Glasgow Math. J. **54** (2012) 647–654. © Glasgow Mathematical Journal Trust 2012.
doi:10.1017/S0017089512000237.

A NOETHER–DEURING THEOREM FOR DERIVED CATEGORIES

ALEXANDER ZIMMERMANN
Université de Picardie, Département de Mathématiques et LAMFA (UMR 7352 du CNRS), 33 rue St Leu, F-80039 Amiens Cedex 1, France
e-mail: alexander.zimmermann@u-picardie.fr

(Received 15 November 2011; revised 28 December 2011; accepted 3 January 2012; first published online 30 March 2012)

Abstract. We prove a Noether–Deuring theorem for the derived category of bounded complexes of modules over a Noetherian algebra.

2010 *Mathematics Subject Classification.* Primary 16E35; Secondary 11S36, 13J10, 18E30, 16G30.

1. Introduction. The classical Noether–Deuring theorem states that given an algebra A over a field K and a finite extension field L of K, two A-modules M and N are isomorphic as A-modules, if $L \otimes_K M$ is isomorphic to $L \otimes_K N$ as an $L \otimes_K A$-module. In 1972, Roggenkamp gave a nice extension of this result to extensions S of local commutative Noetherian rings R and modules over Noetherian R-algebras.

For the derived category of A-modules no such generalisation was documented before. The purpose of this note is to give a version of the Noether–Deuring theorem, in the generalised version given by Roggenkamp, for right bounded derived categories of A-modules. If there is a morphism $\alpha \in Hom_{D(\Lambda)}(X, Y)$, then it is fairly easy to show that for a faithfully flat ring extension S over R the fact that $id_S \otimes \alpha$ is an isomorphism implies that α is an isomorphism. This is done in proposition (1). More delicate is the question if only an isomorphism in $Hom_{D(S\otimes_R\Lambda)}(S \otimes_R X, S \otimes_R Y)$ is given. Then, we need further finiteness conditions on Λ and on R and proceed by completion of R and then a classical going-down argument. This is done in theorem (4) and corollary (8).

For the notation concerning derived categories, we refer to Verdier [**6**]. In particular, $D(A)$ (resp $D^-(A)$, resp $D^b(A)$) denotes the derived category of complexes (resp. right bounded complexes, resp. bounded complexes) of finitely generated A-modules, $K^-(A - proj)$ (resp. $K^b(A - proj)$, resp $K^{-,b}(A - proj)$) is the homotopy category of right bounded complexes (resp. bounded complexes, resp. right bounded complexes with bounded homology) of finitely generated projective A-modules. For a complex Z, we denote by $H_i(Z)$ the homology of Z in degree i, and by $H(Z)$ the graded module given by the homology of Z.

2. The result. We start with an easy observation.

PROPOSITION 1. *Let R be a commutative ring and let Λ be an R-algebra. Let S be a commutative faithfully flat R-algebra. Denote by $D(\Lambda)$ the derived category of complexes of finitely generated Λ-modules. Then, if there is $\alpha \in Hom_{D(\Lambda)}(X, Y)$ so that $id_S \otimes_R^{\mathbb{L}} \alpha \in Hom_{D(S\otimes_R\Lambda)}(S \otimes_R^{\mathbb{L}} X, S \otimes_R^{\mathbb{L}} Y)$ is an isomorphism in $D(S \otimes_R \Lambda)$, then α is an isomorphism in $D(\Lambda)$.*

Proof. Let Z be a complex in $D(\Lambda)$. Since S is flat over R the functor $S \otimes_R - : R-Mod \longrightarrow S-Mod$ is exact, and hence the left derived functor $S \otimes^{\mathbb{L}}_R -$ coincides with the ordinary tensor product functor $S \otimes_R -$. We can therefore work with the usual tensor product and a complex Z of Λ-modules.

We claim that since S is flat, $S \otimes_R -$ induces an isomorphism $S \otimes_R H(Z) \simeq H(S \otimes^{\mathbb{L}}_R Z)$.

