THE RING OF POLYNOMIAL FUNCTORS OF PRIME DEGREE

ALEXANDER ZIMMERMANN

Abstract. Let $\hat{\mathbb{Z}}_p$ be the ring of p-adic integers. We prove in the present paper that the category of polynomial functors from finitely generated free abelian groups to $\hat{\mathbb{Z}}_p$-modules of degree at most p is equivalent to the category of modules over a particularly well understood ring, called Green order. That this is the case was conjectured by Yuri Drozd.

Introduction

Polynomial functors attained a lot of interest in recent years by at least two major discoveries. First, in [19] Henn, Lannes and Schwartz showed that the category of analytic functors from the category of finite dimensional vector spaces to to the category of vector spaces over the same field of characteristic p is equivalent to the category of unstable modules over the mod p Steenrod algebra modulo nilpotent objects. Second, Franjou, Friedlander, Scorichenko and Suslin in [15], Friedlander and Suslin in [16], Touzé [35] as well as Touzé and van der Kallen [36] use strict polynomial functor to prove the finite generation of cohomology of group schemes and to compute Ext-groups of modules over general linear groups. More recently Djament and Vespa studied stable homology of orthogonal, symplectic and unitary groups using some category of polynomial functors [37, 38, 39, 5, 6, 7]. For definitions and more ample remarks of these concepts we refer to section 1.

The category $\mathcal{A}_R$ of polynomial functors $\mathbb{Z}-free \longrightarrow R-mod$ for a commutative ring R is a classical object in algebraic topology (cf Eilenberg, MacLane [10]). Let $\mathcal{A}_R^n$ be the full subcategory of at most degree n polynomial functors in $\mathcal{A}_R$. Quadratic functors were characterized by Baues [2] as modules over a particular algebra. Baues, Dreckmann, Franjou and Pirashvili show in [3] that $\mathcal{A}_R^n$ is a module category of finitely generated R-algebra Γ_R^n as well. This description was used by Drozd to show in [8] that $\mathcal{A}^2_{\hat{\mathbb{Z}}_2}$ and in [9] that $\mathcal{A}^3_{\hat{\mathbb{Z}}_3}$ are two very explicitly given classical orders over $\hat{\mathbb{Z}}_2$ and $\hat{\mathbb{Z}}_3$ respectively, whose representation theory is completely understood. In particular each of them admit only a finite number of indecomposable lattices. Here, and in the sequel, we denote by $\hat{\mathbb{Z}}_p$ the ring of p-adic integers, and by $\mathbb{F}_q$ the field with q elements. These orders were introduced by Roggenkamp in [34]. Recall that an R-order over an integral domain R is an R-algebra Λ, finitely generated projective as R-modules and so that $K\otimes_R\Lambda$ is a semisimple K-algebra, for K being the field of fractions of R.

Drozd conjectures at the end of [9] that $\mathcal{A}^p_{\hat{\mathbb{Z}}_p}$ should be equivalent to the module category of a particular Green order Λ_p over $\hat{\mathbb{Z}}_p$ for all primes p. Drozd proves the case $p=2$ and $p=3$ by explicitly associating the generators of the ring given by Baues or Baues, Dreckmann, Franjou and Pirashvili respectively to matrices in the corresponding matrix rings and computes the kernel and the image of the so-defined mapping. The relations in [3] are sufficiently involved so that going beyond $p=3$ by this method seems to be not realistic.

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In this paper we prove Drozd's conjecture. Our method is conceptual. We develop a recollement diagram of $\mathcal{A}^n_{\mathbb{F}_p}$ by $\mathcal{A}^{n-1}_{\mathbb{F}_p}$ and the module category of the group ring $\mathbb{F}_p\mathfrak{S}_n$, analogous to the one described by Schwartz [40, Section 5.5] for functors $\mathbb{F}_p - mod \longrightarrow \mathbb{F}_p - mod$. This recollement diagram for $\mathcal{A}^n_{\mathbb{F}_p}$ may be of independent interest since it is completely general. It actually appears already in work of Pirashvili [27], as it was indicated by the referee. Comparison of these two diagrams gives many informations. A second ingredient then is the study of various Ext-groups between simple functors, using work of Franjou, Friedlander, Scorichenko and Suslin [15]. The third main ingredient is the explicit projective functor mapping to the reduction modulo p functor. It should be noted that we do not actually use the ring defined by Baues, Dreckmann, Franjou and Pirashvili in [3]. We just use that there is an algebra which is finitely generated, so that the Krull-Schmidt property, lifting of idempotents and similar properties are valid for $\mathcal{A}^n_{\hat{\mathbb{Z}}_p}$. For this reason we do not give a Morita bimodule between the Baues, Dreckmann, Franjou and Pirashvili-ring and the order we get. As an application of our result, we count the number of 'torsion free' indecomposable polynomial functors in $\mathcal{A}^p_{\hat{\mathbb{Z}}_p}$.

Our paper is organized as follows. In Section 1 we give the essential definitions and relate the different concepts. In Section 2 we recall some of the most important discoveries used in the sequel. Section 3 describes the classical recollement diagrams as well as the new one we have to use for $\mathcal{A}_{\mathbb{F}_p}$, and we derive first consequences. The first main result is proved in Section 5. We give the structure of $\mathcal{A}^p_{\mathbb{F}_p}$ there. Finally, in Section 6 we determine $\mathcal{A}^p_{\hat{\mathbb{Z}}_p}$ and prove the second main theorem there.

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1. Generalities on polynomial functors

1.1. **Definitions.** Let $\mathcal{A}$ be a category with direct sums and $\mathcal{B}$ be a category with direct sums and kernels. Then, following Eilenberg and MacLane [10] define the cross effect $F^{(1)}$ of a functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ to be the bifunctor $\mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{B}$ defined on objects by $F^{(1)}(V|W) := ker(F(V \oplus W) \longrightarrow F(V) \oplus F(W))$, and on morphism by the naturality of the construction. For $n \geq 1$, the n-th cross effect $F^{(n)}$ is the cross effect of the $n-1$-st cross effect of F, seen as functor in the first variable. Hence,

$$F^{(n)}(V|W|V_1|\dots|V_{n-1}) := ker\Big(F^{(n-1)}(V \oplus W|V_1|\dots|V_{n-1}) \longrightarrow F^{(n-1)}(V|V_1|\dots|V_{n-1}) \oplus F^{(n-1)}(W|V_1|\dots|V_{n-1})\Big)$$

for objects $V, W, V_1, \dots, V_{n-1}$ in $\mathcal{A}$. Suppose in the sequel $F(0) = 0$. A functor F is said to be polynomial of degree at most n if $F^{(n)} = 0$ for an $n \in \mathbb{N}$ (see Pirashvili [28]). Given a commutative ring R, let $R - mod$ be the category of finitely generated R-modules, let $R - Mod$ be the category of all R-modules, and let $R - free$ be the category of finitely generated free R-modules. Further, call $\mathcal{F}^n_R$ the category of polynomial functors of degree at most n from $R-mod$ to $R-mod$, and $\mathcal{F}_R := \varinjlim_n \mathcal{F}^n_R$. Moreover, let $\mathcal{A}^n_R$ be the category

[1] email to the author from April 12, 2013

of polynomial functors of degree at most n from $\mathbb{Z}-free$ to $R-mod$, and $\mathcal{A}_R := \varinjlim_n \mathcal{A}_R^n$. All these categories are abelian. Observe that additive functors are exactly the degree 1 polynomial functors. The only degree 0 polynomial functor is the trivial functor due to our hypothesis that $F(0) = 0$.

