Conjugacy of Involutions in Integral Group Rings of Certain Dihedral and Semidihedral Groups

by Alexander Zimmermann

There is an extensive literature on conjugacy of torsion units in the group of augmented units in an integral group ring $\mathbb{Z}G$ of a finite group G as long as one looks at conjugacy in the corresponding rational group ring $\mathbb{Q}G$ (cf. for example [LB-83a, LT-90, LP-92, MRSW-87]). H. Zassenhaus conjectured ([Za-74]) that every torsion unit of $\mathbb{Z}G$ is conjugate to an element of $G \cup (-G)$ by a unit of $\mathbb{Q}G$.

However, very few results are known if one restricts to conjugacy in $\mathbb{Z}G$.

The principal motivation for integral conjugacy of torsion units arise from the famous subgroup rigidity theorem of A. Weiss ([W-87]), G. Thompson ([Th-89]) and K. W. Roggenkmap and L. L. Scott ([R-92]) that states in the most simple form that subgroups of finite order and augmentation 1 of the group ring $\hat{\mathbb{Z}}_pG$ are conjugate in the units of $\hat{\mathbb{Z}}_pG$ to subgroups of G as long as G is a p-group. Here $\hat{\mathbb{Z}}_p$ is the ring of p-adic integers. This of course implies that every augmented unit of finite order is part of a group basis of $\hat{\mathbb{Z}}_pG$.

Does this also happen if one replaces $\hat{\mathbb{Z}}_p$ by $\mathbb{Z}$? To be more precise: Is every augmented unit of finite order in $\mathbb{Z}G$ part of a group basis of $\mathbb{Z}G$?

For p-groups G this has been an open question. For non-p groups there are counterexamples but the subgroups under consideration are not conjugate to a subgroup of any group basis even in the units of $\hat{\mathbb{Z}}_qG$ for a suitably chosen prime q. Therefore essentially *a local phenomenon is measured.* The integral case in the above sense has been open up to now.

Out of the results treated in the literature concerning conjugacy classes of units in the integral group ring of a finite group I want to mention the discussion of group rings of abelian groups in G. Higman's thesis [H-39], the direct calculations of Hughes and Pearson [HP-72] on the integral group ring of the dihedral group of order 6, similar calculations of C. Polcino-Milies [PM-74] concerning the dihedral group of order 8 and those of Allen and Hobby on the alternating group of degree 4 ([AH-80]). A. K. Bhandari and I. S. Luthar ([BL-83b]) counted the number of conjugacy classes of elements of finite order in the augmented units of the integral group ring of dihedral groups of order $2p$ for odd primes p. Neither Allen and Hobby nor Luthar and Bhandari determine which of their units are contained in group bases and which are not. The results in [HP-72, PM-74, AH-80, BL-83b] are obtained by complicate calculations and so I think another method has to be applied.

In 1989 K. W. Roggenkamp and L. L. Scott developed a theory similar to A. Fröhlich's theory of invertible bimodules to determine the conjugacy classes of units u in $V(\mathbb{Z}G)$, the group of units of augmentation 1 in the integral group ring $\mathbb{Z}G$ for a finite group G, that are conjugate in $V(\mathbb{Q}G)$ to an element of the fixed group basis G. We refer to [R-89, RT-92] for details. This theory was applied in [Zi-92] to integral group rings of dihedral groups of order 2^{n+1}, n being any natural number greater or equal to 2, and to integral group rings of dihedral groups of order $2p$, p being any odd prime number. Here the difficulties mentioned above do not arise.

The results of [R-89], [RT-92, Part I, Chapter VII] are summarized as follows: Let U be a fixed subgroup of G and let $C_{\mathbb{Z}G}(U)$ be the centralizer of U in $\mathbb{Z}G$. We will assume $C_{\mathbb{Z}G}(U)$ to be abelian and $\mathbb{Z}G$ to be Eichler over $\mathbb{Z}$. Then the kernel $Cl_{\mathbb{Z}G}(C_{\mathbb{Z}G}(U))$ of the natural induction homomorphism $Cl(C_{\mathbb{Z}G}(U)) \longrightarrow Cl(\mathbb{Z}G)$, $Cl(\Lambda)$ being the locally free class group of the ring Λ, parametrizes up to automorphisms of U, the conjugacy classes– conjugate in $V(\mathbb{Z}G)$– of subgroups H of $V(\mathbb{Z}G)$ that are in $V(\mathbb{Q}G)$ conjugate to U. An element of this kernel is induced from an element of the class group of the center of $\mathbb{Z}G$ if and only if the corresponding conjugacy class of groups H are part of some group bases.

