees. Morphisms are complex morphisms modulo homotopy.

The category $K^b(P_\Lambda)$ is the full subcategory of $D^b(\Lambda)$ generated by bounded complexes.

Theorem 1. *(Rickard [9]) Let Λ and Γ be two rings. Then, the following conditions are equivalent:*

(1) $D^b(\Lambda) \simeq D^b(\Gamma)$ *as triangulated categories*
(2) *There is a $T \in K^b(P_\Lambda)$ such that the ring $End_{K^b(P_\Lambda)}(T)$ is isomorphic to Γ as rings and $Hom_{K^b(P_\Lambda)}(T, T[i]) = 0$ for all $i \neq 0$ and the rank one free module is contained in the triangulated category generated by direct summands of finite direct sums of T.*

Remark 1. A complex T as in Theorem 1 is called a *tilting complex from Λ to Γ*. If such a T exists then Λ and Γ are called *derived equivalent.*

The second theorem deals with a different type of complexes making the equivalence of the derived categories more explicit.

Theorem 2. *(Rickard [10]) Let R be a commutative ring and let A and B be two R-projective R–algebras. If A and B are derived equivalent by a functor F, then the functor F induces also a derived equivalence between $D^b(A \otimes_R A^{op})$ and $D^b(A \otimes_R B^{op})$ and the image of A as $A \otimes_R A^{op}$ module under this functor is a complex X in $D^b(A \otimes_R B^{op})$. Furthermore,*

$$X \otimes^L_B - : D^b(B) \longrightarrow D^b(A)$$

is an equivalence of triangulated categories.

Remark 2. The complex X as in Theorem 2 is called a twosided tilting complex. Note that X is defined in the derived category and not in the homotopy category.

We finish with a yet unpublished lemma of Jeremy Rickard.

Lemma 1. *(J. Rickard [12]) Let R be a complete discrete valuation ring and let Λ and Γ be two R–orders in semisimple artinian $frac(R)$–algebras. Let X be a complex of Λ–Γ bimodules bounded from the left and bounded from the right. Let X be isomorphic in $D^b(\Lambda)$ to a tilting complex with endomorphism ring Γ and let X be isomorphic in $D^b(\Gamma^{op})$ to a tilting complex with endomorphism ring Λ, then X is a twosided tilting complex.*

Proof. (J. Rickard)

Let more generally $\mathcal{A}$ and $\mathcal{B}$ be additive categories and let $L_i : \mathcal{A} \longrightarrow \mathcal{B}$ with $i = 1, 2$ be functors with right adjoints R_1 and R_2. Now, a natural transformation

$$L_1 \longrightarrow L_2$$

gives rise to a natural transformation

$$R_2 \longrightarrow R_1L_1R_2 \longrightarrow R_1L_2R_2 \longrightarrow R_1$$

which is induced by the unit $1 \longrightarrow R_1L_1$, the above natural transformation and the counit $L_2R_2 \longrightarrow 1$. This correspondence is called the *conjugate* map. This is characterized by the commutativity of the diagram

$$\begin{array}{ccc} Hom_{\mathcal{B}}(L_2-, -) & \simeq & Hom_{\mathcal{A}}(-, R_2-) \\ \downarrow & & \downarrow \\ Hom_{\mathcal{B}}(L_1-, -) & \simeq & Hom_{\mathcal{A}}(-, R_1-) \end{array}$$

Given a bounded complex of functors

$$L^* : \ldots \longrightarrow L_0 \longrightarrow L_1 \longrightarrow L_2 \longrightarrow \ldots$$

by the commutativity of the diagram defining the adjoint map above, this gives rise to a complex of functors

$$R^* : \ldots \longrightarrow R_2 \longrightarrow R_1 \longrightarrow R_0 \longrightarrow \ldots$$

by taking the conjugate maps. Now, L^* and R^* induce functors $L^* : C^b(\mathcal{A}) \longrightarrow C^b(\mathcal{B})$ and $R^* : C^b(\mathcal{B}) \longrightarrow C^b(\mathcal{A})$ by taking the total complex of the resulting double complex. Of course, homotopies map to homotopies this way such that our functors L^* and R^* carries over to functor

$$L^* : K^b(\mathcal{A}) \longrightarrow K^b(\mathcal{B})$$

and

$$R^* : K^b(\mathcal{B}) \longrightarrow K^b(\mathcal{A})$$

We observe that (L^*, R^*) is an adjoint pair. In fact, for any X and Y in $\mathcal{A}$ we get an isomorphism of the triple complexes

$$Hom_{\mathcal{B}}(L^*(X), Y) \simeq Hom_{\mathcal{A}}(X, R^*(Y))$$

natural in X and in Y. If one takes now total complexes and observes that the zero homology of this complex then gives just the homomorphisms in the homotopy categories, we get by the naturality of the construction in both variables the adjointness property.

We now specialize to $L_i = - \otimes_\Gamma X_i$ for our bimodules X_i which are finitely generated projective on both sides. Then, $R_i = Hom_\Gamma(X_i, -)$. By the fact that X_i is finitely generated projective as Γ–module,

$$Hom_\Gamma(X_i, -) \simeq - \otimes_\Gamma Hom_\Gamma(X_i, \Gamma).$$

We get a unit

$$1_{?(\Lambda)} \longrightarrow - \otimes_\Lambda X \otimes_\Gamma Hom_\Gamma(X, \Gamma)$$

and a counit

$$- \otimes_\Gamma Hom_\Gamma(X, \Gamma) \otimes_\Lambda X \longrightarrow 1_{?(\Gamma)}$$

for ? being C, the complex category or equally well K, the homotopy category. Therefore we get natural maps of complexes of right modules

$$\Lambda \longrightarrow X \otimes_\Gamma Hom_\Gamma(X, \Gamma)$$

and

$$Hom_\Gamma(X, \Gamma) \otimes_\Lambda X \longrightarrow \Gamma$$

which are actually maps of complexes of bimodules by letting the rings act via endomorphisms of the free rank one right modules, using the functoriality of the construction.

Take $D_\Gamma(X) = Hom_\Gamma(X, \Gamma)$ as complex of $\Gamma - \Lambda$–bimodules. Now, $- \otimes_\Lambda X$ is left adjoint to $- \otimes_\Lambda D_\Gamma X$ as functors $C^b(\Lambda) \longrightarrow C^b(\Gamma)$. We get natural maps of complexes of bimodules

$$\Lambda \stackrel{\alpha}{\longrightarrow} X \otimes_\Gamma D_\Gamma X \text{ and } D_\Gamma(X) \otimes_\Lambda X \stackrel{\beta}{\longrightarrow} \Gamma$$

We now show, that α is an isomorphism in the derived category of bounded complexes of projective modules.

Set $k = R/rad\ R$.

We may assume, using Lemma 2, that X consists of bimodules whose restrictions to the left and to the right are projective.

Obviously, $\overline{M} := k \otimes_R (X \otimes_\Gamma D_\Gamma(X))$ is isomorphic to a module in $D^b(\Lambda \otimes \Lambda^{op})$.

We denote the kernel of the mapping induced by α on the degree zero homology by $\overline{K}$. We observe that

$$\begin{array}{ccc} k \otimes_R X \otimes_\Gamma D_\Gamma(X) & \stackrel{\alpha \otimes_\Lambda 1_M}{\longrightarrow} & k \otimes_R X \otimes_\Gamma D_\Gamma(X) \otimes_\Lambda X \otimes_\Gamma D_\Gamma(X) \\ \| & & \downarrow 1_X \otimes_\Gamma \beta \otimes_\Gamma 1_{D_\Gamma(X)} \\ k \otimes_R X \otimes_\Gamma D_\Gamma(X) & = & k \otimes_R X \otimes_\Gamma D_\Gamma(X) \end{array}$$

is a commutative diagram.

Since now $\alpha \otimes 1_{X\otimes D_\Gamma(X)}$ is a split monomorphism, $\overline{K} \otimes_\Lambda X \otimes_\Gamma D_\Gamma(X) = 0$.

Since the complex $D_\Gamma(X)$ is a complex of projective Γ–modules, ${}_\Gamma|D_\Gamma(X)$ is a tilting complex. In fact, $Hom_\Gamma(-, \Gamma)$ is faithfully flat on projective Γ–modules and it is now easy to derive the defining conditions of a tilting complex from the property of X being a tilting complex as complex of Γ–modules.