Friedlander and Suslin define in [16, Definition 2.1] the category of strict polynomial functors $\mathcal{P}_k$ over a field k. A strict polynomial functor F is defined by associating to each finite dimensional k-vector space a k-vector space $F(V)$ and to associate for any two finite dimensional k-vector spaces V and W an element in

$$S^*\left(Hom_k(Hom_k(V,W),k)\right) \otimes Hom_k(F(V),F(W))$$

which in addition satisfy the usual compatibility relations for compositions and the identity. Each of these elements can be interpreted as mapping $Hom_k(V,W) \longrightarrow Hom_k(F(V),F(W))$ by interpreting the formal polynomial

$$S^*\left(Hom_k(Hom_k(V,W),k)\right) \otimes Hom_k(F(V),F(W))$$

as polynomial mapping, so that any strict polynomial functor induces a polynomial functor $k-mod \longrightarrow k-Mod$. Hence, we have a forgetful functor $\mathcal{P}_k \longrightarrow \mathcal{F}_k$. It is shown in [16] that the category of exact degree n strict polynomial functors $\mathcal{P}_k^n$ from $k-mod$ to $k-Mod$ for a field k is equivalent to the category of modules over the Schur algebra $S_k(n,n)$. Moreover, $\mathcal{P}_k = \bigoplus_n \mathcal{P}_k^n$. Finally, the forgetful functor $\mathcal{P}_k \longrightarrow \mathcal{F}_k$ sends a strict polynomial functor of degree at most d to a polynomial functor of degree at most d (cf [29, Remark 4.1]).

A main theme in Section 1 in particular is the question of base change. In Section 5 we shall define for any commutative rings R and S a particular degree d polynomial functor

$$R[-]/(I^{d+1}) : S-free \to R-Mod$$

which assigns to every free S-module V the quotient of the group ring $R[V]$ of the additive group V over R by the $d+1$-th power of the augmentation ideal I. Look at the case $S = \mathbb{Z}$. A key idea, due to Pirashvili, is to replace the morphisms $Hom_{\mathbb{Z}}(\mathbb{Z}^n,\mathbb{Z}^m)$ in the category of free abelian groups by $R[Hom_{\mathbb{Z}}(\mathbb{Z}^n,\mathbb{Z}^m)]/(I^{d+1})$. This construction does not allow base change. Strict polynomial functors basically replace $Hom_{\mathbb{Z}}(\mathbb{Z}^n,\mathbb{Z}^m)$ by $S^d(Hom_{\mathbb{Z}}(\mathbb{Z}^n,\mathbb{Z}^m))$, which does admit base change. Comparison between these two constructions is the main theme in Section 1.

1.2. Functors with values in characteristic 0.

Lemma 1.1. *Let R be an integral domain of characteristic 0. If F is a polynomial functor $F : \mathbb{Z}-free \longrightarrow R-mod$ of degree d, then $Hom_{\mathbb{Z}}(\mathbb{Z}^n,\mathbb{Z}^m) \xrightarrow{F} Hom_R(F(\mathbb{Z}^n),F(\mathbb{Z}^m))$ is polynomial of degree d in the $n \cdot m$ coordinate functions $Hom_{\mathbb{Z}}(\mathbb{Z}^n,\mathbb{Z}^m)$.*

Proof. We shall show by induction on $n+d$ that for any k homomorphisms $f_1, f_2, \ldots, f_k \in Hom_{\mathbb{Z}}(\mathbb{Z}^n,\mathbb{Z}^m)$ and integers $\lambda_1, \lambda_2, \ldots, \lambda_k \in \mathbb{Z}$ one gets $F(\lambda_1 f_1 + \lambda_2 f_2 + \cdots + \lambda_k f_k)$ is a degree d polynomial in the variables $\lambda_1, \lambda_2, \ldots, \lambda_k$.

If the degree of F is 1, there is nothing to show since then the functor is linear.

Let $n > 1$. Now, we know that

$$F(\mathbb{Z}^{n-1} \oplus \mathbb{Z}) = F(\mathbb{Z}^{n-1}) \oplus F(\mathbb{Z}) \oplus F^{(1)}(\mathbb{Z}^{n-1}|\mathbb{Z})\ .$$

Hence, the restriction f_i' of each of the f_i to $\mathbb{Z}^{n-1}$ and the restriction f_i'' to the last component $\mathbb{Z}$ define morphisms $F(\sum_{i=1}^k \lambda_i f_i') : F(\mathbb{Z}^{n-1}) \longrightarrow F(\mathbb{Z}^m)$, $F(\sum_{i=1}^k \lambda_i f_i'') : F(\mathbb{Z}) \longrightarrow F(\mathbb{Z}^m)$ and $F(\sum_{i=1}^k \lambda_i f_i) : F^{(1)}(\mathbb{Z}^{n-1}|\mathbb{Z}) \longrightarrow F(\mathbb{Z}^m)$. In the first two cases, the dimension of the source space is less than n, while the degree of F is unchanged, whereas in the third case the dimension of the source space is n, but the degree of the functor is $d-1$. So, in any of these cases by the induction hypothesis we can express $F(\sum_{i=1}^k \lambda_i f_i)$ a polynomial of degree n in the variables $\lambda_1, \lambda_2, \ldots, \lambda_k$.

We are left with the case $n = 1$. The very same reduction applied to the image and induction on $m + d$ implies that one can suppose that $m = 1$. But then, Eilenberg and MacLane [10, (8.3)] show that for $\lambda \in \mathbb{Z}$ one has

$$F(\lambda\cdot) = F((\lambda - 1)\cdot) + F(1) + F^{(1)}(\lambda \cdot |1),$$

where $F(1)$ is the identity. Now,

$$F(\lambda\cdot) - F((\lambda - 1)\cdot) = F(1) + F^{(1)}(\lambda \cdot |1),$$

where by the induction hypothesis, since the degree of $F^{(1)}$ is less than the degree of F, the right hand side $F(1) + F^{(1)}(\lambda \cdot |1)$ is polynomial of degree $d - 1$ in λ.

We shall now adapt an argument of Kuhn [22, Lemma 4.8] to this slightly more general situation. We claim that a function $f : \mathbb{Z} \longrightarrow R$ is a polynomial if and only if some derivative $f^{(r)}$ vanishes, where $f^{(r)}(n) = f^{(r-1)}(n) - f^{(r-1)}(n - 1)$.