In the author's forthcoming dissertation [Zi-92] this theory is applied to certain dihedral groups $G = D_m$ with subgroup U generated by a non central involution b in D_m.

First we set $m = 2^n$, n being a natural number the following result is obtained:

Theorem 1 *[Zi-92] Let D_{2^n} be the dihedral group of order 2^{n+1}.*

1. *The $V(\mathbb{Q}D_{2^n})$–orbit of a non central involution $b \in D_{2^n}$ splits into $2^{n-1} \cdot \prod_{k\leq n} h_k^+$ conjugacy classes in $\mathbb{Z}D_{2^n}$, h_k^+ being the class number of the maximal real subfield of the field of 2^k–th roots of unity over $\mathbb{Q}$.*

2. *Out of these conjugacy classes 2^{n-1} classes consist of involutions being member of group bases.*

3. *$Cl(C_{\mathbb{Z}D_{2^n}}(b)) \simeq C_{2^{n-1}} \times \prod_{k\leq n} Cl(\mathbb{Z}[\zeta_{2^k} + \zeta_{2^k}^{-1}])^2$, with $\zeta_{2^k}^{2^{k-1}} + 1 = 0$.*

4. *$Cl_{\mathbb{Z}D_{2^n}}(C_{\mathbb{Z}D_{2^n}}(b)) \simeq C_{2^{n-1}} \times \prod_{k\leq n} Cl(\mathbb{Z}[\zeta_{2^k} + \zeta_{2^k}^{-1}])$.*

So H. Cohn's conjecture that $h_k^+ = 1$ for all numbers k (cf. [ACH-65]) is true if and only if every involution in $V(\mathbb{Z}D_{2^n})$ is part of a group basis for all $n \in \mathbb{N}$. The conjecture is proved by J. van der Linden [vdL-82] for $k \leq 7$ and for $k = 8$ using the generalized Riemann hypothesis.

We apply this result to describe the central automorphism group of $\mathbb{Z}D_{2^n}$ explicitly:

Theorem 2 *[Zi-92] The outer central automorphism group $Outcent(\mathbb{Z}D_{2^n})$ of the integral group ring of the dihedral group D_{2^n} of order 2^{n+1} is generated by conjugation by $a - b + 1$, an automorphism of order 2^{n-1} modulo inner automorphisms, and conjugation by $1 + b \cdot (a + a^{-1})$, an automorphism of order 2^{n-2} modulo inner automorphisms. Here a is an element of order 2^n in D_{2^n} and b is a non central involution in D_{2^n}.*

For an abelian group A we denote by $A_{[2]}$ the kernel of the endomorphism of A sending each element to its square. We define $\eta(p) := \zeta_p + \zeta_p^{-1}$ for a primitive p^{th} root of unity ζ_p. If m is a prime p we obtain the following result:

Theorem 3 *[Zi-92] Let D_p be the dihedral group of order $2p$, p being an odd prime.*

1. *In $\mathbb{Z}D_p$ there are exactly*

$$\frac{|Cl(\mathbb{Z}[\eta(p)]C_2)|}{|Cl(\mathbb{Z}[\eta(p)])|}$$

conjugacy classes of involutions, locally and rationally conjugate to b.

2. *An involution $b \in D_p$ is contained in exactly $\frac{p-1}{2}$ representatives of conjugacy classes of group bases.*

3. *Out of the conjugacy classes of involutions rationally and locally conjugate to b in $\mathbb{Z}D_p$ exactly $|(Cl(\mathbb{Z}[\eta(p)]))_{[2]}|$ conjugacy classes consist out of involutions, contained in a group basis.*

4. *If $p \in \{(2nq)^2 + 1 | n \in \mathbb{Z}, n \geq 2, q\mathbb{Z} \in Spec(\mathbb{Z})\}$ is prime, there is a conjugacy class of involutions whose elements are conjugate to b in $V(\hat{\mathbb{Z}}_r D_p)$ for all primes r, but they are not part of any group basis of $\mathbb{Z}D_p$.*

The last part is an immediate application of the result of Ankeney, Chowla and Hasse of [ACH-65].

