Self-equivalences of the derived category of Brauer tree algebras with exceptional vertices

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to Mirela Stefanescu for her 60th birthday

Abstract

Let k be a field and A be a Brauer tree algebra associated with a Brauer tree with possibly non trivial exceptional vertex. In an earlier joint paper with Raphaël Rouquier we studied and defined the group $TrPic_k(\Lambda)$ of standard self-equivalences of the derived category of a k-algebra Λ. In the present note we shall determine a non trivial homomorphism of a group slightly bigger than the pure braid group on $n+1$ strings to $TrPic_k(A)$. This is a generalization of the main result in the joint paper with Raphaël Rouquier. The proof uses the result with Raphaël Rouquier.

Introduction

Let A be a k-algebra for a commutative ring k. Suppose that A is projective as a k-module. An equivalence $F : D^b(A) \longrightarrow D^b(A)$ of the derived category of bounded complexes of A modules to itself is called of standard type if there is a complex X in $D^b(A \otimes_k A^{op})$ so that F is isomorphic to $X \otimes^{\mathbb{L}}_A -$. In an earlier paper with Raphaël Rouquier [10] we introduced the group $TrPic_k(A)$ of isomorphism classes of self-equivalences of standard type of the derived category $D^b(A)$. In case A is a Brauer tree algebra for a Brauer tree with $n+1$ vertices, n edges and no exceptional vertex, we obtained in [10] n self-equivalences $F_1, F_2, \cdots, F_n$ of standard type so that $F_i \circ F_j \simeq F_j \circ F_i$ if $|i-j| \geq 2$ and $F_i \circ F_j \circ F_i = F_j \circ F_i \circ F_j$ if not. In other words, the braid group B_{n+1} on $n+1$ strings maps to $TrPic_k(A)$ by some homomorphism φ_n which identifies the natural braid generators with the self-equivalences above. Moreover, if $n=2$ this morphism was shown to be injective with normal image of index $4 \cdot |Pic_k(A)|$. Khovanov and Seidel proved in [4] that the mapping φ_n is injective for all n. The question of the image remains open for $n > 2$.

The purpose of this note is to see what the construction in the paper [10] has to say for a Brauer tree algebra A of a Brauer tree with n edges, $n+1$ vertices and an exceptional vertex with multiplicity strictly bigger than 1. We shall see that we still get subgroups S_n of the standard braid group B_{n+1}, containing the pure braid group P_{n+1} of B_{n+1} and containing the parabolic subgroup B_n of B_{n+1}.

It should be noted that Schaps and Zakay-Illouz obtained in [11] a homomorphism of the braid group of the affine Dynkin diagram $\tilde{A}_n$ to $TrPic_k(A)$. No statement about the kernel or the image is made. Our approach is completely independent from the approach of

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Schaps and Zakay-Illouz. One should observe that the set of homomorphisms of the braid group of the affine Dynkin diagram $\tilde{A}_n$ to the braid group of the Dynkin diagram A_m for all n and m is infinite, as is the set of homomorphisms from the braid group of the diagram A_m to the braid group of the diagram $\tilde{A}_n$.

Finally I should note that our notation is that of the monography [6] and that of the paper [10]. The present note follows an idea which was sketched in an example in [14] for the case $n = 2$. There the method was used to lift a self-equivalence of the derived category of a group algebra to a self-equivalence of the derived category of the group ring over a complete discrete valuation ring, which reduces to the algebra, subject to the additional condition that if the self-equivalence of the derived category of the algebra fixes the trivial module, then the lifted self-equivalence fixes the trivial lattice.

1 A brief review on self-equivalences of derived categories

(1.1) Let k be a commutative ring and let A be a k-algebra. Rickard proved in [7] that for a k-algebra B the derived category $D^b(A)$ of bounded complexes of A-modules is equivalent as triangulated category to the derived category of bounded complexes of B-modules if and only if there is a so-called tilting complex T over A with endomorphism ring in the derived category isomorphic to B. A *tilting complex* over A is a bounded complex of finitely generated projective A-modules so that $Hom_{D^b(A)}(T, T[n]) = 0$ if $n \neq 0$, and so that $K^b(A-proj)$ is the smallest triangulated category of $D^b(A)$ containing all direct summands of finite direct sums of T. Here $K^b(A - proj)$ denotes the homotopy category of bounded complexes of finitely generated projective A-modules. Given an equivalence $D^b(B) \longrightarrow D^b(A)$ the image of the rank 1 free B-module is a tilting complex.