If ∂_Z is the differential of Z, then

$$0 \longrightarrow \ker(\partial_Z) \longrightarrow Z \stackrel{\partial_Z}{\longrightarrow} im(\partial_Z) \longrightarrow 0$$

is exact in the category of Λ-modules.

Since S is flat,

$$0 \longrightarrow S \otimes_R \ker(\partial_Z) \longrightarrow S \otimes_R Z \stackrel{id_S \otimes_R \partial_Z}{\longrightarrow} S \otimes_R im(\partial_Z) \longrightarrow 0$$

is exact. Hence,

$$\ker(id_S \otimes_R \partial_Z) = S \otimes_R \ker(\partial_Z) \text{ and } im(id_S \otimes_R \partial_Z) = S \otimes_R im(\partial_Z).$$

This shows the claim.

Since $id_S \otimes_R \alpha$ is an isomorphism, its cone $C(id_S \otimes_R \alpha)$ is acyclic. Moreover, $C(id_S \otimes_R \alpha) = S \otimes_R C(\alpha)$ by the very construction of the mapping cone. But now,

$$0 = H(C(id_S \otimes_R \alpha)) = H(S \otimes_R C(\alpha)) = S \otimes_R H(C(\alpha)).$$

Since S is faithfully flat, this implies $H(C(\alpha)) = 0$, and therefore, $C(\alpha)$ is acyclic. We conclude that α is an isomorphism in $D(\Lambda)$ which shows the statement. □

REMARK 2. Observe that we assumed that $X \stackrel{\alpha}{\longrightarrow} Y$ is assumed to be a morphism in $D(\Lambda)$. The question if the existence of an isomorphism $S \otimes_R X \stackrel{\hat{\alpha}}{\longrightarrow} S \otimes_R Y$ in $D(S \otimes_R \Lambda)$ implies the existence of a morphism $\alpha : X \longrightarrow Y$ in $D(\Lambda)$ so that $id_S \otimes^{\mathbb{L}}_R \alpha$ is an isomorphism is left open. Under stronger hypotheses, this is the purpose of Theorem 4 below. The proof follows **[5]** which deals with the module case.

LEMMA 3. *If S is a faithfully flat R-module and Λ is a Noetherian R-algebra, then for all objects X and Y of $D^b(\Lambda)$, we get*

$$Hom_{D^b(S \otimes_R \Lambda)}(S \otimes_R X, S \otimes_R Y) \simeq S \otimes_R Hom_{D^b(\Lambda)}(X, Y).$$

Proof. Since S is flat over R, the functor $S \otimes_R -$ preserves quasi-isomorphisms, and therefore, we get a morphism

$$S \otimes_R Hom_{D^b(\Lambda)}(U, V) \longrightarrow Hom_{D^b(S \otimes_R \Lambda)}(S \otimes_R U, S \otimes_R V)$$

in the following way. Given a morphism ρ in $Hom_{D^b(\Lambda)}(U, V)$ represented by the triple $(U \stackrel{\alpha}{\longleftarrow} W \stackrel{\beta}{\longrightarrow} V)$, for a quasi-isomorphism α and a morphism of complexes β, and $s \in S$ then map $s \otimes \rho$ to $(S \otimes_R U \stackrel{id_S \otimes \alpha}{\longleftarrow} S \otimes_R W \stackrel{s \otimes \beta}{\longrightarrow} S \otimes_R V)$. This is natural in U and V.

We use the equivalence of categories $K^{-,b}(\Lambda - proj) \simeq D^b(\Lambda)$ and suppose, therefore, that X and Y are right bounded complexes of finitely generated projective

Λ-modules. But

$$S \otimes_R Hom_\Lambda(\Lambda^n, U) = S \otimes_R U^n = (S \otimes_R U)^n = Hom_{S\otimes_R\Lambda}((S \otimes_R \Lambda)^n, S \otimes_R U)$$

which proves the statement in case X or Y is in $K^b(A - proj)$ since then a homomorphism is given by a direct sum of finitely many homogeneous mappings in those degrees where the complexes do both have non-zero components. Now, tensor product commutes with direct sums.