Since ${}_\Gamma|D_\Gamma(X)$ is a tilting complex, $\overline{K} \otimes_\Lambda X = 0$. Since $X|_\Lambda$ is a tilting complex, $\overline{K} = 0$.

Hence, we know that α induces an injective mapping on the degree 0–homology. As R–modules we have

$$H_0(X \otimes_\Gamma D_\Gamma(X)) \simeq End_{K^b(P_\Gamma)}(X|_\Gamma).$$

Since every monomorphism between finite dimensional vector spaces of equal dimension is an isomorphism, α induces an isomorphism

$$R/rad\ R \otimes_R \Lambda \simeq R/rad\ R \otimes_R H_0(X \otimes D_\Gamma(X))$$

as $\Lambda - \Lambda$–bimodules. Since we assumed Λ and Γ to be finitely generated over the noetherian complete discrete valuation ring R, we know that α induces an isomorphism between Λ and the degree zero homology of $X \otimes D_\Gamma(X)$..

We have to finish with proving that also β is an isomorphism.

The composition

$$X \stackrel{\alpha\otimes id_X}{\longrightarrow} X \otimes_\Gamma D_\Gamma(X) \otimes_\Lambda X \stackrel{id_X\otimes\beta}{\longrightarrow} X$$

equals the identity and α is an isomorphism, so $id_X \otimes \beta$ is an isomorphism. Forming the triangle in the homotopy category of complexes of finitely generated projective modules

$$D_\Gamma(X) \otimes_\Gamma X \stackrel{\beta}{\longrightarrow} \Gamma \longrightarrow C \rightsquigarrow \dots,$$

we get $X \otimes_\Gamma C$ is acyclic, and since X as complex of right modules is isomorphic to a tilting complex, C is acyclic.

This completes the proof of Lemma 1.

In order to prove that a complex of Λ–Γ–bimodules is a twosided tilting complex, for Λ and Γ being as in the lemma, we only have to compute a complex X of bimodules which restricts on either sides to a tilting complex with correct endomorphism rings.

3. Recapitulation of the onesided situation

3.1. Green orders. K.W. Roggenkamp defined Green orders to explain the structure of blocks of group rings over a complete discrete valuation ring of characteristic 0 with cyclic defect group [13].

We use a suggestion of L. Puig to define Green orders in a different, equivalent, way.

Let R be an integral domain with field of fractions K and let Λ be an indecomposable R–order in the semisimple K–algebra $A = K \otimes_R \Lambda$.

Let I be a complete set of primitive idempotents of Λ (i.e. $End_\Lambda(\oplus_{i\in I}\Lambda \cdot i)$ is Morita equivalent to Λ and the modules $\Lambda \cdot i$ for all $i \in I$ are projective indecomposable left–Λ–modules for all $i \in I$).

Definition 1. *The indecomposable R–order Λ in the semisimple algebra A is called a Green order if*

- *there is a set E of central idempotents of A with $\sum_{e\in E} e = 1$ such that*

$$T := \{(i, e) \in I \times E | i \cdot e \neq 0\}$$

is a tree (i.e. defines a connected relation on $I \times E$, $|E| = |I| + 1$ and for all $i \in I$ we get $|T \cap (\{i\} \times E)| = 2$).

- *noting by $\pi : T \longrightarrow I$ and $\theta : T \longrightarrow E$ the natural projections, there are a transitive permutation ω of T and for all $t \in T$ a Λ–module homomorphism*

$$g_t : \Lambda \cdot \pi(t) \longrightarrow \Lambda \cdot \pi(\omega(t)) \text{ with } g_t(\Lambda \cdot \pi(t)) = ker(g_{\omega(t)}) \stackrel{\sigma_t}{\simeq} \Lambda \cdot \pi(t) \cdot \theta(t) \ .$$

where for all $\lambda \in \Lambda\pi(t)$ we have $\sigma_t g_t(\lambda) = \lambda \cdot \theta(t)$.

Remark 3. We should provide a link to the definition in [13].

(1) It is immediate to see that a finite, connected, unoriented graph T with one edge less than it has vertices does not have cycles (use an induction). Hence, the three conditions on the cardinalities of I and E in relation with T defines in fact a tree.
(2) The transitive permutation ω of T is the 'walk around the Brauer tree' which was invented in Green's paper [4]. Green gives reference to this permutation in the introduction, however, without mentioning the permutation explicitly.
(3) The walk around the Brauer tree manifests in a projective resolution

$$0 \leftarrow \Lambda \cdot \pi(t) \cdot \theta(t) \leftarrow \Lambda\pi(t) \leftarrow \Lambda\pi(\omega(t)) \leftarrow \Lambda\pi(\omega^2(t)) \leftarrow \cdots \leftarrow \Lambda\pi(t) \leftarrow \ldots$$

(4) A tree may be realized by a graph in the plane, meaning a complex of 1-simplices in $\mathbb{R}^2$. One associates to the plane an orientation, and the permutation ω is the 'walk around the tree' as described in [4].

We abbreviate $\Lambda \cdot t = \Lambda\pi(t)\theta(t)$ and $t \cdot \Lambda = \pi(t)\theta(t)\Lambda$. The main theorem of Roggenkamp in [13] may be reformulated as follows.

Theorem 3. *(Roggenkamp* [13]*) We assume that Λ is basic.*

There is an R–torsion R–algebra $\overline{\Omega}$ and a family $(f_t : t\Lambda t \longrightarrow \overline{\Omega})$ of R–algebra homomorphisms with kernel being a principal ideal $a_t t\Lambda t$.

T is totally ordered by ω and a first element. There is a (equivalent) set of primitive idempotents I such that one can choose the first element of T such that the Pierce decomposition

$$\Lambda = \begin{pmatrix} i_1\Lambda i_1 & i_1\Lambda i_2 & \ldots & i_1\Lambda i_k \\ i_2\Lambda i_1 & i_2\Lambda i_2 & \ldots & i_2\Lambda i_k \\ \vdots & \vdots & & \vdots \\ i_k\Lambda i_1 & i_k\Lambda i_2 & \ldots & i_k\Lambda i_k \end{pmatrix}$$

has the following properties

(1) *For all $t < t'$ and $\theta(t) = \theta(t')$ with $t, t' \in T$ we have $t\Lambda t = \pi(t)\Lambda\pi(t') = t'\Lambda t'$.*
(2) *f_t depends only on $\theta(t)$ and we denote $f_{\theta(t)} := f_t$ and $a_{\theta(t)} := a_t$.*
(3) *For all $t > t'$ and $\theta(t) = \theta(t')$ with $t, t' \in T$ we have $a_{\theta(t)} \cdot \pi(t')\Lambda\pi(t) = \pi(t)\Lambda\pi(t')$.*
(4) *If $t, t' \in T$ with $\pi(t) = \pi(t')$ then $\pi(t)\Lambda\pi(t)$ is a pullback*

$$\begin{array}{ccc} \pi(t)\Lambda\pi(t) & \longrightarrow & t\Lambda t \\ \downarrow & & \downarrow f_t \\ t'\Lambda t' & \xrightarrow{f_{t'}} & \overline{\Omega} \end{array}$$

Moreover, if for an R–order Λ' in $K \otimes_R \Lambda$ with I being also a complete set of primitive idempotents of Λ' the tree T and the transitive permutation ω of T from above have the property that there is an element $t_1 \in T$ defining together with ω a linear ordering on T such that the Pierce decomposition has the properties (1)*,* (2)*,* (3) *and* (4) *then Λ' is a Greenorder.*

Remark 4. (1) One should observe that since the e in E are central and pairwise orthogonal, most of the entries in the above matrix are zero. The rest of the entries falls naturally into matrix rings. Furthermore, by the first and the third points of the theorem, the matrix rings in question are upper triangular over $\Omega_{\theta(t)} := t\Lambda t$ which does only depend on $\theta(t)$ und the lower diagonal entries is in a principal ideal $a_t t\Lambda t$, again depending only on $\theta(t)$. These matrix rings are linked among each other by the pullbacks with the main diagonal entries mentioned in point 4 of the theorem. This is the interpretation given by Roggenkamp in [13]. We may hence label the vertices of the tree by the pairs $(\Omega_{\theta(t)}, a_{\theta(t)})$. Mostly we only write down the labels $\Omega_{\theta(t)}$, understanding that the ideal $a_{\theta(t)}\Omega_{\theta(t)}$ is fixed once for all, if this does not cause confusion.