We assume for the moment that R contains $\mathbb{Q}$. Suppose f is a polynomial. Then, it is clear that $f^{(deg(f)+1)} = 0$. Suppose to the contrary that $f^{(r)} = 0$. The polynomials $\binom{X}{k} := \frac{X\cdot(X-1)\cdot\dots\cdot(X-k+1)}{k!}$ for $k \in \{0, 1, \dots, d\}$ form an R-basis of the polynomials of degree at most d in $R[X]$, since $d!$ is invertible in R. Moreover, $\binom{X}{k} - \binom{X-1}{k} = \binom{X}{k-1}$. Now, by induction, $f^{(s)}$ is a polynomial, and hence a linear combination of polynomials $\binom{X}{k}$. The relation $\binom{X}{k} - \binom{X-1}{k} = \binom{X}{k-1}$ gives a polynomial $h^{(s-1)}$ so that $(h^{(s-1)})^{(1)} = f^{(s)}$. By induction on n, the values $g^{(1)}(n) = g(n) - g(n - 1)$ determine the values $g(n)$ up to the value of $g(0)$. Therefore, up to this constant value, $f^{(s-1)} = h^{(s-1)}$.

Now, suppose R an integral domain of characteristic 0. Then, since $R \subseteq frac(R)$, f can be considered as being in values $frac(R)$ which contains $\mathbb{Q}$. This proves the claim.

Now, define $f(\lambda) := F(\lambda\cdot)$ and apply the claim to conclude that $F(\lambda\cdot)$ is polynomial of degree d. ∎

Let K be a field. A priori the category $\mathcal{A}_K$ is different from the category $\mathcal{F}_K$. Nevertheless, in some cases we get one inclusion.

Lemma 1.2. *Let K be either a prime field of finite characteristic or let K be a field of characteristic* 0*. Let $(\circ(K\otimes_{\mathbb{Z}}-))^* : \mathcal{F}_K \longrightarrow \mathcal{A}_K$ be the functor defined by $(\circ(K\otimes_{\mathbb{Z}}-))^*(F) := F \circ (K \otimes_{\mathbb{Z}} -)$. Then, $(\circ(K \otimes_{\mathbb{Z}} -))^*$ induces a fully faithful embedding $\mathcal{F}_K \hookrightarrow \mathcal{A}_K$.*

Proof: The functor $\circ(- \otimes_{\mathbb{Z}} K) : \mathcal{F}_K \longrightarrow \mathcal{A}_K$ induces for any two functors F and G in $\mathcal{F}_K$ a mapping

$$\varphi : Hom_{\mathcal{F}_K}(F, G) \longrightarrow Hom_{\mathcal{A}_K}(F \circ (- \otimes_{\mathbb{Z}} K), G \circ (- \otimes_{\mathbb{Z}} K)) \ .$$

We shall need to show that this mapping is an isomorphism.

Injectivity: Let η_1 and η_2 be two objects in $Hom_{\mathcal{F}_K}(F, G)$. Suppose $\varphi(\eta_1) = \varphi(\eta_2)$. Observe that for any $V \in \mathbb{Z} - free$ we have

$$(\varphi(\eta_1))(V) = \eta_1(K \otimes_{\mathbb{Z}} V) \in Hom_K(F(K \otimes_{\mathbb{Z}} V), G(K \otimes_{\mathbb{Z}} V))$$

and likewise for η_2, satisfying that for any V and W and any $\rho \in Hom_{\mathbb{Z}}(V, W)$ one has $(\eta_1(K \otimes_{\mathbb{Z}} V)) \circ \hat{G}(\rho) = \hat{F}(\rho) \circ (\eta_1(K \otimes_{\mathbb{Z}} W))$.

Since $\circ(- \otimes_{\mathbb{Z}} K)$ is dense, η_1 and η_2 coincide on every object of $K - mod$, and so $\eta_1 = \eta_2$.

As a consequence, without any further hypothesis,

$$\varphi : Hom_{\mathcal{F}_K}(F, G) \hookrightarrow Hom_{\mathcal{A}_K}(F \circ (- \otimes_{\mathbb{Z}} K), G \circ (- \otimes_{\mathbb{Z}} K)) \ .$$

Surjectivity: Let $\eta \in Hom_{\mathcal{A}_K}(F \circ (- \otimes_{\mathbb{Z}} K), G \circ (- \otimes_{\mathbb{Z}} K))$ be a natural transformation. We need to show that there is a natural transformation $\eta' \in Hom_{\mathcal{F}_K}(F, G)$ so that $\varphi(\eta') = \eta$.

In the case K being a prime field of finite characteristic define for any $V \in K - mod$ the mapping $\eta'(V) := \eta(P_V)$, where P_V is a fixed chosen projective cover of V, so that $K \otimes_{\mathbb{Z}} P_V = V$.

In the case K being of characteristic 0, fix for any K-vector space V a free abelian subgroup P_V so that $K \otimes_{\mathbb{Z}} P_V = V$. Define $\eta'(V) := \eta(P_V)$.

We need to show that η' is a natural transformation.

Let $\varphi \in Hom_K(V, W)$.

Consider first the case of K being a prime field of finite characteristic. Since P_V and P_W are projective covers of V and W as abelian groups, there is a $\hat{\varphi} \in Hom_{\mathbb{Z}}(P_V, P_W)$ so that $K \otimes_{\mathbb{Z}} \hat{\varphi} = \varphi$ under the identification $K \otimes_{\mathbb{Z}} P_V = V$ and $K \otimes_{\mathbb{Z}} P_W = W$. Since η is a natural transformation, $\hat{G}(\hat{\varphi}) \circ \eta(P_V) = \eta(P_W) \circ \hat{F}(\hat{\varphi})$. But, by definition, $\hat{G}(\hat{\varphi}) = G(\varphi)$ and $\hat{F}(\hat{\varphi}) = F(\varphi)$, as well as $\eta(P_W) = \eta'(W)$ and $\eta(P_V) = \eta'(V)$. So, η' is a natural transformation.

Suppose now that K is a field of characteristic 0. Since $\mathcal{F}_K = \mathcal{P}_K$ in this case, we know that F (and G resp.) are polynomial laws transforming any linear mapping $V \longrightarrow W$ into a linear mapping $F(V) \longrightarrow F(W)$ (and $G(V) \longrightarrow G(W)$ resp.) which depends polynomially in the coefficients of any matrix representation with respect to any fixed bases. We know that for any $\mathbb{Z}$-linear mapping $\hat{\varphi} : P_V \longrightarrow P_W$ that the equation

$$(\ddagger) : G(K \otimes_{\mathbb{Z}} \varphi) \circ \eta(P_V) = \hat{G}(\hat{\varphi}) \circ \eta(P_V) = \eta(P_W) \circ \hat{F}(\hat{\varphi}) = \eta(P_W) \circ F(K \otimes_{\mathbb{Z}} \varphi).$$

Since this equation holds evaluated in infinitely coefficients, since $\mathbb{Z}$ and K are both infinite, this above equation $(\ddagger)$ holds *as polynomial* equation.

Therefore, the equation hold as well for φ, since there the only difference is that the polynomials are evaluated not only on integer coefficients, but also on coefficients in K. Since the equation holds as polynomials, this equation holds true also evaluated on K.

Therefore again $\eta'_W \circ F(\varphi) = G(\varphi) \circ \eta'_V$. This proves that η' is a natural transformation.