We come to the general case. Recall the so-called stupid truncation τ_N of a complex. Let Z be a complex in $K^{-,b}(\Lambda - proj)$, denoted by ∂ its differential and let $N \in \mathbb{N}$ so that $H_n(Z) = 0$ for all $n \geq N$. We denote the homogeneous components of ∂ so that $\partial_n : Z_n \longrightarrow Z_{n-1}$ for all n. Let $\tau_N Z$ be the complex given by $(\tau_N Z)_n = Z_n$ if $n \leq N$ and $(\tau_N Z)_n = 0$ else. The differential δ on $\tau_N Z$ is defined to be $\delta_n = \partial_n$ if $n \leq N$ and $\delta_n = 0$ else. Now, $\ker(\partial_N) =: C_N(Z)$ is a finitely generated Λ-module. Therefore, we get an exact triangle, called in the sequel the truncation triangle for Z,

$$\tau_N Z \longrightarrow Z \longrightarrow C_N(Z)[N+1] \longrightarrow (\tau_N Z)[1]$$

for all objects Z in $K^{-,b}(A - proj)$. Obviously, $\tau_N(S \otimes_R Z) = S \otimes_R \tau_N Z$ and since S is flat over R also $C_N(S \otimes_R Z) = S \otimes_R C_N(Z)$.

We choose N so that $H_n(X) = H_n(Y) = 0$ for all $n \geq N$. To simplify the notation denote for the moment the bifunctor $Hom_{K^{-,b}(\Lambda-proj)}(-,-)$ by $(-,-)$, the bifunctor $Hom_{K^{-,b}(S\otimes_R\Lambda-proj)}(-,-)$ by $(-,-)_S$ and the bifunctor $S \otimes_R Hom_{K^{-,b}(\Lambda-proj)}(-,-)$ by $S(-,-)$. Further, put $S \otimes_R X =: X_S$ and $S \otimes_R Y =: Y_S$. From the long exact sequence obtained by applying $(X_S, -)_S$ to the truncation triangle of Y_S we get a commutative diagram with exact lines (†)

$$\begin{array}{ccccccccc}
(X_S, C_N(Y_S)[N])_S & \to & (X_S, \tau_N Y_S)_S & \to & (X_S, Y_S)_S & \to & (X_S, C_N(Y_S)[N+1])_S & \to & (X_S, \tau_N Y_S[1])_S \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
S(X, C_N(Y)[N]) & \to & S(X, \tau_N Y) & \to & S(X, Y) & \to & S(X, C_N(Y)[N+1]) & \to & S(X, \tau_N Y[1])
\end{array}$$

Since $\tau_N(Y_S)$ is a bounded complex of projectives,

$$(X_S, \tau_N Y_S)_S = S \otimes_R (X, \tau_N Y) \text{ and } (X_S, \tau_N Y_S[1])_S = S \otimes_R (X, \tau_N Y[1]).$$

We apply $(-, C_N(Y_S)[N+1])_S$ to the truncation triangle for X_S and obtain an exact sequence

$$\begin{aligned}
(\tau_N X_S[1], C_N(Y_S)[N+1])_S &\to (C_N(X_S)[N+1], C_N(Y_S)[N+1])_S \\
&\to (X_S, C_N(Y_S)[N+1])_S \to (\tau_N X_S, C_N Y_S[N+1])_S \\
&\to (C_N(X_S)[N], C_N(Y_S)[N+1])_S
\end{aligned}$$

and a commutative diagram analogous to the diagram (†). Now, for morphisms between finitely presented Λ-modules M and N, we do have that the natural map

$$S \otimes_R Hom_\Lambda(M, N) \longrightarrow Hom_{S\otimes_R\Lambda}(S \otimes_R M, S \otimes_R N)$$

