Tilted symmetric orders are symmetric orders

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Abstract

We prove in this note that a ring which is derived equivalent to a symmetric order is again a symmetric order. Group rings of finite groups over an integral domain of characteristic 0 are symmetric orders.

Introduction and statement of the result

Let R be a noetherian integral domain with field of fractions K. We call an R-algebra Λ with $K \otimes_R \Lambda$ being semi-simple artinian and Λ being finitely generated projective as R-module a (classical) R-order. An R-algebra is called symmetric if $Hom_R(\Lambda, R) \simeq \Lambda$ as $\Lambda \otimes_R \Lambda^{op}$-modules. The bounded derived category of a ring A is denoted by $D^b(A)$. For more ample remarks to the definition see e.g. [3].

Theorem 1 *Assume that R is Dedekind domain and Λ is a symmetric R-algebra, projective as R-module. If $D^b(\Lambda) \simeq D^b(\Gamma)$ as triangulated categories for an R-algebra Γ, then Γ is a symmetric R-algebra as well. Especially, if Λ is an R-order, then Γ is an R-order as well.*

In case Γ is *supposed to be an order as well*, Rickard proved this statement in [4]. His proof does not work if one does not know that Γ is R-projective. We get this fact as a consequence of the symmetry, which we prove first. The existence of a derived equivalence induced by a derived functor follows by a theorem of Keller (cf [1] or [3, Theorem 8.3.2]). König discovered that there are rings which are not orders but derived equivalent to a hereditary order [2]. Theorem 1 answers a question of K. W. Roggenkamp posed at the occasion of the "Darstellungstag Bayreuth 1994".

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The proof

By a result of Keller [1] there are a complex X in $D^b(\Lambda \otimes_R \Gamma^{op})$ such that $X \otimes_\Gamma \Gamma =: T$ and a complex X' in $D^b(\Gamma \otimes_R \Lambda^{op})$ such that $\Gamma \otimes_\Gamma X' \simeq Hom_\Lambda(T, \Lambda)$ and furthermore

$$X \otimes^{\mathbb{L}}_\Gamma - : D^b(\Gamma) \longrightarrow D^b(\Lambda)$$

and

$$- \otimes^{\mathbb{L}}_\Gamma X' : D^b(\Gamma^{op}) \longrightarrow D^b(\Lambda^{op})$$

are equivalences of triangulated categories. The identification of Γ as endomorphism ring of T and as endomorphism ring of $Hom_\Lambda(T, \Lambda)$ is done coherently ; a fact which defines X' in a coherent way once X is chosen. One should observe that since Λ is R-projective of finite type, one can choose a projective resolution of X and of X' as bimodules, as described in [3, Proposition 6.3.1 and Lemma 6.3.12], representing the same object in the derived category. Any projective $\Lambda\otimes_R\Gamma^{op}$ module is projective as Γ^{op}-module since Λ is projective as R-module. The analogous statement is valid for X'. Hence, the left derived tensor product can be replaced by the ordinary tensor product. Then, one gets

$$(X\otimes_\Gamma -)\otimes_\Gamma X'\simeq X\otimes_\Gamma(-\otimes_\Gamma X')$$

as functor $D^b(\Gamma\otimes_R\Gamma^{op})\longrightarrow D^b(\Lambda\otimes_R\Lambda^{op})$. Hence, the functor

$$G:=\ X\otimes^{\mathbb{L}}_\Gamma\ -\ \otimes^{\mathbb{L}}_\Gamma X':\ D^b(\Gamma\otimes_R\Gamma^{op})\longrightarrow D^b(\Lambda\otimes_R\Lambda^{op})$$

is an equivalence.