∎

We are now concerned with the question when a polynomial functor $\mathbb{Z}-free \longrightarrow R-mod$ can be extended to a strict polynomial functor $R-mod \longrightarrow R-mod$ by composing with the 'extending scalars' functor $\mathbb{Z} - free \stackrel{R\otimes_{\mathbb{Z}}-}{\longrightarrow} R - mod$. In other words we study the question when $(\circ(K \otimes_{\mathbb{Z}} -))^*$ is an equivalence $\mathcal{A}_R \simeq \mathcal{P}_R$. In order to prove this, by Lemma 1.2, one needs to show that $(\circ(K \otimes_{\mathbb{Z}} -))^*$ is dense as well.

We have to deal with mainly two cases: the case of R being a field of characteristic 0 and the case of R being a field of characteristic p. We shall see that fields with characteristic 0 behave more like characteristic ∞. The remarks at the beginning of this section on base change properties are particularly visible in the proof of the following lemma.

Lemma 1.3. *Let R be a field of characteristic 0 and let $F : \mathbb{Z} - free \longrightarrow R - mod$ be a polynomial functor of degree d. Then, F extends to a strict polynomial functor $\hat{F}$: $R - free \longrightarrow R - mod$ so that $\hat{F} \circ (R \otimes_{\mathbb{Z}} -) = F$. In particular, $(\circ(R \otimes_{\mathbb{Z}} -))^*$ induces an equivalence $\mathcal{A}_R^d \simeq \mathcal{P}_R^{\leq d}$.*

Proof. By Lemma 1.2 we know that $\mathcal{F}_K^d \hookrightarrow \mathcal{A}_K^d$.

We have to show that this embedding is dense. Let F be a degree d polynomial functor in $\mathcal{A}_K^d$. By Lemma 1.1 we know that for any n and m the functor F induces a degree d polynomial mapping with coefficients in K in the coordinate functions of matrices in $Hom_{\mathbb{Z}}(\mathbb{Z}^n, \mathbb{Z}^m)$. Moreover, F is a functor, that is $F(\alpha\beta) = F(\alpha)F(\beta)$ and $F(id_A) = id_A$ for any free abelian group A and any two composable morphisms of abelian groups α and β.

Let $\alpha : \mathbb{Z}^n \longrightarrow \mathbb{Z}^m$ and $\beta : \mathbb{Z}^k \longrightarrow \mathbb{Z}^n$. The equation $F(\alpha\beta) = F(\alpha)F(\beta)$ translates into an equation between the evaluation of the corresponding polynomials in each degree. Since $\mathbb{Z}$ and K are of characteristic 0, the polynomial equation holds if evaluated on infinitely many values, and so the polynomial equations actually holds *as polynomials.* Friedlander and Suslin remark in [16, remark after Definition 2.1] that this is actually equivalent to saying that F actually is a strict polynomial functor $\hat{G} \in \mathcal{P}_K$ of degree d. This proves the lemma. ∎

1.3. **Functors with values in fields of finite characteristic.**

Lemma 1.4. *Let $\mathbb{F}$ be a field of characteristic p and let $F : \mathbb{Z}-free \longrightarrow \mathbb{F}-mod$ be polynomial functor of degree at most $p-1$ which preserves the initial object, i.e. $F(0)=0$. Then, for any homomorphism $\alpha : \mathbb{Z}^n \longrightarrow \mathbb{Z}^m$ one gets $F(p\cdot\alpha)=0$.*

Proof. Let $M=\mathbb{Z}^n$ and $N=\mathbb{Z}^m$. We write $p\cdot\alpha$ in the diagram

$$\begin{array}{ccccc} M & \stackrel{\delta_M}{\longrightarrow} & M\oplus M\oplus\cdots\oplus M & \stackrel{\sigma_M}{\longrightarrow} & M \\ \alpha\downarrow & & \downarrow \alpha\oplus\alpha\oplus\cdots\oplus\alpha & & \downarrow\alpha \\ N & \stackrel{\delta_N}{\longrightarrow} & N\oplus N\oplus\cdots\oplus N & \stackrel{\sigma_N}{\longrightarrow} & N \end{array}$$

where δ is the diagonal mapping and σ is the summation mapping, on M or on N respectively.

Denote by $F^{(i)}$ the i^{th} cross effect of the functor F. Since F is polynomial of degree at most p,

$$F(A_1\oplus A_2\oplus\cdots\oplus A_p)=\bigoplus_{i=1}^{p}\bigoplus_{j_1<\cdots<j_i}F^{(i-1)}(A_{j_1}|\dots|A_{j_i})$$

for p abelian groups $A_1,\dots,A_p$, and this decomposition is functorial with respect to these groups. Moreover, since $deg(F)=p-1$, and since $F(0)=0$, one gets $F^{(p-1)}=0$.

Since $F(p\cdot\alpha)=F(\alpha\circ\sigma_M\circ\delta_M)=F(\alpha)\circ F(\sigma_M\circ\delta_M)$, it is necessary and sufficient to show that $F(\sigma_M\circ\delta_M)=0$. Now,

$$F(M\oplus M\oplus\cdots\oplus M)=\bigoplus_{i=1}^{p}\bigoplus_{j_1<\cdots<j_i}F^{(i-1)}(M_{j_1}|\dots|M_{j_i})$$

where $M_l=M$ is the copy of M in the l-th position of $\oplus_{i=1}^{p}M$, for all $l\le p$. The mapping $F(\sigma_M\circ\delta_M)=F(\sigma_M)\circ F(\delta_M)$ factors as a sum $\sum_{i=1}^{p}F(\sigma_M)\circ\iota_i\circ\pi_i\circ F(\delta_M)$ where ι_i is the embedding of $\bigoplus_{j_1<\cdots<j_i}F^{(i-1)}(M_{j_1}|\dots|M_{j_i})$ into $F(M\oplus M\oplus\cdots\oplus M)$ and π_i is the projection of $F(M\oplus M\oplus\cdots\oplus M)$ to this direct factor. But now, for $i<p$, $F(\sigma_M)|_{(\bigoplus_{j_1<\cdots<j_i}F^{(i-1)}(M_{j_1}|\dots|M_{j_i}))}\circ\pi_i\circ F(\delta_M)$ sums up of p identical mappings, which sum up to 0 in characteristic p. Hence, $F(\sigma_M\circ\delta_M)=F(\sigma_M)\circ F^{(p-1)}(\alpha|\alpha|\dots|\alpha)\circ F(\delta_M)=0$ using that $F^{(p-1)}=0$. ∎

Corollary 1.5. *Let $\mathbb{F}$ be a field of characteristic p and let $F : \mathbb{Z}-free \longrightarrow \mathbb{F}-mod$ be polynomial functor of degree at most $p-1$ which preserves the initial object, i.e. $F(0)=0$. Then, for any two homomorphisms $\alpha : \mathbb{Z}^n \longrightarrow \mathbb{Z}^m$ and $\beta : \mathbb{Z}^n \longrightarrow \mathbb{Z}^m$ so that $\alpha-\beta\in p\cdot Hom_{\mathbb{Z}}(\mathbb{Z}^n,\mathbb{Z}^m)$, one gets $F(\alpha)=F(\beta)$.*