is an isomorphism (cf. [**2**, Proposition 2.10]). Given a projective resolution $P_\bullet \longrightarrow M$ of M, denote by $\partial_n : \Omega^n M \hookrightarrow P_{n-1}$ the embedding of the n-th syzygy of M into the degree $n-1$ homogeneous component of the projective resolution. Then

$$Ext^n_\Lambda(M, N) = Hom_\Lambda(\Omega^n M, N)/\left(Hom_\Lambda(P_{n-1}, N) \circ \partial_n\right)$$

and therefore,

$$\begin{aligned} S\otimes_R Ext^n_\Lambda(M,N) &= S\otimes_R (Hom_\Lambda(\Omega^n M,N)/Hom_\Lambda(P_{n-1},N)\circ\partial_n)\\ &= (S\otimes_R Hom_\Lambda(\Omega^n M,N))\,/\,(S\otimes_R (Hom_\Lambda(P_{n-1},N)\circ\partial_n))\\ &= Hom_{S\otimes_R\Lambda}(S\otimes_R \Omega^n M, S\otimes_R N)/Hom_{S\otimes_R\Lambda}(S\otimes_R P_{n-1}, S\otimes_R N)\\ &\quad \circ(1_S\otimes\partial_n)\\ &= Ext^n_{S\otimes_R\Lambda}(S\otimes_R M, S\otimes_R N) \end{aligned}$$

for all $n\in\mathbb{N}$, natural in M and N. This shows

$$(C_N(X_S)[N+1], C_N(Y_S)[N+1])_S = S\otimes_R (C_N(X)[N+1], C_N(Y)[N+1])$$

and

$$(C_N(X_S)[N], C_N(Y_S)[N+1])_S = S\otimes_R (C_N(X)[N], C_N(Y)[N+1])$$

By the case for bounded complex of projectives, we get that the natural morphism is an isomorphism for

$$(\tau_N X_S[1], C_N(Y_S)[N+1])_S \simeq S\otimes_R (\tau_N X[1], C_N(Y)[N+1])$$

and

$$(\tau_N X_S, C_N(Y_S)[N+1])_S \simeq S\otimes_R (\tau_N X, C_N(Y)[N+1]).$$

Therefore, also

$$(X_S, C_N(Y_S)[N+1])_S \simeq S\otimes_R (X, C_N(Y)[N+1])$$

and by the very same arguments

$$(X_S, C_N(Y_S)[N])_S \simeq S\otimes_R (X, C_N(Y)[N]).$$

This shows that, we get, isomorphisms in the two left and the two right vertical morphisms of (†) and hence also the central vertical morphism is an isomorphism. Hence

$$(X_S, Y_S)_S \simeq S\otimes_R (X, Y)$$

and the lemma is proved. □

THEOREM 4. *Let R be a commutative Noetherian ring, let S be a commutative Noetherian R-algebra and suppose that S is a faithfully flat R-module. Suppose $S\otimes_R rad(R) = rad(S)$. Let Λ be a Noetherian R-algebra, let X and Y be two objects of of $D^b(\Lambda)$ and suppose that $End_{D^b(\Lambda)}(X)$ is a finitely generated R-module. Then,*

$$S\otimes^{\mathbb{L}}_R X \simeq S\otimes^{\mathbb{L}}_R Y \Leftrightarrow X\simeq Y.$$

REMARK 5. We observe that, if R is local and $S=\hat{R}$ is the $rad(R)$-adic completion, then S is faithfully flat as R-module and $S\otimes_R rad(R) = rad(S)$.

Proof of theorem 4. According to the hypotheses, we now suppose that $End_{D^b(\Lambda)}(X)$ and $End_{D^-(\Lambda)}(Y)$ are finitely generated R-module and that $S\otimes_R rad(R) = rad(S)$. Since

S is flat over R, tensor product of S over R is exact and we may replace the left derived tensor product by the ordinary tensor product. We only need to show "⇒" and assume, therefore, that X and Y are in $K^{-,b}(\Lambda - proj)$, and that $S \otimes_R X$ and $S \otimes_R Y$ are isomorphic.