Remark 1 In [BL-83b] A. K. Bhandari and I. S. Luthar determined the total number of conjugacy classes of involutions in $V(\mathbb{Z}D_p)$ without minding if a conjugacy class is locally conjugate to the fixed involution b of D_p. A. K. Bhandari and I. S. Luthar do not discuss at all questions concerning the relation of elements of finite order and group bases they lie in. They do, however, also look at elements of order p in $V(\mathbb{Z}D_p)$ and they are able to give the generators explicitely if the class number of $\mathbb{Z}[\eta(p)]$ is equal to 1.

I am very grateful to Professor Dr. J. Ritter to bring this result of A. K. Bhandari and I. S. Luthar to my attention at the Erfurt conference.

Remark 2 The proof of Theorem 3 is incomparatively easier than that of Theorem 1 since in case of dihedral groups of order $2p$, p being an odd prime, there are only three Wedderburn components for $\mathbb{Q}D_p$ and two of them are abelian. In contrary the Wedderburn decomposition of $\mathbb{Q}D_{2^n}$ is much more involved.

The method used in the proofs of the Theorems 1 and 3 are illustrated by another example, the semidihedral group of order 16. In fact we will prove here the

Theorem 4 *Let S be the semidihedral group of order 16. In the group of augmented units of $\mathbb{Z}S$ there is a conjugacy class $\mathcal{C}$ of involutions such that each representative of $\mathcal{C}$ is not part of a group basis of $\mathbb{Z}S$.*

Remark 3 Because of the subgroup rigidity theorem every involution of $\mathbb{Z}S$ is conjugate by a unit of the 2-adic group ring $\hat{\mathbb{Z}}_2 S$ to an involution in S (cf. [W-87, Th-89, RT-92, R-92]).

Remark 4 Since S is a 2-group this solves an open problem: In case of p-groups G the coefficient domain $\mathbb{Z}$ is definitively worse than $\hat{\mathbb{Z}}_p$ in spite of subgroup rigidity!

We have to consider various pullback diagrams arising all in the same manner: If A is a K-algebra for a field K of characteristic 0 and if Γ is an R-order in A for R being the ring of algebraic integers in K. Then let $e^2 = e \in Z(A)$ be a central idempotent. It is well known (see [CR-82,7, (2.12)] that

$$\begin{array}{ccc} \Gamma & \xrightarrow{\cdot e} & \Gamma e \\ \downarrow \cdot(1-e) & & \downarrow \\ \Gamma(1-e) & \longrightarrow & \Gamma e/(\Gamma \cap \Gamma e) \end{array}$$

is a pullback diagram. We call this diagram induced by e.

Let $S =< a, b | a^8 = b^2 = baba^{-3} = 1 >$ be the semidihedral group of order 16. We prove that there is an involution u in $V(\mathbb{Z}S)$, the group of augmented units, not contained in a group basis.

Let $C' := C_{\mathbb{Z}S}(b)$ be the centralizer of the involution b in $\mathbb{Z}S$. Then obviously C' is additively generated by the set

$$\{1, b, a + a^3, (a + a^3)b, a^2 + a^6, (a^2 + a^6)b, a^5 + a^7, (a^5 + a^7)b, a^4, a^4 b\}.$$

Since C' is abelian we have to calculate the kernel $Cl_{\mathbb{Z}S}(C')$ of the homomorphism $Cl(C')$ to $Cl(\mathbb{Z}S)$ and prove that there is an element $(\mathcal{A})$ in $Cl_{\mathbb{Z}S}(C')$ not being an image of an element of $Cl(Z(\mathbb{Z}S))$ under the induction map, $Z(\mathbb{Z}S)$ being the center of $\mathbb{Z}S$.