Proof. This is a consequence of the previous lemma and [10, p. 76, formula 8.5] and [10, Theorem 9.3]. Indeed, $F(\sum_{n=1}^{p+1}\rho_n)=\sum_{n=1}^{p+1}\sum_{i_1<\cdots<i_n}F^{(n-1)}(\rho_{i_1}|\dots|\rho_{i_n})$ implies

$$F(\alpha+p\cdot\gamma)=F(\alpha)+F^{(p-1)}(\underbrace{\gamma|\gamma|\dots|\gamma}_{p\text{ factors}})+F^{(p)}(\alpha|\underbrace{\gamma|\gamma|\dots|\gamma}_{p\text{ factors}})=F(\alpha)+F(p\gamma)=F(\alpha)$$

by Lemma 1.4. ∎

Lemma 1.6. *Let $\mathbb{F}$ be the prime field of characteristic p and let $F : \mathbb{Z}-free \longrightarrow \mathbb{F}-mod$ be a polynomial functor of degree less or equal to $p-1$. Then, F factors through the functor $\mathbb{F}\otimes_{\mathbb{Z}} - : \mathbb{Z}-free \longrightarrow \mathbb{F}-mod$. Moreover, if $F=F'\circ(\mathbb{F}\otimes_{\mathbb{Z}} -)$, then F is polynomial of degree m if and only if F' is polynomial of degree m. Hence, $(\circ(\mathbb{F}\otimes_{\mathbb{Z}} -))^*$ induces an equivalence $\mathcal{A}_{\mathbb{F}}^{\le p-1}\simeq\mathcal{F}_{\mathbb{F}}^{\le p-1}$.*

Proof. Since $(\mathbb{F} \otimes_{\mathbb{Z}} -)^* : \mathcal{F}_{\mathbb{F}} \hookrightarrow \mathcal{A}_{\mathbb{F}}$ is a fully faithful embedding, we need to show that $(\mathbb{F} \otimes_{\mathbb{Z}} -)^*$ is dense. So, given a functor $F : \mathbb{Z} - free \longrightarrow \mathbb{F} - mod$. One has to show that there is a functor $\hat{F} : \mathbb{F} - mod \longrightarrow \mathbb{F} - mod$ with $\hat{F} \circ (\mathbb{F} \otimes_{\mathbb{Z}} -) = F$.

For any $V \in \mathbb{F} - mod$ choose P_V a projective cover as abelian group. Then, P_V is in $\mathbb{Z}$-free. By the universal property of projective covers one has for any $\alpha \in Hom_{\mathbb{F}}(V, W)$ a (non-unique) $\hat{\alpha} \in Hom_{\mathbb{Z}}(P_V, P_W)$ so that

$$\begin{array}{ccc} P_V & \stackrel{\hat{\alpha}}{\longrightarrow} & P_W \\ \downarrow & & \downarrow \\ V & \stackrel{\alpha}{\longrightarrow} & W \end{array}$$

is commutative. Put

$$\hat{F}(V) := F(P_V) \ \text{ and } \hat{F}(\alpha) := F(\hat{\alpha}) \ .$$

We need to show that this gives a functor $\hat{F} : \mathbb{F} - mod \longrightarrow \mathbb{F} - mod$. Let $\hat{\alpha}$ and $\hat{\alpha}'$ be two different lifts of $\alpha : V \longrightarrow W$, then $\hat{\alpha} - \hat{\alpha}'$ lifts the 0-mapping, and so $\hat{\alpha} - \hat{\alpha}' \in p \cdot Hom_{\mathbb{Z}}(P_V, P_W)$. Corollary 1.5 implies that $\hat{F}(\hat{\alpha}) = \hat{F}(\hat{\alpha}')$. Using Corollary 1.5 again one gets that $\hat{F}(id_V) = id_{\hat{F}(V)}$ since id_{P_V} is a lift of id_V. Moreover, let $\alpha : U \longrightarrow V$ and $\beta : V \longrightarrow W$, then choosing lifts $\hat{\alpha} : P_U \longrightarrow P_V$ and $\beta : P_V \longrightarrow P_W$, one gets $\widehat{\alpha\beta} - \hat{\alpha}\hat{\beta}$ lifts the 0-mapping. So, $\widehat{\alpha\beta} - \hat{\alpha}\hat{\beta} \in p \cdot Hom_{\mathbb{Z}}(P_U, P_W)$ and again by Corollary 1.5 one has $F(\widehat{\alpha\beta}) = F(\hat{\alpha}\hat{\beta})$. ■

2. A review on polynomial functors and functor cohomology

2.1. **Polynomial functors are modules.** Let R be a commutative ring. We know that by a result of Baues, Dreckmann, Franjou and Pirashvili [3] a polynomial functor of degree at most n from free abelian groups to R-modules is defined by giving R-modules $F_m(\mathbb{Z}|\mathbb{Z}|\dots|\mathbb{Z})$ for all $m \le n$ and mappings

$$h_k^m : F_m(\mathbb{Z}|\mathbb{Z}|\dots|\mathbb{Z}) \longrightarrow F_{m+1}(\mathbb{Z}|\mathbb{Z}|\dots|\mathbb{Z})$$

for $k \le m \le n-1$ and

$$p_k^{m+1} : F_{m+1}(\mathbb{Z}|\mathbb{Z}|\dots|\mathbb{Z}) \longrightarrow F_m(\mathbb{Z}|\mathbb{Z}|\dots|\mathbb{Z})$$

for $k \le m \le n-1$ satisfying the relations :

$$(*) \ \ h_j^m p_i^m = \left\{ \begin{array}{ll} p_i^{m+1} h_{j+1}^{m+1} & \text{for } j < i \\ p_{i+1}^{m+1} h_j^{m+1} & \text{for } j > i \\ 1 + t_i^m + p_i^{m+1} h_{i+1}^{m+1} + p_{i+1}^{m+1} h_i^{m+1} + p_{i+1}^{m+1} t_i^{m+1} h_{i+1}^{m+1} & \\ +p_i^{m+1} t_{i+1}^{m+1} h_i^{m+1} + p_{i+1}^{m+1} p_i^{m+2} t_{i+1}^{m+2} h_i^{m+2} h_{i+1}^{m+1} & \text{for } j = i \end{array} \right\}$$

and relations coding functoriality of h, t and p. Define the algebra Γ_R^n over R by a quiver with n vertices $F_m(\mathbb{Z}|\mathbb{Z}|\dots|\mathbb{Z})$ for any m with $1 \le m \le n$ and arrows $h_k^m : F_m(\mathbb{Z}|\mathbb{Z}|\dots|\mathbb{Z}) \longrightarrow F_{m+1}(\mathbb{Z}|\mathbb{Z}|\dots|\mathbb{Z})$ and $p_k^{m+1} : F_{m+1}(\mathbb{Z}|\mathbb{Z}|\dots|\mathbb{Z}) \longrightarrow F_m(\mathbb{Z}|\mathbb{Z}|\dots|\mathbb{Z})$ subject to the relations $(*)$. Observe that the relations above do not form a set of admissible relations. The third relation though should be read as the defining equation for the symbols t_i^n, and this way the relations $(*)$ is a set of admissible relations. The result [3] of Baues et alii implies that $\Gamma_R^n - mod$ is equivalent to the category of polynomial functors of degree at most n.