Lemma 1 *Let R be a commutative noetherian ring and let Λ be an R-algebra which is finitely generated projective as R-module. Let G be a derived equivalence as described above. Then,*

$$G(Hom_R(\Gamma,R))\simeq Hom_R(\Lambda,R).$$

Proof. This is a consequence of adjointness formulas and the fact that T is perfect over Λ. Indeed,

$$\begin{aligned}
&Hom_{D^b(\Gamma\otimes_R\Gamma^{op})}(\Gamma\otimes_R\Gamma^{op},G^{-1}(Hom_R(\Lambda,R)[n]))\\
&\quad\simeq\ Hom_{D^b(\Lambda\otimes_R\Lambda^{op})}(T\otimes_R Hom_\Lambda(T,\Lambda),Hom_R(\Lambda,R)[n])\\
&\quad\simeq\ Hom_{D^b(\Lambda)}(T,Hom_{\Lambda^{op}}(Hom_\Lambda(T,\Lambda),Hom_R(\Lambda,R))[n])\\
&\quad\simeq\ Hom_{D^b(\Lambda)}(T[-n],Hom_R(Hom_\Lambda(T,\Lambda)\otimes_\Lambda\Lambda,R))\\
&\quad\simeq\ Hom_{D^b(\Lambda)}(T[-n],Hom_R(Hom_\Lambda(T,\Lambda),R))\\
&\quad\simeq\ Hom_R(RHom_{D^b(\Lambda)}(T,\Lambda)\otimes_\Lambda T[-n],R)\\
&\quad\simeq\ Hom_R(Hom_{D^b(\Lambda)}(T,T[-n]),R)\\
&\quad\simeq\ \begin{cases}Hom_R(\Gamma,R) & \text{if } n=0\\ 0 & \text{if } n\neq 0\end{cases}
\end{aligned}$$

Here the first isomorphism is applying G, the second is an adjointness formula between $Hom_{\Lambda^{op}}(Hom_\Lambda(T,\Lambda),-)$ and $-\otimes_\Lambda Hom_\Lambda(T,\Lambda)$, the third is again a similar adjointness formula, the fourth is the fact that $-\otimes_\Lambda\Lambda$ is equivalent to the identity functor, and then we use again the adjointness of $Hom_R(Hom_\Lambda(T,\Lambda),-)$ and $Hom_\Lambda(T,\Lambda)\otimes_\Lambda -$. Finally we use the fact that T is a tilting complex. This proves that the homology of $G^{-1}(Hom_R(\Lambda,R))$ is concentrated in degree 0 and isomorphic to $Hom_R(\Gamma,R)$. This finishes the proof of the lemma. ■

Proof of theorem 1. Since Λ is symmetric, $\Lambda\simeq Hom_R(\Lambda,R)$ in $D^b(\Lambda\otimes_R\Lambda^{op})$. Moreover,

$$\begin{aligned}
Hom_{D^b(\Gamma\otimes_R\Gamma^{op})}(\Gamma\otimes_R\Gamma^{op},G^{-1}(\Lambda)[n]))\ &=\ Hom_{D^b(\Lambda\otimes_R\Lambda^{op})}(T\otimes_R Hom_\Lambda(T,\Lambda),\Lambda)[n])\\
&=\ Hom_{D^b(\Lambda)}(T,Hom_\Lambda(Hom_\Lambda(T,\Lambda),\Lambda[n]))\\
&=\ Hom_{D^b(\Lambda)}(T,T[n])\\
&=\ \begin{cases}\Gamma & \text{if } n=0\\ 0 & \text{else}\end{cases}
\end{aligned}$$

applying G^{-1} gives that $\Gamma\simeq Hom_R(\Gamma,R)$ in $D^b(\Gamma\otimes_R\Gamma^{op})$.

If R is an integral domain, $Hom_R(\Gamma,R)$ is R-torsion free, hence so is Γ. Tensoring the derived equivalence induced by T with the field of fractions K of R (see [3, Proposition 6.2.4]), one gets a Morita equivalence between $A:=K\otimes_R\Lambda$ and $B:=K\otimes_R\Gamma$. Since Γ is R-torsion free, Γ is an R-order in B. This proves the theorem. ■

References

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