(1.2) In case A is flat as k-module, there exists a complex X in $D^b(A \otimes_R B^{op})$ so that $X \otimes^{\mathbb{L}}_B - : D^b(B) \longrightarrow D^b(A)$ is an equivalence (cf Keller [3] for the general case or Rickard [8] for A and B being projective as k-modules). The complex X is called a twosided tilting complex and the equivalence $X \otimes^{\mathbb{L}}_B -$ is called an equivalence of standard type. Moreover, $X \otimes^{\mathbb{L}}_B B$ is a tilting complex over A, and for any tilting complex T over A with endomorphism ring B there exists a twosided tilting complex X with $T \simeq X \otimes^{\mathbb{L}}_B B$ (cf Keller [3]). Any two tilting complexes X_1 and X_2 with

$$X_2 \otimes^{\mathbb{L}}_B B \simeq T \simeq X_1 \otimes^{\mathbb{L}}_B B$$

differ by an automorphism of B (cf [10]). This automorphism of B is induced by possibly different identifications of B with the endomorphism ring of T.

(1.3) In case B and A are even projective as k-modules, a quasi-inverse of a standard equivalence between the derived categories of A and B is standard as well. Moreover, (cf [10]) if $B = A$, the set of isomorphism classes of self-equivalences of standard type of the derived category of bounded complexes of A-modules forms a group $TrPic_k(A)$. This group contains the ordinary Picard group $Pic_k(A)$ as subgroup. $Pic_k(A)$ is in general not normal in $TrPic_k(A)$.

(1.4) For the theory of representations of finite groups an important class of examples of algebras is the class of Brauer tree algebras (cf e.g. Feit [2] or in connection with derived equivalences [6]). Two Brauer tree algebras A_1 and A_2 defined over the same field and with respect to two Brauer trees with the same number of edges and vertices and with the same multiplicity of the exceptional vertex have equivalent derived categories. As consequence we get $TrPic_k(A_1) \simeq TrPic_k(A_2)$, even though $Pic_k(A_1)$ and $Pic_k(A_2)$ are in general different.

If A is a Brauer tree algebra for a Brauer tree with n edges and $n+1$ vertices and without exceptional vertex, in [10] Raphaël Rouquier and the author defined a homomorphism

$$\varphi_n : B_{n+1} \longrightarrow TrPic_R(A).$$

The restriction of φ_n to any standard parabolic subgroup B_3 is injective. Moreover, Khovanov and Seidel show in [4] that φ_n is always injective.

(1.5) We have to work explicitly with φ_n and so we shall give some details on its construction. By the above we may assume that the Brauer tree of A is a line with n edges. The projective indecomposable modules are numbered $P_1, P_2, \dots, P_n$ so that $Hom_A(P_i, P_{i+1}) \neq 0$ for all $i \in \{1, 2, \dots n-1\}$. We use the convention that the vertices of the Brauer tree are numbered so that the projective P_i is supported by the ordinary characters with the numbers i and $i-1$. Then, (cf [10]) for any $i \in \{1, 2, \dots, n\}$ the complexes X_i defined by

$$\dots \longrightarrow 0 \longrightarrow P_i \otimes_k Hom_A(P_i, A) \longrightarrow A \longrightarrow 0 \longrightarrow \dots$$

are twosided tilting complexes with ismorphism class in $TrPic_k(A)$. They satisfy the relations

$$X_i \otimes_A X_{i+1} \otimes_A X_i \simeq X_{i+1} \otimes_A X_i \otimes_A X_{i+1}$$

and

$$X_i \otimes_A X_j \simeq X_j \otimes_A X_i\ .$$

This means that mapping the standard braid generators s_i of

$$\begin{array}{rl} B_{n+1} =< s_1, s_2, \dots, s_n \quad | & s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}\ ; s_i s_j = s_j s_i \\ & \forall i = 1, \dots, n-1 \forall j = 2, \dots, n : |i-j| \geq 2 > \end{array}$$

to the self-equivalence induced by X_i defines a homomorphism of groups.