Since the ring homomorphism $\hat{Z}_p \longrightarrow \mathbb{F}_p$ induces an embedding $\mathbb{F}_p - mod \longrightarrow \hat{\mathbb{Z}}_p - mod$, we get an induced embedding $\Gamma_{\mathbb{F}_p}^n - mod \longrightarrow \Gamma_{\hat{\mathbb{Z}}_p}^n - mod$ which is also induced by the surjective ring homomorphism $\Gamma_{\hat{\mathbb{Z}}_p}^n \longrightarrow \Gamma_{\mathbb{F}_p}^n$.

Remark 2.1. We should mention that the description of [3] was recently generalised by Hartl, Pirashvili and Vespa [18] to show an isomorphism of the category of functors $free - groups \longrightarrow \mathbb{Z} - Mod$ with the category of what they call pseudo-Mackey functors.

2.2. **Some facts on functor cohomology.** We shall give some facts that we will need from Franjou, Friedlander, Scorichenko and Suslin [15]. Basically, these results reduce the computation of extension groups between polynomial functors to questions between extension groups between strict polynomial functors.

Let k be a commutative ring. Let $\mathcal{F}(k)$ be the category of functors $k-mod \longrightarrow k-mod$ and let $\mathcal{P}(k)$ the category of strict polynomial functors between k-modules. Let $\mathcal{F}^n = \mathcal{F}^n(k)$ be the category of degree n polynomial functors from $k-mod$ to $k-mod$. If $k = \mathbb{F}_q$ for $q = p^s$, the field with q elements, we write $\mathcal{F}^n(k) = \mathcal{F}^n(q)$. In this case, for any strict polynomial functor P in $\mathcal{P}(k)$ let $P^{(m)}$ be the functor twisted by the Frobenius endomorphism defined by $\mathbb{F}_q \ni x \mapsto x^{p^m} \in \mathbb{F}_q$.

Theorem 1. [15] *Given any two homogeneous strict polynomial functors P and Q between $\mathbb{F}_q$-vector spaces for $q = p^s$. If the degrees of P and Q are different and strictly smaller than q, then $Ext^*_{\mathcal{F}(k)}(P, Q) = 0$. Moreover, if the degree of P and Q coincide, then*

$$\lim_{\longrightarrow m} Ext^*_{\mathcal{P}(k)}(P^{(m)}, Q^{(m)}) \simeq Ext^*_{\mathcal{F}(k)}(P, Q)$$

Frobenius twisting decreases the 'degree of homological triviality' as is shown in a result of H.H. Andersen.

Proposition 2.2. *(H. H. Andersen; see* [15, Corollary 1.3]*) For two homogeneous strict polynomial functors P and Q between $\mathbb{F}_q$-vector spaces of the same degree, for $q = p^s$ and for $m \in \mathbb{N} \cup \{0\}$ we get*

$$Ext^*_{\mathcal{P}(k)}(P^{(m)}, Q^{(m)}) \leq Ext^*_{\mathcal{P}(k)}(P^{(m+1)}, Q^{(m+1)})$$

The first of the two statements in Theorem 1 actually is due to Kuhn:

Lemma 2.3. [22] *Any functor $F \in \mathcal{F}(q)$ decomposes into a direct sum $F = \oplus_{i=0}^{q-1} F_i$ where $F_i(V) := \{x \in F(V) |\ F(\lambda\cdot)(x) = \lambda^i \cdot x \quad \forall \lambda \in \mathbb{F}_q\}$. This induces a decomposition of the category of functors between $\mathbb{F}_q$-vector spaces $\mathcal{F}(q) = \prod_{i=0}^{q-1} \mathcal{F}(q)_i$.*

Finally, a result due to Kuhn will be essential in the sequel.

Theorem 2. *(N. Kuhn* [23, 24]*) The injective envelope $I_{\mathbb{F}_p}$ of the trivial module in the category of analytic functors $\mathcal{F}^\omega(\mathbb{F}_p)$ from finite dimensional $\mathbb{F}_p$-vector spaces to $\mathbb{F}_p$-vector spaces is uniserial and the only composition factors in $\mathcal{F}^p(\mathbb{F}_p)$ are the two composition factors of $soc_2(I_{\mathbb{F}_p}))$, where, as usual, soc_2 denotes the second layer in the socle series* [4, Definition 1.2.1].

The next result of Franjou, Lannes and Schwartz implies that the categories $\mathcal{A}^p_{\mathbb{F}_p}$ and $\mathcal{F}^p_{\mathbb{F}_p}$ are different.

Theorem 3. *(Franjou-Lannes-Schwartz* [12]*; Franjou-Pirashvili* [13]*)*

$$Ext_{\mathcal{A}^p_{\mathbb{F}_p}}(id, \mathbb{F}_p \otimes id) \simeq \mathbb{F}_p[e_1, e_2, \dots]/(e_h^p; h \geq 0) \otimes \Lambda(\xi_1)$$

where $\Lambda(\xi_1)$ is the exterior algebra in one variable with generator in degree $2p-1$ and e_h are generators in degree $2p^h$. Moreover,

$$Ext_{\mathcal{F}^p_{\mathbb{F}_p}}(id, id) \simeq \mathbb{F}_p[e_0, e_1, \dots]/(e_h^p; h \geq 0).$$

Remark that $\mathbb{F}_p \otimes_{\mathbb{Z}} id = id$ as functors on the category $\mathbb{F}_p - mod$. As a consequence, $Ext^2_{\mathcal{F}^p_{\mathbb{F}_p}}(id, id) \neq 0$. Indeed, by [12, 7.3] the following four term sequence is a non zero element:

$$0 \longrightarrow id \longrightarrow S_p \longrightarrow S^p \longrightarrow id \longrightarrow 0;$$

where S_p is the degree p homogeneous part of the coinvariants under the $\mathfrak{S}_p$ action on the tensor algebra, and where S^p is the degree p homogeneous part of the invariants of the tensor algebra.

3. On recollement diagrams

We remind the reader to the notion of a recollement diagram. A recollement diagram is given by three categories $\mathcal{A},\mathcal{B}$ and $\mathcal{C}$ with functors

$$\begin{matrix} & \overset{q}{\leftarrow} & & \overset{l}{\leftarrow} & \\ \mathcal{A} & \overset{i}{\rightarrow} & \mathcal{B} & \overset{e}{\rightarrow} & \mathcal{C} \\ & \overset{p}{\leftarrow} & & \overset{r}{\leftarrow} & \end{matrix}$$

so that

1. (l,e) and (e,r) are adjoint pairs.
2. (q,i) and (i,p) are adjoint pairs.
3. i is a full embedding and $e(B')=0 \Leftrightarrow B'\simeq i(A')$ for an $A'\in\mathcal{A}$.
4. the adjointness morphisms $e\circ r\longrightarrow id_{\mathcal{C}}$ and $id_{\mathcal{C}}\longrightarrow e\circ l$ are isomorphisms.