(2) With $I = \{i_1, i_2, \ldots, i_k\}$ being the above set of primitive idempotents, one obtains that g_t is just multiplication by t and σ_t is the identity.

(3) By the last of the 4 properties one can see that the the projective indecomposable modules Λi for $i \in I$ are pullbacks, where $t \cdot i \neq 0 \neq t' \cdot i$ for two different t, t' in T,

$$\begin{array}{ccc} \Lambda i & \stackrel{g_t}{\rightarrow} & \Lambda t \\ \downarrow g_{t'} & & \downarrow \tilde{f}_t \\ \Lambda t' & \stackrel{\tilde{f}_{t'}}{\rightarrow} & (\overline{\Omega}_j)_{j=1}^k \end{array}$$

where

$$\tilde{f}_t|_{t''\Lambda t} = \left\{ \begin{array}{ccc} f_t & \text{if} & t'' = t \\ 0 & \text{if} & t'' \neq t \end{array} \right. \text{ and } \tilde{f}_{t'}|_{t''\Lambda t'} = \left\{ \begin{array}{ccc} f_{t'} & \text{if} & t'' = t' \\ 0 & \text{if} & t'' \neq t' \end{array} \right.$$

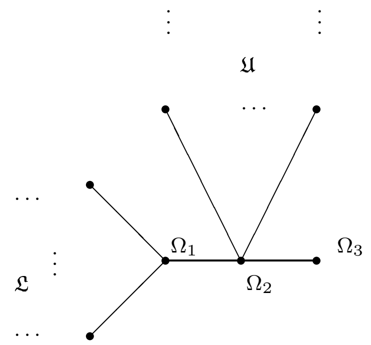
Moreover,

$$\overline{\Omega}_j = \left\{ \begin{array}{ccc} \overline{\Omega} & \text{if} & i = i_j \\ 0 & \text{if} & i \neq i_j \end{array} \right.$$

As a consequence, we get that

$$ker(\tilde{f}_t) \simeq ker(\sigma_{t'} g_{t'}) \simeq ker(g_{t'})$$

3.2. Tilting Green orders. We recall the following facts from [16] and from the refinement [6]. We are given a Green order Λ with Brauer tree



and data (Ω_i, f_i), where we denote

$$\mathfrak{L} := \{ \text{ vertices } v | \text{ the shortest path in the tree from } v \text{ to } \Omega_2 \text{ passes } \Omega_1\}$$

$$\mathfrak{U} := \{ \text{ vertices } v | \text{ the shortest path in the tree from } v \text{ to } \Omega_1 \text{ passes } \Omega_2\} \setminus \{\Omega_3\}$$

The orientation in the plane is meant to be counterclockwise.

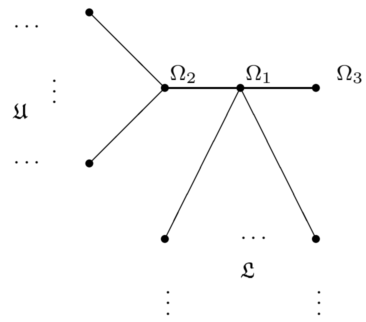
The indecomposable projective corresponding to the edge linking Ω_1 and Ω_2 is denoted by $P = \Lambda i_{1,2}$, for an idempotent $i_{1,2}$ of Λ and the indecomposable projective corresponding to the edge linking Ω_2 and Ω_3 is denoted by $Q = \Lambda i_{2,3}$ for an idempotent $i_{2,3}$ of Λ. We identify edges with indecomposable projectives and denote the indecomposable projective corresponding to the edge e by P_e. Set

$$L := \bigoplus_{\binom{\text{edges } e \text{ involving}}{\text{only vertices in } \mathfrak{L}}} P_e \quad \text{and} \quad U := \bigoplus_{\binom{\text{edges } f \text{ involving}}{\text{only vertices in } \mathfrak{U}}} P_f$$

The complex

$$T := (0 \longrightarrow L \oplus U \oplus P \oplus P \stackrel{(0,0,0,g_{i_{1,2}e_2})}{\longrightarrow} Q \longrightarrow 0)$$

is a tilting complex. Its endomorphism ring is a Green order Γ associated to the following tree with obvious notations.



The argument applied to right Γ–modules implies that, denoting by P'^* the indecomposable projective corresponding to the edge linking Ω_1 and Ω_2 and by Q'^* the projective indecomposable corresponding to the edge linking Ω_1 and Ω_3, defining L'^* and U'^* analogously as above, the complex

$$T' := (0 \longrightarrow L'^* \oplus U'^* \oplus P'^* \oplus P'^* \stackrel{(0,0,0,g_{j_{1,2}e_1})}{\longrightarrow} Q'^* \longrightarrow 0)$$

is a tilting complex with endomorphism ring Λ.

We denote by i_U the idempotent corresponding to U, by i_L the idempotent corresponding to L, by $i_{1,2}$ the idempotent corresponding to P and by $i_{2,3}$ the idempotent corresponding to Q.

Analogously, we denote by j_U the idempotent corresponding to U'^*, by j_L the idempotent corresponding to L'^*, by $j_{1,2}$ the idempotent corresponding to P'^* and by $j_{1,3}$ the idempotent corresponding to Q'^*.

By abuse of language we denote *both for* Λ *and for* Γ the central idempotents corresponding to Ω_1 by e_1, those corresponding to Ω_2 by e_2, those corresponding to U by e_U and those corresponding to L by e_L.

We choose the 'beginning element' of T_Λ and of T_Γ such that

$$T_\Lambda = \{i_U e_2 < i_{1,2}e_1 < i_L e_L < i_L e_1 < i_{1,2}e_2 < i_{2,3}e_3 < i_{2,3}e_2 < i_U e_U\}$$

and

$$T_\Gamma = \{j_{1,3}e_1 < j_{1,3}e_3 < j_U e_2 < j_U e_U < j_{1,2}e_2 < j_{1,2}e_1 < j_L e_L < j_L e_1\}$$

Then, the result in [16] affirms that

$$\begin{array}{rcl}
i_U\Lambda i_U &=& j_U\Lambda j_U \text{ with } f^{\Lambda}_{i_U,e_2} = f^{\Gamma}_{i_U,e_2} \text{ and } f^{\Lambda}_{i_U,e_U} = f^{\Gamma}_{i_U,e_U},\\
i_{1,2}\Lambda i_U &=& j_{1,2}\Gamma j_U,\\
i_L\Lambda i_L &=& j_L\Gamma j_L \text{ with } f^{\Lambda}_{i_L,e_1} = f^{\Gamma}_{i_L,e_1} \text{ and } f^{\Lambda}_{i_L,e_L} = f^{\Gamma}_{i_L,e_L},\\
i_{1,2}\Lambda i_L &=& j_{1,2}\Gamma j_L,\\
i_{1,2}\Lambda i_{1,2} &=& j_{1,2}\Gamma j_{1,2} \text{ with } f^{\Lambda}_{i_{1,2}e_1} = f^{\Gamma}_{j_{1,2}e_1} \text{ and } f^{\Lambda}_{i_{1,2}e_2} = f^{\Gamma}_{j_{1,2}e_2},\\
i_{1,2}\Lambda i_L &=& j_{1,2}\Gamma j_L,\\
i_U\Lambda i_{1,2} &=& j_U\Gamma j_{1,2},\\
i_L\Lambda i_{1,2} &=& j_L\Gamma j_{1,2},\\
i_L\ ker(g_{(i_{1,2},e_2)}) &=& j_L\Gamma j_{1,3},\\
i_{1,2}\ ker(g_{(i_{1,2},e_2)}) &=& j_{1,2}\Gamma j_{1,3},\\
i_{2,3}\Lambda i_U &=& ker(g_{(j_{1,2},e_1)})\ j_U,\\
i_{2,3}\Lambda i_{1,2} &=& ker(g_{(j_{1,2},e_1)})\ j_{1,2}\ .
\end{array}$$

In section 4 we shall construct a complex X of Λ–Γ–bimodules which is isomorphic in the derived category of Λ–left modules to T and which is isomorphic in the derived category of Γ–right modules to T'.