(1.6) We shall derive the tilting complex associated to X_i. If $2 \leq i \leq n-1$, then

$$X_i \otimes_A P_j \simeq \begin{cases} P_j & \text{if} \quad |j-i| \geq 2 \\ P_i[1] & \text{if} \quad j = i \\ \dots \longrightarrow 0 \longrightarrow P_i \longrightarrow P_j \longrightarrow 0 \longrightarrow \dots & \text{if} \quad |j-i| = 1 \end{cases}$$

For $i = 1$ one gets

$$X_1 \otimes_A P_j \simeq \begin{cases} P_j & \text{if} \quad j \geq 3 \\ P_1[1] & \text{if} \quad j = 1 \\ \dots \longrightarrow 0 \longrightarrow P_1 \longrightarrow P_2 \longrightarrow 0 \longrightarrow \dots & \text{if} \quad j = 2 \end{cases}$$

For $i = n$ one gets

$$X_n \otimes_A P_j \simeq \begin{cases} P_j & \text{if} \quad j \leq n-2 \\ P_n[1] & \text{if} \quad j = n \\ \dots \longrightarrow 0 \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow 0 \longrightarrow \dots & \text{if} \quad j = n-1 \end{cases}$$

The complexes

$$T_i := \bigoplus_{j=1}^{n} X_i \otimes_A P_j$$

are tilting complexes with endomorphism ring isomorphic to A.

2 Tilting with an exceptional vertex

(2.1) We study $A(l)$, the Brauer tree algebra associated to a Brauer tree being a line with non trivial exceptional vertex associated to the vertex l for $0 \leq l \leq n$. The indecomposable projective module of $A(l)$ supported by the ordinary characters associated to the vertex i and $i-1$ is called $P_i(l)$. Then we can define complexes T_i as above, using the convention that if we do not make precise the differential, then the differential is a mapping with maximal possible image. This condition uniquely determines the differential. Then, the complexes T_i are still tilting complexes as is easily verified. The endomorphism ring is not isomorphic to $A(l)$. Indeed, by [7] the endomorphism ring is again a Brauer tree algebra with the same exceptional multiplicity, and comparing the dimensions of the homomorphisms between the projective indecomposable modules of such an algebra with the dimensions of the homomorphisms of the indecomposable direct summands of $T(i)$, we find easily

$$End_{D^b(A(l))}(T_l) \simeq A(l-1) \ \forall l \in \{1, \dots, n\},$$

$$End_{D^b(A(l))}(T_{l+1}) \simeq A(l+1) \ \forall l \in \{0, \dots, n-1\}$$

and

$$End_{D^b(A(l))}(T_i) \simeq A(l)$$

for $i \notin \{l, l+1\}$ and all $l \in \{0, \dots, n\}$.

(2.2) Let X be a complex in $D^b(B_1 \otimes_k B_2^{op})$ for two k-algebras B_1 and B_2. If now $X \otimes_{B_2} B_2$ is a tilting complex over B_1 with endomorphism ring B_2 and if $B_1 \otimes_{B_1} X$ is a tilting complex over B_2 with endomorphism ring B_1, then X is a twosided tilting complex. This lemma due to Rickard is proved in [12].

(2.3) The most easy case is where the tilting complexes T_i induce a self-equivalence. A twosided tilting complex X_i associated to the tilting complex T_i over $A(l)$ is now

$$X_i \ : \ \dots \longrightarrow 0 \longrightarrow P_i \otimes_k Hom_{A(l)}(P_i, A(l)) \longrightarrow A(l) \longrightarrow 0 \longrightarrow \dots$$

if $i \notin \{l, l+1\}$.

(2.4) Instead of finding a twosided tilting complex X_l associated to the tilting complex T_l over $A(l)$ we determine the projective indecomposable module $X_l \otimes_{A(l)} P_i$ under the isomorphism $X_l \otimes_{A(l)} A(l) \simeq A(l-1)$. First, we have to get around the ambiguity in choosing X_l. For a Brauer tree algebra A associated to a line, let $Aut_0(A)$ be the subgroup of the automorphism group $Aut_k(A)$ of A which fixes each projective indecomposable module. Now, it is clear that $Aut_k(A)/Aut_0(A)$ is cyclic of order 2. In fact, the quotient is generated by the graph automorphism of the Brauer tree, neglecting the exceptional vertex. This follows from the discussion in [10], but also from Frauke Bleher's thesis [1]. Since the choice of a twosided tilting complex is unique up to an (outer) automorphism of the algebra and since we only ask for the images of the projective indecomposable modules, only this cyclic group of order 2 matters. This cyclic group of order 2 now corresponds to numbering the vertices in the graph from 0 to n or from n to 0. On the other hand, we may modify a given twosided tilting complex with tensoring by the $A(n-l)-A(l)$-bimodule ${}_{\alpha}A(l)_1$, where α is the isomorphism $A(n-l) \longrightarrow A(l)$ induced by the graph automorphism.