We denote a recollement diagram as above by $(\mathcal{A},\mathcal{B},\mathcal{C},(e,l,r),(i,q,p))$.

We shall give a result which is a special case of a recent result of Chrysostomos Psaroudakis and Jorge Vitoria [32]. We shall give our original proof below, since in our special case the proof is much easier than the proof for the general statement from [32].

Proposition 3.1. *Let $(\mathcal{A},\mathcal{B},\mathcal{C},(e,l,r),(i,q,p))$ be a recollement diagram. Suppose $\mathcal{B}\simeq B-mod$ and $\mathcal{C}=C-mod$ are module categories and that $\mathcal{B}$ satisfies the Krull-Schmidt theorem on projective modules. Suppose that e is representable. Then, $\mathcal{A}=A-mod$ again is a module category, there is an idempotent e' in B so that C is Morita equivalent to $e'Be'$, and A is Morita equivalent to $B/Be'B$.*

Proof. Since e has a left and a right adjoint, e is exact. Therefore, $e=Hom_{\mathcal{B}}(P,-)$ where P is a projective object. Since l and r are left and right adjoints to e, we get that $l=P\otimes_{End_B(P)}-$, that $r=Hom_{End_B(P)}(Hom_B(P,B),-)$ and that $C\simeq End_B(P)$. Since $\mathcal{B}$ is a Krull-Schmidt category, then up to Morita equivalence, one can choose $P=Be'$ for an idempotent $e'^2=e'\in B$ and we get C is Morita equivalent to $e'Be'$.

The third condition in the definition of a recollement diagram implies that $\mathcal{A}$ can be identified with those B-modules M for which $e'M=0$. Hence,

$$\mathcal{A}\simeq\{M\in B-mod|\ e'M=0\}\simeq B/Be'B-mod.$$

This proves the proposition. ∎

Another important observation is that, by the adjointness properties, l maps projective object in $\mathcal{C}$ to projective objects in $\mathcal{B}$, and that r maps injective objects in $\mathcal{C}$ to injective objects in $\mathcal{B}$.

3.1. **Analyzing Schwartz' recollement for polynomial functors.** Let $q=p^s$ for a prime p. Then, using the notation of Section 2.2, following Kuhn [25, Theorem 1.3] or Schwartz [40, §5.5] we have a recollement diagram

$$\begin{matrix} & \leftarrow & & \leftarrow & \\ \mathcal{F}^{n-1}(q) & \rightarrow & \mathcal{F}^n(q) & \rightarrow & \prod_{n(\lambda)=n}\mathbb{F}_q\mathfrak{S}_\lambda-mod \\ & \leftarrow & & \leftarrow & \end{matrix}$$

where $\mathfrak{S}_\lambda=\mathfrak{S}_{\lambda_1}\times\cdots\times\mathfrak{S}_{\lambda_{s-1}}$ where $\mathfrak{S}_k$ is the symmetric group on k elements and where $n(\lambda):=\lambda_0+\cdots+\lambda_{s-1}$. Moreover, the functor $\mathcal{F}^n(q)\longrightarrow\prod_{n(\lambda)=n}\mathbb{F}_q\mathfrak{S}_\lambda-mod$ is representable by id^λ and for a partition $\lambda=(\lambda_0\geq\cdots\geq\lambda_{s-1})$, we set $id^\lambda:=\oplus_{j=1}^{s-1}id^{\otimes\lambda_j}$.

Remark 3.2. Hence, in case $s=1$ and $n<p$, the recollement becomes

$$\begin{matrix} & \leftarrow & & \leftarrow & \\ \mathcal{F}^{n-1}(p) & \rightarrow & \mathcal{F}^n(p) & \rightarrow & \prod_{\text{partitions of } n}(\mathbb{F}_p-mod) \\ & \leftarrow & & \leftarrow & \end{matrix}$$

since $\mathbb{F}_p\mathfrak{S}_n$ is semisimple, since $\mathbb{F}_p$ is a splitting field, and therefore its module category is equivalent to a direct product of copies $\mathbb{F}_p - mod$.

In case $s = 1$ and $n = p$, the recollement becomes

$$\begin{array}{ccccc} & \leftarrow & & \leftarrow & \\ \mathcal{F}^{p-1}(p) & \rightarrow & \mathcal{F}^p(p) & \rightarrow & \mathbb{F}_p\mathfrak{S}_p - mod \\ & \leftarrow & & \leftarrow & \end{array}$$

We have an immediate consequence of the above result.

Lemma 3.3. *For any simple polynomial functor F there is a strict polynomial functor $\hat{F}$ such that the forgetful functor, which assigns to every strict polynomial functor its polynomial functor by evaluating the polynomial as mapping, maps $\hat{F}$ to F.*

Proof. This is done by induction on the degree. The simple objects in $\mathcal{F}^d(q)$ are in bijection with the union of the simple objects in $\mathcal{F}^{d-1}(q)$ and the simple objects in $\mathbb{F}_q\mathfrak{S}_d - mod$. Now, any simple $\mathbb{F}_q\mathfrak{S}_d$-module is image of a simple module of the Schur algebra $S_{\mathbb{F}_q}(d, d)$ under the Schur functor as is a classical fact. Since the category of degree d strict polynomial functors is equivalent to the category of modules over the Schur algebra $S(d, d)$, the simple objects in $\mathcal{F}^d(q)$ of degree d are images of a strict polynomial functor. By induction, the simple objects of degree less than d are images of strict polynomial functors. ∎

3.2. **Recollement for $\mathcal{A}_{\mathbb{F}_p}$.** For the category $\mathcal{A}_{\mathbb{F}_p}$ we get a similar recollement diagram.

Remark 3.4. The treatment we give for Section 3.2 was suggested by the referee, and follows paths which are not easily documented. They appeared in the special case of degree 2 in introductory remarks in [11, 12]. An alternative proof can be obtained, and actually this was our initial approach, using Piriou's thesis [30, 31] along its first chapter. I am very grateful to the referee for this useful hint.

The recollement diagram was proved in a more general situation in the meantime by Djament and Vespa [7, Theorem 2.2], so that the result Proposition 3.8 is a special case of [7, Theorem 2.2].

We shall mainly work with properties of adjoint functors. Lemma 3.5 below seems to be a well-known result. It appears maybe for the first time in the proof of [12, Lemma 0.4] without further reference. In order to keep the presentation as self-contained as possible, we shall provide a short proof.