The combinatorics. In [6] it is proven that iterating the procedure above, or otherwise said, taking as 'new' Λ the 'old' order Γ and choosing a numeration for Ω_1, Ω_2 and Ω_3, it is possible to end up after a certain number of steps as Γ a Green order associated to a tree being a star with the same data we started with, and any association of the vertices of the star to the data (Ω_i, f_i).

In other words, it is possible to 'deform' the tree to a star and to interchange any two vertices.

So, tensoring the complexes we construct, this gives a twosided tilting complex between any two Green orders with the same data, no matter the tree looks like as long as the data and the number of vertices are fixed.

3.3. An observation. In our onesided construction we have to deal with two terms tilting complexes. Let

$$X = (0 \longrightarrow X_1 \longrightarrow X_0 \longrightarrow 0)$$

be a twosided two term tilting complex and we may and will assume that X has homology concentrated in the degrees 0 and 1.

Using the arguments in the appendix we may furthermore assume that X_0 is a projective bimodule.

We denote by $\Omega_1(H_0(X))$ the first syzygy of $H_0(X)$ *as bimodule*. We get short exact sequences

$$0 \longrightarrow \Omega_1(H_0(X)) \longrightarrow X_0 \longrightarrow H_0(X) \longrightarrow 0$$

and

$$0 \longrightarrow H_1(X) \longrightarrow X_1 \longrightarrow \Omega_1(H_0(X)) \longrightarrow 0.$$

Moreover, $K = frac(R)$ being the field of fractions of R,

$$K \otimes_R X_1 \simeq K \otimes_R H_1(X) \oplus K \otimes_R \Omega_1'(H_0(X)).$$

We are given two onesided tilting complexes T and S over Λ and Γ^{op} resp. and we want to find a twosided tilting complex X in $D^b(\Lambda \otimes_R \Gamma^{op})$ with

$$X \simeq T \text{ in } D^b(\Lambda) \text{ and } X \simeq S \text{ in } D^b(\Gamma^{op})$$

It is therefore necessary to find a bimodule $H_1(X)$ such that

$${}_{\Lambda}|H_1(X) \simeq H_1(T) \text{ and } H_1(X)|_{\Gamma} \simeq H_1(S)$$

Furthermore, the restriction of the composition

$$X_1 \longrightarrow \Omega_1'(H_0(X)) \longrightarrow X_0$$

to the left and to the right have to coincide with the onesided complex, in the derived category.

This small observation will allow us to construct X.

4. Constructing the complex

We shall give a twosided tilting complex in the situation above.

Again we have to compute the homology in degree 1 and find a bimodule which restricts to these homologies on either side.

We remind the reader that we denote

$$g_\Lambda := g_{i_{1,2}e_2} : \Lambda i_{1,2} \longrightarrow \Lambda i_{2,3}$$

and

$$g_\Gamma := g_{j_{1,2}e_1} : j_{1,2}\Gamma \longrightarrow j_{1,3}\Gamma$$

There is only one canonical way to produce a bimodule $H_1(X)$: The bimodule is isomorphic to

$$\begin{pmatrix} i_U\Lambda i_U & i_U\Lambda i_{1,2} & i_U ker(g_\Lambda) & i_U\Lambda i_L \\ i_{1,2}\Lambda i_U & i_{1,2}\Lambda i_{1,2} & i_{1,2}ker(g_\Lambda) & i_{1,2}\Lambda i_L \\ i_L\Lambda i_U & i_L\Lambda i_{1,2} & i_L ker(g_\Lambda) & i_L\Lambda i_L \\ i_{2,3}\Lambda i_U & i_{2,3}\Lambda i_{1,2} & i_{2,3}ker(g_\Lambda) & i_{2,3}\Lambda i_L \end{pmatrix}$$

which is equal to

$$\begin{pmatrix} j_U\Gamma j_U & j_U\Gamma j_{1,2} & j_U\Gamma j_{1,3} & j_U\Gamma j_L \\ j_{1,2}\Gamma j_U & j_{1,2}\Gamma j_{1,2} & j_{1,2}\Gamma j_{1,3} & j_{1,2}\Gamma j_L \\ j_L\Gamma j_U & j_L\Gamma j_{1,2} & j_L\Gamma j_{1,3} & j_L\Gamma j_L \\ ker(g_\Gamma)j_U & ker(g_\Gamma)j_{1,2} & ker(g_\Gamma)j_{1,3} & ker(g_\Gamma)j_L \end{pmatrix}$$

where we used the relations mentioned in Subsection 3.2 and that we know, by the fact that the idempotents e_1, e_2, e_U and e_L are central and pairwise orthogonal (both for Λ and for Γ), that

$$\begin{aligned} i_U ker(g_\Lambda) &= j_U\Gamma j_{1,3} = 0, \\ i_U\Lambda i_L &= j_U\Gamma j_L = 0, \\ i_L\Lambda i_U &= j_L\Gamma j_U = 0, \\ i_{2,3}ker(g_\Lambda) &= ker(g_\Gamma)j_{1,3} = 0. \end{aligned}$$

On this bimodule Λ acts by multiplication by

$$\begin{pmatrix} i_U\Lambda i_U & i_U\Lambda i_{1,2} & i_U\Lambda i_L & i_U\Lambda i_{2,3} \\ i_{1,2}\Lambda i_U & i_{1,2}\Lambda i_{1,2} & i_{1,2}\Lambda i_L & i_{1,2}\Lambda i_{2,3} \\ i_L\Lambda i_U & i_L\Lambda i_{1,2} & i_L\Lambda i_L & i_L\Lambda i_{2,3} \\ i_{2,3}\Lambda i_U & i_{2,3}\Lambda i_{1,2} & i_{2,3}\Lambda i_L & i_{2,3}\Lambda i_{2,3} \end{pmatrix}$$

from the left and Γ acts by matrix multiplication by

$$\begin{pmatrix} j_U\Gamma j_U & j_U\Gamma j_{1,2} & j_U\Gamma j_{1,3} & j_U\Gamma j_L \\ j_{1,2}\Gamma j_U & j_{1,2}\Gamma j_{1,2} & j_{1,2}\Gamma j_{1,3} & i_{1,2}\Gamma i_L \\ j_{1,3}\Gamma j_U & j_{1,3}\Gamma j_{1,2} & j_{1,3}\Gamma j_{1,3} & j_{1,3}\Gamma j_L \\ j_L\Gamma j_U & j_L\Gamma j_{1,2} & j_L\Gamma j_{1,3} & j_L\Gamma j_L \end{pmatrix}$$

The homology in degree 0 is easy to compute, it is $\Lambda e_3 = e_3\Gamma$ as $\Lambda \otimes \Gamma^{op}$–module. The projective cover of this module as bimodule is

$$X_0 := \Lambda i_{2,3} \otimes_R j_{1,3}\Gamma.$$

For constructing the bimodule extension of the homology with the first syzygy of

$$\Omega_3 = e_3 i_{2,3}\Lambda i_{2,3} = e_3 j_{1,3}\Lambda j_{1,3}$$

we have to explicit this module. (Again, not to overload the notation, we write "$\otimes$" for "$\otimes_R$".)