Let $X_S := S \otimes_R X$ and $S \otimes_R Y =: Y_S$ in $D^b(S \otimes_R \Lambda)$ to shorten the notation and denote by φ_S the isomorphism $X_S \longrightarrow Y_S$. Since then X_S is a direct factor of Y_S by means of φ_S, the mapping

$$\varphi_S = \sum_{i=1}^{n} s_i \otimes \varphi_i : X_S \longrightarrow Y_S$$

for $s_i \in S$ and $\varphi_i \in Hom_{D^b(\Lambda)}(X, Y)$ has a left inverse $\psi : Y_S \longrightarrow X_S$, so that,

$$\psi \circ \varphi_S = id_{X_S}.$$

Then,

$$0 \longrightarrow rad(R) \longrightarrow R \longrightarrow R/rad(R) \longrightarrow 0$$

is exact and since S is flat over R, we get that

$$0 \longrightarrow S \otimes_R rad(R) \longrightarrow S \longrightarrow S \otimes_R (R/rad(R)) \longrightarrow 0$$

is exact. This shows that,

$$S \otimes_R (R/rad(R)) \simeq S/(S \otimes_R rad(R)).$$

By hypothesis, we have $S \otimes_R rad(R) = rad(S)$, identifying canonically $S \otimes_R R \simeq S$. Then, there are $r_i \in R$ so that $1_S \otimes r_i - s_i \in rad(S)$ for all $i \in \{1, \ldots, n\}$.

Put

$$\varphi := \sum_{i=1}^{n} r_i \varphi_i \in Hom_{D^b(\Lambda)}(X, Y).$$

Then,

$$\begin{aligned}\sum_{i=1}^{n} \psi \circ (1_S \otimes (r_i\varphi_i)) - 1_S \otimes id_X &= \sum_{i=1}^{n} (\psi \circ (1_S \otimes r_i\varphi_i) - \psi \circ (s_i \otimes \varphi_i)) \\ &= \sum_{i=1}^{n} (1_S \otimes r_i - s_i) \cdot (\psi \circ (id_S \otimes \varphi_i)) \\ &\in \big(rad(S) \otimes_R End_{D^b(\Lambda)}(X)\big)\end{aligned}$$

and since $End_{D^b(\Lambda)}(X)$ is a Noetherian R-module, using Nakayama's lemma, we obtain that $\psi \circ (\sum_{i=1}^{n} 1_S \otimes r_i\varphi_i)$ is invertible in $S \otimes_R End_{D^b(\Lambda)}(X)$. Hence, $id_S \otimes_R \varphi$ is left split, and therefore,

$$X_S \stackrel{id_S \otimes_R \varphi}{\longrightarrow} Y_S \longrightarrow C(id_S \otimes_R \varphi) \stackrel{0}{\longrightarrow} X_S[1]$$

is a distinguished triangle, with $C(id_S \otimes_R \varphi)$ being the cone of $id_S \otimes_R \varphi$. However,

$$C(id_S \otimes_R \varphi) = S \otimes_R C(\varphi)$$

and hence,

$$X_S \overset{id_S \otimes_R \varphi}{\longrightarrow} Y_S \longrightarrow S \otimes_R C(\varphi) \overset{0}{\longrightarrow} X_S[1]$$

is a distinguished triangle.