(2.5) Now, determining the image of the projective indecomposable modules is an easy task since we know that both algebras are Brauer tree algebras (cf [7]), associated to a line and the endomorphism algebra is isomorphic to $A(l-1)$. If we compare the dimension of homomorphisms between the complexes $X_l \otimes_{A(l)} P_i$ and the dimensions of the indecomposable projective $A(l-1)$-modules, we find that in $D^b(A(l-1))$ we have

$$X_l \otimes_{A(l)} P_i \simeq P_i$$

for all $i \in \{1, \ldots, n\}$.

(2.6) Similarly arguments yield that a twosided tilting complex X_{l+1} associated to T_{l+1} over $A(l)$ has the property

$$X_{l+1} \otimes_{A(l)} P_i \simeq P_i$$

in $D^b(A(l+1))$ for all $i \in \{1, \ldots, n\}$.

3 Deducing self-tilting complexes

We are now ready to determine a group induced by the tilting complexes X_i for $i \in \{1, \ldots, n\}$ mapping to $TrPic_k(A(l))$ for any l.

(3.1) The braid group B_n maps onto the Weyl group $\mathfrak{S}_n$ by adding the relations that the standard generators have order 2. Let π_n be this morphism. Let $\mathfrak{S}_{n-1}(l)$ be the subgroup of $\mathfrak{S}_n$ which fixes the letter l. Of course, $\mathfrak{S}_{n-1}(l) \simeq \mathfrak{S}_{n-1}$. Now, $\pi_n^{-1}(\mathfrak{S}_{n-1}(l))$ is a subgroup of B_n.

(3.2) We observe that

$$D^b(\prod_{l=0}^{n} A(l)) \simeq \prod_{i=0}^{n} D^b(A(l)) \ .$$

Since X_i is a complex of $A(l) \otimes_k A(l)^{op}$-modules if $i \notin \{l, l+1\}$ and X_l is a complex of $A(l) \otimes_k A(l+1)^{op}$-modules, while X_{l+1} is a complex of $A(l+1) \otimes_k A(l)^{op}$-modules, an element $s_i \in B_{n+1}$ acts on $D^b(\prod_{l=0}^n A(l))$ as tensor product with the twosided tilting complex $X_i^{\prod} := \prod_{l=0}^n X_i$ in $D^b((\prod_{l=0}^n A(l)) \otimes_k (\prod_{l=0}^n A(l))^{op})$. Moreover, the tensor product with $X_i^{\prod}$ induces a permutation of the index set $\{0, \ldots, n\}$.

(3.3) Hence, this way s_i permutes the $l+1$ factors in the product $\prod_{l=0}^n D^b(A(l))$ as the involution $(i-1, i)$ permutes the set $\{0, \ldots, n\}$. This is exactly the image of s_i under the mapping π_{n+1}.

Moreover, let $w \in B_{n+1}$. Then, the self-equivalence Ω induced by w on $\prod_{l=0}^n D^b(A(l))$ induces a self-equivalence Ω_l on the l-th factor if and only if the permutation $\pi_{n+1}(w)$ of $\{0, 1, \ldots, n\}$ fixes l:

$$\begin{array}{ccc} \prod_{l=0}^n D^b(A(l)) & \xrightarrow{\Omega} & \prod_{l=0}^n D^b(A(l)) \\ \downarrow & & \downarrow \\ D^b(A(l)) & \xrightarrow{\Omega_l} & D^b(A(l)) \end{array}$$

Observe that if one defines a functor $D^b(A(i)) \longrightarrow \prod_{l=0}^n D^b(A(l))$ and composes Ω with it and the projection $\prod_{l=0}^n D^b(A(l)) \longrightarrow D^b(A(i))$, the result will not be an equivalence unless the permutation $\pi_{n+1}(w)$ of $\{0, 1, \ldots, n\}$ fixes i.

(3.4) Now we have proved the main result.

Theorem 1 *Let k be a field. Let $A(l)$ be the Brauer tree algebra associated to a Brauer tree being a line with n edges, with exceptional vertex at the l-th vertex, and with exceptional multiplicity $m \geq 2$. Let B_{n+1} be the (ordinary) braid group on $n+1$ strings and let $\pi_{n+1} : B_{n+1} \longrightarrow \mathfrak{S}_{n+1}$ be the canonical projection. Then, there is a group homomorphism*

$$\varphi_{n+1} : \pi_{n+1}^{-1}(Stab_{\mathfrak{S}_{n+1}}(l)) \longrightarrow TrPic_k(A(l)) \ .$$

Moreover, $s_i \in \pi_{n+1}^{-1}(Stab_{\mathfrak{S}_{n+1}}(l))$ for $i \in \{1, 2, \ldots, l-1, l+2, l+3, \ldots, n\}$ and then, the self-equivalence $\varphi_{n+1}(s_i)$ is infinite cyclic.