$$\Lambda i_{2,3} \otimes_R j_{1,3}\Gamma =$$

$$= \begin{pmatrix} i_U\Lambda i_{2,3} \otimes j_{1,3}\Gamma j_U & i_U\Lambda i_{2,3} \otimes j_{1,3}\Gamma j_{1,2} & i_U\Lambda i_{2,3} \otimes j_{1,3}\Gamma j_{1,3} & i_U\Lambda i_{2,3} \otimes j_{1,3}\Gamma j_L \\ i_{1,2}\Lambda i_{2,3} \otimes j_{1,3}\Gamma j_U & i_{1,2}\Lambda i_{2,3} \otimes j_{1,3}\Gamma j_{1,2} & i_{1,2}\Lambda i_{2,3} \otimes j_{1,3}\Gamma j_{1,3} & i_{1,2}\Lambda i_{2,3} \otimes j_{1,3}\Gamma j_L \\ i_L\Lambda i_{2,3} \otimes j_{1,3}\Gamma j_U & i_L\Lambda i_{2,3} \otimes j_{1,3}\Gamma j_{1,2} & i_L\Lambda i_{2,3} \otimes j_{1,3}\Gamma j_{1,3} & i_L\Lambda i_{2,3} \otimes j_{1,3}\Gamma j_L \\ i_{2,3}\Lambda i_{2,3} \otimes j_{1,3}\Gamma j_U & i_{2,3}\Lambda i_{2,3} \otimes j_{1,3}\Gamma j_{1,2} & i_{2,3}\Lambda i_{2,3} \otimes j_{1,3}\Gamma j_{1,3} & i_{2,3}\Lambda i_{2,3} \otimes j_{1,3}\Gamma j_L \end{pmatrix}$$

$$= \begin{pmatrix} i_U\Lambda i_{2,3} \otimes j_{1,3}\Gamma j_{1,2} & i_U\Lambda i_{2,3} \otimes j_{1,3}\Gamma j_{1,3} & i_U\Lambda i_{2,3} \otimes j_{1,3}\Gamma j_L \\ i_{1,2}\Lambda i_{2,3} \otimes j_{1,3}\Gamma j_{1,2} & i_{1,2}\Lambda i_{2,3} \otimes j_{1,3}\Gamma j_{1,3} & i_{1,2}\Lambda i_{2,3} \otimes j_{1,3}\Gamma j_L \\ i_{2,3}\Lambda i_{2,3} \otimes j_{1,3}\Gamma j_{1,2} & i_{2,3}\Lambda i_{2,3} \otimes j_{1,3}\Gamma j_{1,3} & i_{2,3}\Lambda i_{2,3} \otimes j_{1,3}\Gamma j_L \end{pmatrix}$$

since $j_{1,3}\Gamma j_U = 0$ and $i_L\Lambda i_{2,3} = 0$.

This module maps by a mapping $\hat{\pi}$ onto Ω_3 by mapping $i_{2,3}\Lambda i_{2,3} \otimes j_{1,3}\Gamma j_{1,3}$ onto its e_3–component. We denote this last mapping by π. We hence have an exact sequence

$$0 \longrightarrow \begin{pmatrix} i_U\Lambda i_{2,3} \otimes j_{1,3}\Gamma j_{1,2} & i_U\Lambda i_{2,3} \otimes j_{1,3}\Gamma j_{1,3} & i_U\Lambda i_{2,3} \otimes j_{1,3}\Gamma j_L \\ i_{1,2}\Lambda i_{2,3} \otimes j_{1,3}\Gamma j_{1,2} & i_{1,2}\Lambda i_{2,3} \otimes j_{1,3}\Gamma j_{1,3} & i_{1,2}\Lambda i_{2,3} \otimes j_{1,3}\Gamma j_L \\ i_{2,3}\Lambda i_{2,3} \otimes j_{1,3}\Gamma j_{1,2} & ker(\pi) & i_{2,3}\Lambda i_{2,3} \otimes j_{1,3}\Gamma j_L \end{pmatrix} \stackrel{\iota}{\longrightarrow}$$

$$\stackrel{\iota}{\longrightarrow} \begin{pmatrix} i_U\Lambda i_{2,3} \otimes j_{1,3}\Gamma j_{1,2} & i_U\Lambda i_{2,3} \otimes j_{1,3}\Gamma j_{1,3} & i_U\Lambda i_{2,3} \otimes j_{1,3}\Gamma j_L \\ i_{1,2}\Lambda i_{2,3} \otimes j_{1,3}\Gamma j_{1,2} & i_{1,2}\Lambda i_{2,3} \otimes j_{1,3}\Gamma j_{1,3} & i_{1,2}\Lambda i_{2,3} \otimes j_{1,3}\Gamma j_L \\ i_{2,3}\Lambda i_{2,3} \otimes j_{1,3}\Gamma j_{1,2} & i_{2,3}\Lambda i_{2,3} \otimes j_{1,3}\Gamma j_{1,3} & i_{2,3}\Lambda i_{2,3} \otimes j_{1,3}\Gamma j_L \end{pmatrix} \stackrel{\hat{\pi}}{\longrightarrow} \Omega_3 \longrightarrow 0$$

Now, we have to find the common extension of the kernel of this sequence and the homology.

We modify the homology-bimodule to do so. The special structure of the Green order Λ (namely the first three properties in Roggenkamp's theorem) and this choice of the first element as done above implies that $i_U\Lambda i_{2,3} = i_U\Lambda i_{1,2}$ and $i_{2,3}\Lambda i_{1,2} = e_2 i_{1,2}\Lambda i_{1,2}$.

We shall use

$$\begin{pmatrix} i_U\Lambda i_U & i_U\Lambda i_{2,3} & e_1 i_U\Lambda i_{1,2} & i_U\Lambda i_L \\ i_{1,2}\Lambda i_U & e_1 i_{1,2}\Lambda i_{1,2} \oplus i_{1,2}\Lambda i_{2,3} & e_1 i_{1,2}\Lambda i_{1,2} & i_{1,2}\Lambda i_L \\ i_L\Lambda i_U & i_L\Lambda i_{1,2} & e_1 i_L\Lambda i_{1,2} & i_L\Lambda i_L \\ i_{2,3}\Lambda i_U & e_2 i_{2,3}\Lambda i_{2,3} & e_1 i_{2,3}\Lambda i_{1,2} & i_{2,3}\Lambda i_L \end{pmatrix}$$

as Λ–left–module which is equal to

$$\begin{pmatrix} j_U\Gamma j_U & j_U\Gamma j_{1,2} & j_U\Gamma j_{1,3} & j_U\Gamma j_L \\ j_{1,2}\Gamma j_U & e_1 j_{1,2}\Gamma j_{1,2} \oplus e_2 j_{1,2}\Gamma j_{1,2} & j_{1,2}\Gamma j_{1,3} & j_{1,2}\Gamma j_L \\ j_L\Gamma j_U & j_L\Gamma j_{1,2} & j_L\Gamma j_{1,3} & j_L\Gamma j_L \\ e_2 j_{1,2}\Gamma j_U & e_2 j_{1,2}\Gamma j_{1,2} & e_2 j_{1,2}\Gamma j_{1,3} & e_2 j_{1,2}\Gamma j_L \end{pmatrix}$$

as Γ–right–module, using the ordering and the special choice of a first element in T_Γ.

We need one more observation concerning the construction of the differential.

The theorem of Roggenkamp gives us a canonical Λ-module structure on $\begin{pmatrix} 0 \\ \overline{\Omega} \\ 0 \\ \overline{\Omega} \end{pmatrix}$ by the following fact. Since in the chosen ordering of T_Λ we have $i_{2,3}e_2 > i_{1,2}e_2 > i_U e_2$, there are inclusions

$$g_{i_U e_2}(\Lambda i_U) < e_2\Lambda i_{1,2} \text{ and } g_{i_{1,2}e_2}(\Lambda i_{1,2}) < e_2\Lambda i_{2,3}\ .$$

Hence we get an exact sequence

$$0 \longrightarrow e_2\Lambda i_U \stackrel{g_{i_U e_2} g_{i_{1,2} e_2}}{\longrightarrow} e_2\Lambda i_{2,3} \stackrel{\begin{pmatrix} 0 \\ f_{e_2} \\ 0 \\ f_{e_2} \end{pmatrix}}{\longrightarrow} \begin{pmatrix} 0 \\ \overline{\Omega} \\ 0 \\ \overline{\Omega} \end{pmatrix} \longrightarrow 0\ .$$

Likewise one finds a Λ–module structure by transport of structure of $\Lambda i_{1,2}e_1$ via $\tilde{f}_{i_{1,2}e_1}$ on $\begin{pmatrix} 0 \\ \overline{\Omega} \\ 0 \\ 0 \end{pmatrix}$.

This gives

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \overline{\Omega} & \overline{\Omega} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \overline{\Omega} & 0 & 0 \end{pmatrix}$$

the structure of a Λ-left–module. The structure from the right as Γ-right-module is given by the fact that

$$f^{\Lambda}_{e_2} = f^{\Gamma}_{e_2} \text{ and } f^{\Lambda}_{e_1} = f^{\Gamma}_{e_1}$$

The second line in the matrix gets its right-Γ-module structure by transport of structure of $e_1 j_{1,2}\Gamma$ via $(0, -f^{\Gamma}_{e_1}, -f^{\Gamma}_{e_1}, 0)$ and the fourth line gets its Γ-right-module structure by transport of structure of $e_2 j_{1,2}\Gamma$ via $\tilde{f}_{j_{1,2}e_2}$.