Since φ_S is an isomorphism, φ_S has a right inverse $\chi : Y_S \longrightarrow X_S$ as well. Now, since $X_S \simeq Y_S$, S is faithfully flat over R, and $End_{D^b(\Lambda)}(X)$ is finitely generated as R-module; using Lemma 3, we obtain that $End_{D^b(\Lambda)}(Y)$ is finitely generated as R-module as well. The same argument as for the left inverse ψ shows that $(id_S \otimes \varphi) \circ \chi$ is invertible in $S \otimes_R End_{D^b(\Lambda)}(Y)$. Hence,

$$X_S \overset{id_S \otimes_R \varphi}{\longrightarrow} Y_S \overset{0}{\longrightarrow} S \otimes_R C(\varphi) \overset{0}{\longrightarrow} X_S[1]$$

is a distinguished triangle. This shows that $S \otimes_R C(\varphi)$ is acyclic, and hence,

$$0 = H(S \otimes_R C(\varphi)) = S \otimes_R H(C(\varphi)).$$

Since S is faithfully flat over R also $H(C(\varphi)) = 0$, which implies that $C(\varphi)$ is acyclic, and therefore, φ is an isomorphism.

This proves the theorem. □

Let A be an algebra over a complete discrete valuation ring R which is finitely generated as a module over R. We shall need a Krull–Schmidt theorem for the derived category of bounded complexes over A. This fact seems to be well-known, but for the convenience of the reader we give a proof.

PROPOSITION 6. *Let R be a complete discrete valuation ring and let A be an R-algebra, finitely generated as R-module. Then, the Krull–Schmidt theorem holds for $K^{-,b}(A - proj)$.*

Proof. We first show a Fitting lemma for $K^{-,b}(A - proj)$.

Let X be a complex in $K^{-,b}(A - proj)$ and let u be an endomorphism of the complex X. Then, $X = X' \oplus X''$ as graded modules, by Fitting's lemma in the version for algebras over complete discrete valuation rings [**1**, Lemma 1.9.2]. The restriction of u on X' is an automorphism in each degree and the restriction of u on X'' is nilpotent modulo $rad(R)^m$ for each m. Therefore, u is a diagonal matrix $\begin{pmatrix} \iota & 0 \\ 0 & \nu \end{pmatrix}$ in each degree where $\iota : X' \longrightarrow X'$ is invertible, and $\nu : X'' \longrightarrow X''$ is nilpotent modulo $rad(R)^m$ for each m in each degree. The differential ∂ on X is given by $\begin{pmatrix} \partial_1 & \partial_2 \\ \partial_3 & \partial_4 \end{pmatrix}$ and the fact that u commutes with ∂ shows that $\partial_3 \iota = \nu \partial_3$ and $\partial_2 \nu = \iota \partial_2$. Therefore, $\partial_3 \iota^s = \nu^s \partial_3$ and $\partial_2 \nu^s = \iota^s \partial_2$ for all s. Since ν is nilpotent modulo $rad(R)^m$ for each m in each degree, and ι is invertible, $\partial_2 = \partial_3 = 0$. Hence, the differential of X restricts to a differential on X' and a differential on X''. Moreover, X' and X'' are both projective modules, since X is projective.

Now, X, and therefore, also X'' is exact in degrees higher than N, say. We fix $m \in \mathbb{N}$ and obtain, therefore, that u is nilpotent modulo $rad(R)^m$ in each degree lower than N. Let M_m be the nilpotency degree. Then, since X'' is exact in degrees higher than N, modulo $rad(R)^m$ the restriction of the endomorphism u^{M_m} to X'' is homotopy equivalent to 0 in degrees higher than N. We get, therefore, that the restriction of u to X'' is actually nilpotent modulo $rad(R)^m$ for each m.

Hence, the endomorphism ring of an indecomposable object is local and the Krull–Schmidt theorem is an easy consequence by the classical proof as in [**4**] or in [**1**].

This shows the proposition. □

REMARK 7. If R is a field and A is a finite dimensional R-algebra, then, we would be able to argue more directly. Indeed, $X' = \mathrm{im}(u^N)$ and $X'' = \ker(u^N)$ for large enough N. Then, it is obvious that X' and X'' are both subcomplexes of X. Observe that R may be a field in proposition 6.

For the next Corollary, we follow closely [**5**].