We use Reiner-Ullom's version of Milnor's Mayer-Vietoris-Sequence (cf. [RU-74], [CR-82,7, 49.28)]). The idempotent $e_4 = (1 + a^4)/2$ induces a pullback diagramm

$$\begin{array}{ccc} C' & \xrightarrow{\cdot e_4} & B_2 \\ \downarrow \cdot(1-e_4) & & \downarrow \\ \mathbb{Z}[\sqrt{-2}] < b > & \longrightarrow & \overline{B}_2 \end{array}$$

with B_2 (cf. [Zi-92]) being the image of the residue modulo $(1 - a^4)C'$ in $\mathbb{Z}D_4$, the group ring of the dihedral group of order 8. Moreover $\overline{B}_2$ is the image of B_2 under the map $\mathbb{Z}D_2 \longrightarrow \mathbb{F}_2D_2$ [1] induced by reduction modulo 2. Since in $\mathbb{Z}[\sqrt{-2}]$ only 1 and -1 are units as is well known, we see that in $\mathbb{Z}[\sqrt{-2}] < b >$ only $\pm 1, \pm b$ are units (an element of $\mathbb{Z}[\sqrt{-2}] < b >$ is a unit if and only if it is a unit in its maximal order $\mathbb{Z}[\sqrt{-2}] \oplus \mathbb{Z}[\sqrt{-2}]$ induced by the two one-dimensional representations (cf. [R-83, Lemma 3])). Since B_2 is a subring of $\mathbb{Z}(C_4 \times C_2)^2$, it has only trivial units $\pm 1, \pm b$ (cf.[H-39]). Next we see that $U(\overline{B}_2)$ is generated by

$$\{b, 1 + a + a^3, b + a + a^3, \}$$

and is elementary abelian of order 8. Therefore

$$U(\overline{B}_2)/(U^*(\mathbb{Z}[\sqrt{-2}] < b >) \cdot U^*(B_2)) \simeq C_2 \times C_2,$$

if we denote by * the images of the unit groups in $\overline{B}_2$ under the natural map. Hence the following sequence of groups is exact:

$$1 \longrightarrow C_2 \times C_2 \longrightarrow Cl(C') \longrightarrow Cl(B_2) \oplus Cl(\mathbb{Z}[\sqrt{-2}] < b >) \longrightarrow 1$$

We are therefore concerned with the calculation of the group $Cl(B_2)$ and also of $Cl(\mathbb{Z}[\sqrt{-2}] < b >)$.

First we have a pullback diagram

$$\begin{array}{ccc} \mathbb{Z}[\sqrt{-2}] < b > & \longrightarrow & \mathbb{Z}[\sqrt{-2}] \\ \downarrow & & \downarrow \\ \mathbb{Z}[\sqrt{-2}] & \longrightarrow & \mathbb{Z}[\sqrt{-2}]/2\mathbb{Z}[\sqrt{-2}] \end{array}$$

by the idempotent $(1 + b)/2$. Second we observe that $\mathbb{Z}[\sqrt{-2}]/2\mathbb{Z}[\sqrt{-2}] \simeq \mathbb{F}_2C_2$. And third we realize that

$$1 \longrightarrow C_2 \longrightarrow Cl(\mathbb{Z}[\sqrt{-2}] < b >) \longrightarrow Cl(\mathbb{Z}[\sqrt{-2}]) \oplus Cl(\mathbb{Z}[\sqrt{-2}]) \longrightarrow 1$$

is an exact sequence of abelian groups. Since $Cl(\mathbb{Z}[\sqrt{-2}]) = 1$, a result known to Gauß, we see that

$$Cl(\mathbb{Z}[\sqrt{-2}] < b >) \simeq C_2.$$

[1] $\mathbb{F}_2$ is the field with 2 elements

[2] C_n denotes the cyclic group of order n

Though $Cl(B_2)$, and more generally $Cl(B_n)$, is calculated in [Zi-92] we repeat the few necessary arguments for the special case $Cl(B_2)$. The idempotent $(1+a^2)/2$ induces the pullback diagram

$$\begin{array}{ccc} B_2 & \longrightarrow & B_1 \\ \downarrow & & \downarrow \\ \mathbb{Z} < b > & \longrightarrow & \mathbb{Z}/4\mathbb{Z} < b > \end{array}$$

which in turn induces the Mayer-Vietoris-sequence

$$1 \longrightarrow C_2 \longrightarrow Cl(B_2) \longrightarrow Cl(B_1) \oplus Cl(\mathbb{Z}) \longrightarrow 1.$$

Since $B_1 = \mathbb{Z} < b > +2\mathbb{Z} < a, b > \subseteq \mathbb{Z}V_4$, the group ring of Klein's 4-group, the following pullback diagram is induced by the idempotent $(1+a)/2$:

$$\begin{array}{ccc} B_1 & \longrightarrow & \mathbb{Z} < b > \\ \downarrow & & \downarrow \\ \mathbb{Z} < b > & \longrightarrow & \mathbb{Z}/4\mathbb{Z} < b > \end{array}$$

Therefore again by the Mayer-Vietoris-sequence we obtain the following exact sequence:

$$1 \longrightarrow C_2 \longrightarrow Cl(B_1) \longrightarrow Cl(\mathbb{Z} < b >) \oplus Cl(\mathbb{Z} < b >) \longrightarrow 1$$

Therefore $Cl(B_2)$ is of order 4, it is cyclic as is shown in [Zi-92], but we don't need this here and so we don't prove it.