As a whole, by the analogous arguments as in the case of the Λ-structure, we get a right Γ-structure *using the same mapping*

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & (-f_{e_1}, f_{e_2}) & -f_{e_1} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & f_{e_2} & 0 & 0 \end{pmatrix}$$

for the left and for the right.

With these preparations we can define our X_1 and the differential as pullback diagram

$$\begin{pmatrix} j_U\Gamma j_U & j_U\Gamma j_{1,2} & j_U\Gamma j_{1,3} & j_U\Gamma j_L \\ j_{1,2}\Gamma j_U & e_1 j_{1,2}\Gamma j_{1,2} \oplus e_2 j_{1,2}\Gamma j_{1,2} & j_{1,2}\Gamma j_{1,3} & j_{1,2}\Gamma j_L \\ j_L\Gamma j_U & j_L\Gamma j_{1,2} & j_L\Gamma j_{1,3} & j_L\Gamma j_L \\ e_2 j_{1,2}\Gamma j_U & e_2 j_{1,2}\Gamma j_{1,2} & e_2 j_{1,2}\Gamma j_{1,3} & e_2 j_{1,2}\Gamma j_L \end{pmatrix} \longleftarrow X_1$$

$$\| \qquad\qquad\qquad \|$$

$$\begin{pmatrix} i_U\Lambda i_U & i_U\Lambda i_{2,3} & e_1 i_U\Lambda i_{1,2} & i_U\Lambda i_L \\ i_{1,2}\Lambda i_U & e_1 i_{1,2}\Lambda i_{1,2} \oplus i_{1,2}\Lambda i_{2,3} & e_1 i_{1,2}\Lambda i_{1,2} & i_{1,2}\Lambda i_L \\ i_L\Lambda i_U & i_L\Lambda i_{1,2} & e_1 i_L\Lambda i_{1,2} & i_L\Lambda i_L \\ i_{2,3}\Lambda i_U & e_2 i_{2,3}\Lambda i_{2,3} & e_1 i_{2,3}\Lambda i_{1,2} & i_{2,3}\Lambda i_L \end{pmatrix} \longleftarrow X_1$$

$$\downarrow \chi \qquad\qquad\qquad \downarrow \varphi$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \overline{\Omega} & \overline{\Omega} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \overline{\Omega} & 0 & 0 \end{pmatrix} \stackrel{\psi}{\longleftarrow} \begin{pmatrix} 0 & i_U\Lambda i_{2,3} \otimes j_{1,3}\Gamma j_{1,2} & i_U\Lambda i_{2,3} \otimes j_{1,3}\Gamma j_{1,3} & i_U\Lambda i_{2,3} \otimes j_{1,3}\Gamma j_L \\ 0 & i_{1,2}\Lambda i_{2,3} \otimes j_{1,3}\Gamma j_{1,2} & i_{1,2}\Lambda i_{2,3} \otimes j_{1,3}\Gamma j_{1,3} & i_{1,2}\Lambda i_{2,3} \otimes j_{1,3}\Gamma j_L \\ 0 & 0 & 0 & 0 \\ 0 & i_{2,3}\Lambda i_{2,3} \otimes j_{1,3}\Gamma j_{1,2} & ker(\pi) & i_{2,3}\Lambda i_{2,3} \otimes j_{1,3}\Gamma j_L \end{pmatrix}$$

where

$$\psi = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & f_{e_2}^{\Lambda} \cdot f_{e_1}^{\Gamma} & f_{e_2}^{\Lambda} \cdot f_{e_1}^{\Gamma} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & f_{e_2}^{\Lambda} \cdot f_{e_1}^{\Gamma} & 0 & 0 \end{pmatrix}$$

and

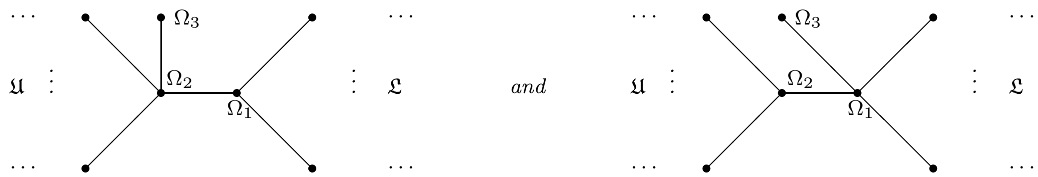
$$\chi = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & (-f_{e_1}, f_{e_2}) & -f_{e_1} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & f_{e_2} & 0 & 0 \end{pmatrix}$$

We define the complex

$$X := (0 \longrightarrow X_1 \xrightarrow{\iota\circ\phi} \Lambda i_{2,3} \otimes_R j_{1,3}\Gamma \longrightarrow 0)$$

Proposition 1. *The complex X in $D^b(\Lambda)$ is isomorphic to a tilting complex with endomorphism ring Γ and X in $D^b(\Gamma^{op})$ is isomorphic to a tilting complex with endomorphism ring Λ.*

Theorem 4. *Let Λ and Γ be Green orders associated to the Brauer trees*



for Λ *for Γ*

with the same data $(\Omega_i, f_i)_{i\in I}$ for the same suitable index set I of vertices.

Let e_3^{Λ} be the central idempotent of $K \otimes_R \Lambda$ corresponding to the leaf Ω_3 of Λ and let e_3^{Γ} be the central idempotent of $K \otimes_R \Gamma$ corresponding to the leaf Ω_3 of Γ.

- *Then, the complex X as defined above is a twosided tilting complex.*
- *The term X_0 of X in degree 0 is the projective cover of Ω_3 as Λ–Γ–bimodule, where Λ acts as multiplication by $\Lambda \cdot e_3^{\Lambda}$ on the left and Γ acts as multiplication by $\Gamma \cdot e_3^{\Gamma}$ on the right.*
- *In case Λ and Γ are Gorenstein orders, the homogeneous component X_1 of X in degree 1 is a Λ–Γ–bimodule which induces a stable equivalence of Morita type.*

Remark 5. We should remind the reader to Broué's definition of a stable equivalence of Morita type. Let R be a commutative ring and let A and B be two R–algebras. If there are a finitely generated $A \otimes_R B^{op}$-module M and a finitely generated $B \otimes_R A^{op}$-module N such that

$$M \otimes_B N \simeq A \oplus P_A \text{ and } N \otimes_A M \simeq B \oplus P_B$$

as $A \otimes_R A^{op}$-modules (or as $B \otimes_R B^{op}$-modules resp.) for a projective $A \otimes_R A^{op}$-module P_A and a projective $B \otimes_R B^{op}$-module P_B, then M is said to induce a stable equivalence of Morita type.

Theorem 4 follows from Proposition 1 and Lemma 1.

The proof of Proposition 1 will cover the two following subsections.

4.1. Restricting to the left. Certainly, the first column in X_1 is isomorphic to Λi_U and the last column in X_1 is isomorphic to Λi_L. On these both modules the differential is zero.

The third column gives rise to a a complex defined by the following pullback diagram.

$$\begin{array}{ccc} e_1\Lambda i_{1,2} & \stackrel{\alpha}{\longleftarrow} & P_0 \\ \downarrow -\tilde{f}_{i_{1,2},e_1} & & \downarrow \beta \\ \begin{pmatrix} 0 \\ \overline{\Omega} \\ 0 \\ 0 \end{pmatrix} & \longleftarrow & \begin{pmatrix} i_U\Lambda i_{2,3}\otimes j_{1,3}\Gamma j_{1,3} \\ i_{1,2}\Lambda i_{2,3}\otimes j_{1,3}\Gamma j_{1,3} \\ i_L\Lambda i_{2,3}\otimes j_{1,3}\Gamma j_{1,3} \\ ker(\pi) \end{pmatrix} \\ & & \downarrow \\ & & \Lambda i_{2,3}\otimes_R j_{1,3}\Gamma j_{1,3} \end{array}$$

The module in degree 1 is P_0, the module in degree 0 is $\Lambda i_{2,3}\otimes_R j_{1,3}\Gamma$ and the differential is given by the composition of the two right hand mappings.