COROLLARY 8. *Let R be a commutative semilocal Noetherian ring, let S be a commutative R-algebra so that $\hat{S} := \hat{R} \otimes_R S$ is a faithful projective $\hat{R}$-module of finite type. Let Λ be a Noetherian R-algebra, finitely generated as R-module, and let X and Y be two objects of $D^b(\Lambda)$ and suppose that $End_{D^b(\Lambda)}(X)$ and $End_{D^b(\Lambda)}(Y)$ are finitely generated R-module. Then,*

$$S \otimes^{\mathbb{L}}_R X \simeq S \otimes^{\mathbb{L}}_R Y \Leftrightarrow X \simeq Y.$$

Proof. If $S \otimes^{\mathbb{L}}_R X \simeq S \otimes^{\mathbb{L}}_R Y$ in $D^b(S \otimes_R \Lambda)$, we get $\hat{S} \otimes^{\mathbb{L}}_R X \simeq \hat{S} \otimes^{\mathbb{L}}_R Y$ in $D^b(\hat{S} \otimes_R \Lambda)$. Since R is semilocal with maximal ideals $m_1, \ldots, m_s$, we get $\hat{R} = \prod_{i=1}^s \hat{R}_{m_i}$ for the completion $\hat{R}_{m_i}$ of R at m_i. Now, $\hat{S}$ is projective faithful of finite type, and so, there are $n_1, \ldots, n_s$ with

$$\hat{S} \simeq \prod_{i=1}^{s} (\hat{R}_{m_i})^{n_i}$$

and therefore, $\hat{S} \otimes^{\mathbb{L}}_R X \simeq \hat{S} \otimes^{\mathbb{L}}_R Y$ implies

$$\prod_{i=1}^{s} (\hat{R}_{m_i})^{n_i} \otimes^{\mathbb{L}}_R X \simeq \prod_{i=1}^{s} (\hat{R}_{m_i})^{n_i} \otimes^{\mathbb{L}}_R Y.$$

Hence,

$$(\hat{R}_{m_i} \otimes^{\mathbb{L}}_R X)^{n_i} \simeq (\hat{R}_{m_i} \otimes^{\mathbb{L}}_R Y)^{n_i}$$

for each i, and therefore by Proposition 6

$$\hat{R}_{m_i} \otimes^{\mathbb{L}}_R X \simeq \hat{R}_{m_i} \otimes^{\mathbb{L}}_R Y$$

for each i. By Theorem 4, we obtain $X \simeq Y$. □

We get cancellation of factors from this statement.

COROLLARY 9. *Under the hypothesis of theorem 4 or of corollary 8, we get $X \oplus U \simeq Y \oplus U$ in $D^b(\Lambda)$ implies $X \simeq Y$.*

Proof. This is clear by corollary 8 in combination with proposition 6. □

REMARK 10. In [**3**], we developed a theory to roughly speaking parameterise geometrically objects in $D^b(A)$ by orbits of a group action on a variety. For this purpose, we need to assume that A is a finite dimensional algebra over an algebraically closed field K, so that it is possible to use arguments and constructions from algebraic geometry. Using theorem 4, we can extend the theory to non algebraically closed fields K as well.

ACKNOWLEDGEMENT. I wish to thank the referee for useful suggestions which lead to substantial improvements.

REFERENCES

1. D. Benson, *Representations and Cohomology I* (Cambridge University Press, Cambridge, 1997).

2. D. Eisenbud, *Commutative algebra, with a view toward algebraic geometry* (Springer, New York, 2004).

3. B. T. Jensen, X. Su and A. Zimmermann, Degenerations for derived categories, *J. Pure Appl. Algebra*. **198** (2005), 281–295

4. S. Lang, *Algebra* (Addison-Wesley, Menlo Park California, 1993).

5. K. W. Roggenkamp, An extension of the Noether–Deuring theorem, *Proc. Am. Math. Soc.* **31** (1972), 423–426.

6. J.-L. Verdier, Des catégories dérivées des catégories abéliennes, *Astérisque.* **239** (1996).