Next we have to calculate the class group of the center of $\mathbb{Z}S$. It is generated additively by

$$\{1, a + a^3, a^2 + a^6, a^5 + a^7, (1 + a^2 + a^4 + a^6)b, (1 + a^2 + a^4 + a^6)ab\}$$

The idempotent e_4 induces the pullback diagram

$$\begin{array}{ccc} Z(\mathbb{Z}S) & \xrightarrow{\cdot e_4} & Z_2(1) \\ \downarrow & & \downarrow \\ \mathbb{Z}[\sqrt{-2}] & \longrightarrow & \mathbb{Z}[\sqrt{-2}]/2\mathbb{Z}[\sqrt{-2}]. \end{array}$$

Since

$$Z_2(1) = \mathbb{Z} + \mathbb{Z}(a + a^3) + 2\mathbb{Z}a^2 + \mathbb{Z}(1 + a^2)b + \mathbb{Z}(a + a^3)b \subset \mathbb{Z}(C_4 \times C_2),$$

the group of units of $Z_2(1)$ contains only ± 1. Therefore the Mayer-Vietoris-sequence becomes there

$$1 \longrightarrow C_2 \longrightarrow Cl(Z(\mathbb{Z}S)) \longrightarrow Cl(Z_2(1)) \oplus Cl(\mathbb{Z}[\sqrt{-2}]) \longrightarrow 1.$$

We omit the proof that $Cl(Z_2(1)) \simeq C_4 \times C_2$, since it is not important here, and refer to [Zi-92]. What is important is the wellknown fact that $Cl(\mathbb{Z}[\sqrt{-2}]) = 1$. Let $\overline{\mathcal{A}}$ be an ideal of $\mathbb{Z}[\sqrt{-2}] < b >$ locally but not globally free. The existence of $\overline{\mathcal{A}}$ is guaranteed by the fact that the class group $Cl(\mathbb{Z}[\sqrt{-2}] < b >) = C_2$. By [Ta-84] we see that

$$1 \longrightarrow C_2 \longrightarrow Cl(\mathbb{Z}S) \longrightarrow Cl(\mathbb{Z}D_2) \oplus Cl(\begin{pmatrix} \mathbb{Z}[\sqrt{-2}] & \mathbb{Z}[\sqrt{-2}] \\ \mathbb{Z}[\sqrt{-2}] & \mathbb{Z}[\sqrt{-2}] \end{pmatrix}) \longrightarrow 1$$

is the Mayer-Vietoris-sequence corresponding to the idempotent e_4. Now,

$$(1) \qquad Cl(\begin{pmatrix} \mathbb{Z}[\sqrt{-2}] & \mathbb{Z}[\sqrt{-2}] \\ \mathbb{Z}[\sqrt{-2}] & \mathbb{Z}[\sqrt{-2}] \end{pmatrix}) \simeq Cl(\mathbb{Z}[\sqrt{-2}]) \simeq 1$$

by Swan's theorem (cf. [S-62]).

Let $(\mathcal{A})$ be an arbitrary preimage of the isomorphism class $(\overline{\mathcal{A}})$ in $Cl(C')$. (Observe that $(\overline{\mathcal{A}})$ maps to 1 if it is induced to an isomorphism class of an ideal of $\begin{pmatrix} \mathbb{Z}[\sqrt{-2}] & \mathbb{Z}[\sqrt{-2}] \\ \mathbb{Z}[\sqrt{-2}] & \mathbb{Z}[\sqrt{-2}] \end{pmatrix}$.)