We shall prove that this is isomorphic (in $D^b(\Lambda)$) to the natural mapping

$$g_{i_{1,2},e_1}:\Lambda i_{1,2}\longrightarrow\Lambda i_{2,3}.$$

Let

$$\begin{array}{rcl} j_{1,3}:\Lambda i_{1,2} & \longrightarrow & j_{1,3}\Gamma j_{1,3} \\ \lambda i_{1,2} & \longrightarrow & j_{1,3} \end{array}$$

be the constant mapping. Then, we have a homomorphism

$$\begin{array}{rcl} g_{i_{1,2},e_2}\otimes j_{1,3}:\Lambda i_{1,2} & \rightarrow & \Lambda i_{2,3}\otimes_R j_{1,3}\Gamma j_{1,3} \\ \lambda & \mapsto & g_{i_{1,2},e_2}(\lambda)\otimes j_{1,3} \end{array}$$

Since $e_2\cdot e_3=0$ and $g_{i_{1,2},e_2}(\Lambda i_{1,2})\subseteq\Lambda e_2$, we get that

$$(g_{i_{1,2},e_2}\otimes j_{1,3})(\Lambda i_{2,3})\subseteq ker\hat{\pi}=\begin{pmatrix} i_U\Lambda i_{2,3}\otimes j_{1,3}\Gamma j_{1,3} \\ i_{1,2}\Lambda i_{2,3}\otimes j_{1,3}\Gamma j_{1,3} \\ i_L\Lambda i_{2,3}\otimes j_{1,3}\Gamma j_{1,3} \\ ker(\pi) \end{pmatrix}$$

Now, Roggenkamp's classification of a Green order together with our choice of a first element in T_Λ ensure that $i_U\Lambda i_{2,3}=i_U\Lambda i_{1,2}$ and that $i_{1,2}\Lambda i_{2,3}=e_2 i_{1,2}\Lambda i_{1,2}$.

The lower horizontal mapping however is $(0,f_{e_2}\cdot f_{e_1},0,0)^{tr}$. Since $j_{1,3}$ is the identity element in $j_{1,3}\Gamma j_{1,3}$, the square

$$\begin{array}{ccc} e_1\Lambda i_{1,2} & \longleftarrow & \Lambda i_{1,2} \\ \downarrow & & \downarrow -(g_{i_{1,2},e_2}\otimes j_{1,3}) \\ \begin{pmatrix} 0 \\ \overline{\Omega} \\ 0 \\ 0 \end{pmatrix} & \longleftarrow & ker\ \hat{\pi} \end{array}$$

is commutative. The universal property of the pullback yields a unique mapping

$$\rho:\Lambda i_{2,3}\longrightarrow P_0$$

such that $\alpha\rho(\lambda)=\lambda e_1$ and $\beta\rho(\lambda)=(\lambda e_2)\otimes j_{1,3}$ for any $\lambda\in\Lambda i_{1,2}$. If we define the mapping

$$\begin{array}{rcl} 1\otimes j_{1,3}:\Lambda i_{2,3} & \rightarrow & \Lambda i_{2,3}\otimes j_{1,3}\Gamma j_{1,3} \\ \lambda & \mapsto & \lambda\otimes j_{1,3} \end{array}$$

we get a commutative diagram

$$\begin{array}{ccccc}
e_1\Lambda i_{1,2} & \stackrel{\alpha}{\longleftarrow} & P_0 & \stackrel{\rho}{\longleftarrow} & \Lambda i_{1,2} \\
\downarrow -f_{i_{1,2}e_1} & & \downarrow \beta & & \| \\
\begin{pmatrix} 0 \\ \overline{\Omega} \\ 0 \\ 0 \end{pmatrix} & \longleftarrow & \begin{pmatrix} i_U\Lambda i_{2,3}\otimes j_{1,3}\Gamma j_{1,3} \\ i_{1,2}\Lambda i_{2,3}\otimes j_{1,3}\Gamma j_{1,3} \\ i_L\Lambda i_{2,3}\otimes j_{1,3}\Gamma j_{1,3} \\ ker(\pi) \end{pmatrix} & \stackrel{-g_{i_{1,2}e_2}\otimes j_{1,3}}{\longleftarrow} & e_2\Lambda i_{1,2} \\
 & & \downarrow & & \downarrow -g_{i_{1,2}e_2}\otimes j_{1,3} \\
 & & \Lambda i_{2,3}\otimes_R j_{1,3}\Gamma j_{1,3} & \stackrel{\cdot 1\otimes j_{1,3}}{\longleftarrow} & \Lambda i_{2,3}
\end{array}$$

By Remark 4 one gets that

$$ker(\beta)\simeq ker(\tilde{f}_{i_{1,2}e_1})\simeq ker(g_{i_{1,2}e_2})\ .$$

Moreover, the cokernel of $\alpha\iota$ is Ω_3 by definition of $\hat{\pi}$. Clearly, by construction of our Green order, $coker(\Lambda i_{1,2}\longrightarrow\Lambda i_{2,3})=\Omega_3$ and $1\otimes j_{1,3}$ provides this isomorphism.

Hence, the complex, which is given by the composite of the two middle vertical mappings is isomorphic to the complex which is given by the two right most vertical mappings. This is what we claimed.

The second column gives a complex arising from the following pullback diagram:

$$\begin{array}{ccccc}
e_1\Lambda i_{1,2}\oplus e_2\Lambda i_{2,3} = & \begin{pmatrix} i_U\Lambda i_{2,3} \\ e_1 i_{1,2}\Lambda i_{1,2}\oplus i_{1,2}\Lambda i_{2,3} \\ i_L\Lambda i_{1,2} \\ e_2 i_{2,3}\Lambda i_{2,3} \end{pmatrix} & \longleftarrow & P_1 \\
 & \downarrow \chi & & \downarrow \\
 & \begin{pmatrix} 0 \\ \overline{\Omega} \\ 0 \\ \overline{\Omega} \end{pmatrix} & \stackrel{\psi}{\longleftarrow} & \Lambda i_{2,3}\otimes_R j_{1,3}\Gamma j_{1,2}
\end{array}$$

The mapping $P_1\longrightarrow\Lambda i_{2,3}\otimes_R j_{1,3}\Gamma j_{1,2}$ is surjective, the mapping to the artinian quotient being so, and $\Lambda i_{2,3}\otimes_R j_{1,3}\Gamma j_{1,2}$ is a projective Λ–module, such that the two term complex formed by this mapping is isomorphic in $D^b(\Lambda)$ to its homology. This homology is isomorphic to the kernel of (the restriction of) χ, using the universal property of the pullback. We claim that this kernel is isomorphic to $\Lambda i_{1,2}$.

Observe that the lowest entry in the column giving the mapping χ is just the structure giving mapping

$$f_{i_{2,3},e_2}: i_{2,3}\Lambda i_{2,3}e_2\longrightarrow\overline{\Omega}$$

in Roggenkamp's theorem.

Moreover, the defining permutation ω of T_Λ is such that $\omega(i_{1,2}e_2)=i_{2,3}e_2$. Hence, we have the defining map

$$g_{(i_{1,2},e_2)}:\Lambda i_{1,2}\longrightarrow\Lambda i_{2,3}$$

of a Green order with image being $\Lambda i_{1,2}e_2$.