Proof. Let $w \in \pi_{n+1}^{-1}(Stab_{\mathfrak{S}_{n+1}}(l))$, then $\varphi_{n+1}(w)$ will be defined to be the derived equivalence denoted by Ω_l in (3.3). Since Ω is standard, it is not difficult to show that Ω_l is standard as well.

Let $|i-j| \geq 2$. Then, virtually the same computation as in [10] yields

$$X_i^{\prod} \otimes_{\prod A(l)} X_j^{\prod} \simeq X_j^{\prod} \otimes_{\prod A(l)} X_i^{\prod} .$$

Actually, this may be verified in each component separately.

Similarly, looking at each of the factors of the product category separately one gets

$$X_i^{\prod} \otimes_{\prod A(l)} X_{i+1}^{\prod} \otimes_{\prod A(l)} X_i^{\prod} \simeq X_{i+1}^{\prod} \otimes_{\prod A(l)} X_i^{\prod} \otimes_{\prod A(l)} X_{i+1}^{\prod}$$

as complex of

$$\prod_{\substack{l=0 \\ (l \notin \{i-1,i+1\})}}^{n} A(l) \otimes_k A(l)^{op} \prod A(i-1) \otimes_k A(i+1)^{op} \prod A(i+1) \otimes_k A(i-1)^{op} \text{ -modules.}$$

Hence, φ_n extends to a homomorphism $\varphi_n : B_{n+1} \longrightarrow TrPic_k(\prod_{l=0}^n A(l))$.

Let now $w_1, w_2 \in \pi_{n+1}^{-1}(Stab_{\mathfrak{S}_{n+1}}(l))$. By (3.3) one has $\varphi_n(w_1)$ and $\varphi_n(w_2)$ induce elements in $TrPic_k(A(l))$. Moreover, $\varphi_n(w_1 w_2) = \varphi_n(w_1)\varphi_n(w_2)$ is an easy consequence. The fact that $\varphi_n(s_i)$ is infinite cyclic is easily seen to be true in $TrPic_k(\prod_{l=0}^n A(l))$ and as well in $TrPic_k(A(l))$. This finishes the proof. ■

Corollary 3.1 *Let A be a Brauer tree algebra associated to a Brauer tree with n edges and a non trivial exceptional vertex. Then, there is a group homomorphism*

$$\pi_{n+1}^{-1}(Stab_{\mathfrak{S}_{n+1}}(l)) \longrightarrow TrPic_k(A) .$$

Under this group homomorphism the image of the remaining s_i is infinite cyclic.

Indeed, such an algebra is derived equivalent to the Brauer tree algebra $A(0)$. The rest follows. ■

Remark 3.2 The p-adic analogue of a Brauer tree algebra is a Green order as defined by Roggenkamp in [9]. At the points (2.1) and (2.5) which were used in the proof of Theorem 1 we used that only a Brauer tree algebra with the same number of simple modules and with the same exceptional multiplicity can be derived equivalent to a Brauer tree algebra. This is not known for Green orders. Nevertheless, the tilting complexes T_i used in the proof are all of a certain type, which were studied in a joint paper with Steffen König (cf [5] and its generalization in [6, Chapter 4]). If one uses this paper at that point one sees that also there the endomorphism ring of such a tilting complex is a Green order again, associated to a Brauer tree being a line and with the same structural constants. Moreover, for these tilting complexes an explicit twosided tilting complex was determined in [12, 13]. The result holds in the same way for Green orders associated to a line with one exceptional vertex and so that the combinatorial data are the same for all the vertices except for the exceptional vertex. The result which takes the place of the main result of [10] is [15]. It is not difficult to fill the details for the proof of this statements. We leave this task to the reader.

Remark 3.3 φ_{n+1} maps any subgroup $B_3 =< s_i, s_{i+1} >$ of $\pi_{n+1}^{-1}(Stab_{\mathfrak{S}_{n+1}}(l))$ injectively to $TrPic_k(A(l))$. This follows from the method of proof in [10], where it is proved that a non trivial word in B_3 is mapped to a non trivial complex in P_i and P_{i+1}. Here, we get the same result with the same proof for the image of P_i and P_{i+1} by this word.

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