If $(\mathcal{A})$ would be induced by an ideal of $Z(\mathbb{Z}S)$, say $\mathcal{B}$, we calculate

$$\begin{aligned}
\overline{\mathcal{A}} &\simeq \mathcal{A} \otimes_{C'} C'(1-e_4) \\
&\simeq \mathcal{B} \otimes_{Z(\mathbb{Z}S)} C' \otimes_{C'} C'(1-e_4) \\
&\simeq \mathcal{B} \otimes_{Z(\mathbb{Z}S)} C'(1-e_4) \\
&\simeq \mathcal{B} \otimes_{Z(\mathbb{Z}S)} Z(\mathbb{Z}S)(1-e_4) \otimes_{Z(\mathbb{Z}S)(1-e_4)} C'(1-e_4) \\
&\simeq Z(\mathbb{Z}S)(1-e_4) \otimes_{Z(\mathbb{Z}S)(1-e_4)} C'(1-e_4) \\
&\simeq C'(1-e_4);
\end{aligned}$$

the fifth isomorphism holds because of (1). This however is a contradiction to the choice of $\overline{\mathcal{A}}$ and of $\mathcal{A}$.

Let us turn to the question if there is an isomorphism class of an ideal in $Cl_{\mathbb{Z}S}(C')$ such that none of the representatives are induced by an ideal of the center of $\mathbb{Z}S$ rather than an isomorphism class of an ideal in $Cl(C')$.

Let $\hat{\mathcal{B}}$ be an ideal generating the class group of $\mathbb{Z}S$. It has order 2 and is isomorphic to a Swan-module by a theorem of Endo (cf. [Ta-84, 3. Theorem 2.5 & Theorem 2.6]). Those, however, are induced by an ideal of the center of $\mathbb{Z}S$: For the proof of this fact it is convenient to prove a little lemma analogous to [GRH-88, Theorem 2.7].

Lemma 1 *[Zi-92] K is the quotient field of the Dedekind domain R of characteristic 0. Let Λ be an R–order in a finite dimensional K–algebra A and let Δ be an R–order in a finite dimensional K–algebra B, such that $\Lambda \leq \Delta$ and $A \leq B$ and let J be a locally free Λ–right ideal. Then $J \otimes_\Lambda \Delta \simeq J \cdot \Delta$ evaluated in B.*

Proof.

$$0 \longrightarrow \Delta \longrightarrow B \longrightarrow B/\Delta \longrightarrow 0$$

is an exact sequence of Δ–modules and therefore also of Λ-modules. Then an exact sequence of Λ–modules is induced:

$$Tor_1^\Lambda(J, B/\Delta) \longrightarrow J \otimes_\Lambda \Delta \longrightarrow J \otimes_\Lambda B$$

Since J is projective,

$$Tor_1^\Lambda(J, B/\Delta) = 0.$$

Hence

$$\begin{aligned} J \otimes_\Lambda B &\simeq (J \otimes_\Lambda \Delta) \otimes_R K \\ &\simeq K \otimes_R J \otimes_\Lambda \Delta \\ &\simeq (J \otimes_R K) \otimes_\Lambda \Delta \\ &\simeq A \otimes_\Lambda \Delta \\ &\simeq K \otimes_R \Lambda \otimes_\Lambda \Delta \\ &\simeq K \otimes_R \Delta \\ &\simeq B \end{aligned}$$

and the lemma is proved if one follows the isomorphisms. q.e.d.

Since for Swan-modules one can choose idèles with coefficients in the center of $\mathbb{Z}S$ representing them ([Ta-84, 1 (3.13)]), by the lemma they are induced by an ideal of the center of $\mathbb{Z}S$.

Let φ denote the map $Cl(C') \longrightarrow Cl(\mathbb{Z}S)$ and ψ denote the map $Cl(Z(\mathbb{Z}S)) \longrightarrow Cl(C')$. If $\varphi(\mathcal{A})$ is not equal to 1, it is equal to the isomorphism class of the above Swan-module $\mathcal{D} = \varphi(\psi(\hat{\mathcal{D}}))$, for an ideal $\hat{\mathcal{D}}$ of the center of $\mathbb{Z}S$, since induction from the center to the group ring factors through $Cl(C')$. Therefore there is an ideal $\mathcal{E}$ of C' in the kernel of φ with

$$(\mathcal{A}) = (\mathcal{E}) \cdot \psi(\hat{\mathcal{D}}).$$

Since $im\,\psi$ is a group and $\psi(\hat{\mathcal{D}})$ is in $im\,\psi$ we conclude that

$$(\mathcal{E}) \in ker\,\varphi \ \setminus \ im\,\psi.$$

In either case there is an ideal, $\mathcal{A}$ if it is in $ker\,\varphi$, or $\mathcal{E}$, if it is not, inducing an involution in $V(\mathbb{Z}S)$ not being part of a group basis but rationally conjugate to b.