The Λ–module homomorphism

$$\Lambda i_{1,2}\stackrel{\begin{pmatrix}\cdot e_1 \\ e_2 g_{(i_{1,2},e_2)}\end{pmatrix}}{\longrightarrow} e_1\Lambda i_{1,2}\oplus e_2\Lambda i_{2,3}$$

is injective. In fact, since $g_{(i_{1,2},e_2)}(\Lambda i_{1,2}) = e_2\Lambda i_{1,2}$, an element λ in the kernel has to be in the kernel of the mapping $(\cdot e_1, \cdot e_2)$ which is impossible since

$$\Lambda i_{1,2} \subseteq \Lambda i_{1,2}e_1 \oplus \Lambda i_{1,2}e_2.$$

On the other hand, the composite

$$\Lambda i_{1,2} \stackrel{\binom{\cdot e_1}{e_2 g_{(i_{1,2},e_2)}}}{\longrightarrow} e_1\Lambda i_{1,2} \oplus e_2\Lambda i_{2,3} \stackrel{\chi}{\longrightarrow} \begin{pmatrix} 0 \\ \overline{\Omega} \\ 0 \\ \overline{\Omega} \end{pmatrix}$$

is zero. In fact, in the lowest entry of the module on the right the composition yields 0: We have a commutative diagram

$$\begin{array}{ccccc} \Lambda i_{1,2} & \stackrel{g_{(i_{1,2},e_2)}}{\longrightarrow} & \Lambda i_{2,3} & \stackrel{\cdot e_2}{\longrightarrow} & \Lambda i_{2,3}e_2 \\ & & \downarrow \cdot e_3 & & \downarrow \tilde{f}_{i_{2,3},e_2} \\ & & \Lambda i_{2,3}e_3 & \stackrel{\tilde{f}_{i_{2,3},e_3}}{\longrightarrow} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ \overline{\Omega} \end{pmatrix} \end{array}$$

Now,

$$\chi \circ g_{(i_{1,2},e_2)} = f_{i_{2,3},e_2} \circ \cdot e_2 \circ g_{(i_{1,2},e_2)} = f_{i_{2,3},e_3} \circ \cdot e_3 \circ g_{(i_{1,2},e_2)}$$

and $g_{(i_{1,2},e_2)}$ has image in Λe_2. Since $e_2e_3 = 0$, we obtain the result.

The second entry is dealt by the following argument.

We have to deal with

$$i_{1,2}\Lambda i_{1,2} \stackrel{\binom{\cdot e_1}{e_2 g_{(i_{1,2},e_2)}}}{\longrightarrow} i_{1,2}\Lambda i_{1,2}e_1 \oplus e_2 i_{1,2}\Lambda i_{2,3} = i_{1,2}\Lambda i_{1,2}e_1 \oplus e_2 i_{2,3}\Lambda i_{2,3} \stackrel{(-f_{i_{1,2}e_1}, f_{i_{2,3}e_2})}{\longrightarrow} \overline{\Omega}$$

We know that $f_{i_{2,3}e_2} = f_{i_{1,2}e_2}$ by the second property of a Green order in Roggenkamp's theorem. Hence, since we started with an element in $\Lambda i_{1,2}$, the composite is 0.

Now, since $i_{1,2}e_2 < i_{2,3}e_2$ in T_Λ, we get that $\begin{pmatrix} 0 \\ \overline{\Omega} \\ 0 \\ 0 \end{pmatrix}$ is a Λ-submodule of $\begin{pmatrix} 0 \\ \overline{\Omega} \\ 0 \\ \overline{\Omega} \end{pmatrix}$ and hence we get a short exact sequence of Λ-modules

$$0 \longrightarrow \begin{pmatrix} 0 \\ \overline{\Omega} \\ 0 \\ 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 \\ \overline{\Omega} \\ 0 \\ \overline{\Omega} \end{pmatrix} \stackrel{\nu}{\longrightarrow} \begin{pmatrix} 0 \\ 0 \\ 0 \\ \overline{\Omega} \end{pmatrix} \longrightarrow 0$$

Clearly,

$$ker\ \chi \subseteq ker\ \nu\chi\ .$$

Moreover,

$$\nu\chi|_{\Lambda i_{1,2}e_1} = 0 \text{ and } \nu\chi|_{\Lambda i_{2,3}e_2} = \tilde{f}_{i_{2,3}e_2}\ .$$

Now,

$$ker\ \tilde{f}_{i_{2,3}e_2} = ker\ g_{i_{2,3}e_3} = im\ g_{i_{1,2}e_2} = (\Lambda i_{1,2}e_2)\ .$$

Moreover,

$$\begin{array}{ccc} \Lambda i_{1,2} & \stackrel{e_2 g_{i_{1,2}e_2}}{\longrightarrow} & \Lambda i_{2,3}e_2 \\ \downarrow \cdot e_2 & & \downarrow \begin{pmatrix} 0 \\ f_{e_2} \\ 0 \\ f_{e_2} \end{pmatrix} \\ \Lambda i_{1,2}e_2 & \stackrel{\tilde{f}_{i_{1,2}e_2}}{\longrightarrow} & \begin{pmatrix} 0 \\ \overline{\Omega} \\ 0 \\ \overline{\Omega} \end{pmatrix} \end{array}$$

is commutative; the image of $\tilde{f}_{i_{1,2}e_2}$ being $ker\ \nu$. What remains now to prove is that

$$0 \longrightarrow \Lambda i_{1,2} \stackrel{\binom{e_1}{e_2}}{\to} \Lambda i_{1,2}e_1 \oplus \Lambda i_{1,2}e_2 \stackrel{(-\tilde{f}_{i_{1,2}e_1}, \tilde{f}_{i_{1,2}e_2})}{\longrightarrow} \begin{pmatrix} 0 \\ \overline{\Omega} \\ 0 \\ 0 \end{pmatrix} \longrightarrow 0$$

is exact. But this follows from the definition.

Collecting the pieces, we get that

$$ker(\chi) \simeq \Lambda i_{1,2}.$$

4.2. Restricting to the right. The restriction to the right is treated completely analogously as the restriction to the left. We leave the verification to the reader.

Remark 6. It might be a interesting to try to see if the functor obtained by tensoring with our module X_1 differs from the functor defined by P. Gabriel and Chr. Riedtmann in [3].

5. Appendix

We shall repeat a lemma which is well known to the experts[1], however, as far as it is known to the author, it was never written down.

Lemma 2.
- *Let R be a commutative ring and let Λ and Γ be R–algebras which are projective as R–modules.*
- *Let X be a complex in $D^b(\Lambda \otimes_R \Gamma^{op})$ such that X in $D^b(\Lambda)$ is isomorphic to a tilting complex T and X in $D^b(\Gamma^{op})$ is a isomorphic to a tilting complex S.*
- *Let m_Λ be the smallest natural number, such that there is a $T' \simeq T$ in $K^b(P_\Lambda)$ with ${T_k}' = 0$ for all $k \leq m_\Lambda$*
- *and let n_Λ be the smallest natural number such that there is a $T'' \simeq T$ in $K^b(P_\Lambda)$ with ${T_k}'' = 0$ for all $k \geq n_\Lambda$.*
- *Similarly, m_Γ and n_Γ are defined.*

Then, $m_\Lambda = m_\Gamma =: m$ and for $n = max(n_\Gamma, n_\Lambda)$ there is a $\tilde{X} \simeq X$ in $D^b(\Lambda \otimes_R \Gamma^{op})$ with $\tilde{X}_k$ being a projective $\Lambda \otimes_R \Gamma^{op}$ module for $k = n, n-1, ..., m$ and X_{n+1} is a module which is projective if restricted to Λ and is projective when restricted to Γ^{op}.

Proof. Since we are dealing with projective modules, $m_\Lambda = m_\Gamma$.

Without loss of generality we may, and will assume that $m = 0$. We take a projective resolution as complex of bimodules of X:

$$\ldots \longrightarrow P_3 \stackrel{d_3}{\longrightarrow} P_2 \stackrel{d_2}{\longrightarrow} P_1 \stackrel{d_1}{\longrightarrow} P_0 \longrightarrow 0 \longrightarrow \ldots$$

We truncate as follows:

$$X' := (0 \longrightarrow ker(d_n) \longrightarrow P_n \longrightarrow \ldots \longrightarrow P_0 \longrightarrow 0)$$

[1] in fact, it was mentioned to the author by J.Rickard

and observe that, since both rings are projective as R–modules, $P_0, \dots, P_n$ are both projective when restricted to either side. Clearly, $X \simeq X'$ in $D^b(\Lambda \otimes_R \Gamma^{op})$ since the homology in the degree $n+1$ is 0. Since $n = max\{n_\Lambda, n_\Gamma\}$, the complex $(0 \longrightarrow ker(d_n) \longrightarrow P_n \longrightarrow 0)$ decomposes as $L \stackrel{(id,0)}{\longrightarrow} L \oplus L'$ as complex of Λ–modules. Since $P_n = L \oplus L'$ we conclude that L is projective. Likewise, $ker(d_n)$ is projective as right Γ-module.

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