References

[ACH-65] N.C. Ankeney, S. Chowla, H. Hasse: On the class number of the maximal real subfield of a cyclotomic field; J. reine angew. Math. 217 (1965), 217–220.

[AH-80] P. J. Allen, C. Hobby: A Characterization of Units in $\mathbb{Z}[A_4]$; J. of Algebra 66, 534–543, (1980).

[BL-83b] A. K. Bhandari, I. S. Luthar: Conjugacy classes of torsion units of the integral group ring of D_p; Comm. Alg. 11 (14), 1607–1627 (1983); Erratum in Comm. Alg. 18 (1990).

[CR-82,7] C. W. Curtis, I. Reiner: Methods of Representation theory, Vol. 1 & 2; John Wiley Interscience 1982 & 1987.

[GRH-88] W. Gustafson, K.W. Roggenkamp: Automorphisms and Picard groups for Hereditary Orders; Visiting Scholar's Lectures (1988), Texas Tech Univ.

[H-39] G. Higman: The Units in Group Rings; Dr. phil. thesis, Oxford University (1939).

[HP-72] I. Hughes, K. E. Pearson: The group of units in the group ring $\mathbb{Z}S_3$; Can. Math. Bull. 15 (1972), 529–534.

[LB-83a] I. S. Luthar, A. K. Bhandari: Torsion Units of Integral Group Rings of Metacyclic Groups; J. Number Theory 17 (1983), 270–283.

[LT-90] I.S. Luthar, P. Trama: Zassenhaus-conjecture for certain integral group rings; J. Indian Math. Soc. 55 (1990) no. 1–4, 199–212.

[LP-92] I.S. Luthar, I.B.S. Passi: Torsion Units in Matrix Group Rings; Comm. Alg. 20 (4) (1992), 1223–1228.

[MRSW-87] Z. Marciniak, J. Ritter, S.K. Sehgal, A. Weiss: Torsion Units in Integral Group Rings of Some Metabelian Groups, II; J. Number Theory 25 (1987), 340–352.

[PM-74] C. Polcino Milies: The Group of Units of the Integral Group Ring $\mathbb{Z}D_4$; Bol. Soc. Brasil. Math. 4 (1974), 85–92.

[RU-74] I. Reiner, S. Ullom: A Mayer-Vietoris Sequence for Class Groups; J. Algebra 31 (1974), 305–342.

[R-83] K. W. Roggenkamp: Units in metabelian group rings: non splitting examples for normalized units; J. of Pure and Appl. Algebra 27 (1983), 299–314.

[R-89] K.W. Roggenkamp: The isomorphism problem and related topics; Bayreuther Mathematische Schriften 33 (1989), 173–196.

[R-92] K. W. Roggenkamp: Subgroup rigidity of p–adic group rings (Weiss' arguments revisited); to appear J. Lon. Math. Soc. (1992).

[RT-92] K.W. Roggenkamp, M.J. Taylor: Group Rings and Class Groups; Basel, 1992.

[S-62] R.G. Swan: Projective modules over group rings and maximal orders; Annals of Mathematics 76 (1962), 55-61.

[Ta-84] M. J. Taylor: Classgroups of Group Rings; Cambridge 1984.

[Th-89] G. Thompson: On the conjugacy of group bases, Ph.D. theses, Univ. of Virginia, 1989.

[vdL-82] F. J. van der Linden: Class Number Computations of Real Abelian Number Fields; Math. Comp. 39 no.160 (1982), 693–707.

[W-87] A. Weiss: Rigidity of p–adic p–torsion; Annals of Mathematics (1987), 317–332.

[Za-74] H. Zassenhaus: On the torsion units of finite group rings, Estudos de Mathematica, Institut de alta cultura, Lisboa (1974), 119–126.

[Zi-92] A. Zimmermann: Endliche Untergruppen der Einheitengruppe ganzzahliger Gruppenringe; Dissertation, Stuttgart 1992.