Lemma 3.5. *Let $\mathcal{C}$, $\mathcal{C}'$ and $\mathcal{D}$ be categories, let $L : \mathcal{C} \longrightarrow \mathcal{C}'$ be a functor admitting a right adjoint R, and let $\hat{L}$ be the induced functor $Funct(\mathcal{C}', \mathcal{D}) \longrightarrow Funct(\mathcal{C}, \mathcal{D})$ on the functor categories given by precomposition with L, and let likewise $\hat{R}$ be the induced functor $Funct(\mathcal{C}, \mathcal{D}) \longrightarrow Funct(\mathcal{C}', \mathcal{D})$ given by precomposition with R. Then $\hat{R}$ is left adjoint to $\hat{L}$.*

Proof. We get natural transformations

$$id_{\mathcal{C}} \xrightarrow{\eta} RL \text{ and } LR \xrightarrow{\epsilon} id_{\mathcal{C}'}$$

so that the compositions

$$L \longrightarrow LRL \longrightarrow L \text{ and } R \longrightarrow RLR \longrightarrow R$$

are the identity on the respective ctegories (cf e.g. Maclane [26, IV.1.Theorem 2]).

Now, given any functor $X : \mathcal{C} \longrightarrow \mathcal{D}$, then we obtain a natural transformation

$$id_{Funct(\mathcal{C},\mathcal{D})} \longrightarrow (\hat{L}\hat{R})$$

induced from η by

$$X \xrightarrow{X(\eta)} XRL = (\hat{L}\hat{R})(X)$$

and likewise a natural transformation

$$(\hat{R}\hat{L}) \longrightarrow id_{Funct(\mathcal{C}',\mathcal{D})}.$$

Since the compositions $L \longrightarrow LRL \longrightarrow L$ and $R \longrightarrow RLR \longrightarrow R$ are the identity, this holds as well for the compositions $\hat{L} \longrightarrow \hat{L}\hat{R}\hat{L} \longrightarrow \hat{L}$ and $\hat{R} \longrightarrow \hat{R}\hat{L}\hat{R} \longrightarrow \hat{R}$. Again by [26, IV.1.Theorem 2] we obtain the statement. ∎

We shall need a very simple observation.

Lemma 3.6. *Let A and B be two polynomial functors of degree at most n and let*

$$0 \longrightarrow A \longrightarrow C \longrightarrow B \longrightarrow 0$$

be an exact sequence of functors. Then, the degree of C is at most n as well. Moreover, taking cross effects is exact.

Proof: We get a commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A^{(2)}(U|V) & \longrightarrow & C^{(2)}(U|V) & \longrightarrow & B^{(2)}(U|V) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A(U\oplus V) & \longrightarrow & C(U\oplus V) & \longrightarrow & B(U\oplus V) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A(U)\oplus A(V) & \longrightarrow & C(U)\oplus C(V) & \longrightarrow & B(U)\oplus B(V) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

and the snake lemma implies that

$$0 \longrightarrow A^{(2)}(U|V) \longrightarrow C^{(2)}(U|V) \longrightarrow B^{(2)}(U|V) \longrightarrow 0$$

is exact. Induction on the degree gives the result. ∎

We shall now prove a general statement on functor categories which is an adaption from Franjou [11, Section 1].

Lemma 3.7. *Let R be a commutative ring and let G is an object in $\mathcal{A}_R^n$. Then we get $Hom_{\mathcal{A}_R}((R\otimes id^{\otimes n}, G) = G^{(n-1)}$.*

Proof. Let

$$n-\mathbb{Z}-free = \underbrace{(\mathbb{Z}-free)\times\cdots\times(\mathbb{Z}-free)}_{n\ copies}$$

be the category with objects $F_1\times\cdots\times F_n$ for F_i being an object in $\mathbb{Z}-free$ for each $i\in\{1,\ldots,n\}$ and morphisms being

$$Hom_{n-\mathbb{Z}-free}(F_1\times\cdots\times F_n, G_1\times\cdots\times G_n) := Hom_{\mathbb{Z}}(F_1,G_1)\times\cdots\times Hom_{\mathbb{Z}}(F_n,G_n),$$

for all objects F_i, G_j of $\mathbb{Z}-free$. Composition of morphisms is given by composition of mappings in $\mathbb{Z}-free$. We shall define

$$\Pi_n : n-\mathbb{Z}-free \longrightarrow \mathbb{Z}-free$$

and

$$\Delta_n : \mathbb{Z}-free \longrightarrow n-\mathbb{Z}-free$$

by $\Pi_n(F_1,\ldots,F_n) := F_1\oplus\cdots\oplus F_n$ and $\Delta_n(F) := (F,\ldots,F)$. Then Π_n is left and right adjoint to Δ_n as is easily seen, almost by definition.

Now, we consider the category $n-\mathcal{A}_R$ of functors $n-\mathbb{Z}-free \longrightarrow R-Mod$ with morphisms being natural transformations. We get functors

$$\begin{array}{ccc}
n-\mathcal{A}_R & \stackrel{\Delta^n}{\longrightarrow} & \mathcal{A}_R \\
F & \mapsto & F\circ\Delta_n
\end{array}$$

and

$$\begin{array}{ccc} \mathcal{A}_R & \stackrel{\Pi^n}{\longrightarrow} & n-\mathcal{A}_R \\ F & \mapsto & F\circ\Pi_n \end{array}$$

By Lemma 3.5, we see that the functor Π^n is left and right adjoint to Δ^n. Put

$$\begin{array}{ccc} n-\mathbb{Z}-free & \stackrel{\boxtimes^n}{\longrightarrow} & \mathbb{Z}-free \\ (M_1,\dots,M_n) & \mapsto & M_1\otimes\dots\otimes M_n \end{array}$$

and obtain $id^{\otimes n}=\boxtimes^n\circ\Delta_n$ and $R\otimes id^{\otimes n}=(R\otimes\boxtimes^n)\circ\Delta_n$. But then

$$\begin{array}{rcl} Hom_{\mathcal{A}_R}((R\otimes id^{\otimes n},G) & = & Hom_{\mathcal{A}_R}((R\otimes\boxtimes^n)\circ\Delta_n,G) \\ & = & Hom_{n-\mathcal{A}_R}((R\otimes\boxtimes^n),G\circ\Pi_n) \end{array}$$

Let $I:=\{i_1,\dots,i_m\}\subseteq\{1,\dots,n\}$ be an m-element subset of $\{1,\dots,n\}$, and suppose $i_1<\dots<i_m$. Then we may consider $\pi_I^n: n-\mathbb{Z}-free\longrightarrow m-\mathbb{Z}-free$ to be the functor given by

$$\pi_I^n(M_1,\dots,M_n):=(M_{i_1},\dots,M_{i_m})$$

and the functor $\iota_I^n: m-\mathbb{Z}-free\longrightarrow n-\mathbb{Z}-free$ given by injection into the corresponding coordinates so that $\pi_I^n\circ\iota_I^n=id$. Then, again by definition, ι_I^n is left and right adjoint to π_I^n. Using Lemma 3.5 we get that the functors

$$\widehat{\iota_I^n}: n-\mathcal{A}_R\longrightarrow m-\mathcal{A}_R$$

and

$$\widehat{\pi_I^n}: m-\mathcal{A}_R\longrightarrow n-\mathcal{A}_R$$

obtained by pre-composition with the functors ι_I^n and π_I^n form a pair of left and right adjoint functors.

By definition of the cross effect of a functor we get

$$G(V_1\oplus\dots\oplus V_n)=\bigoplus_{m=1}^n\bigoplus_{i_1<\dots<i_m}G^{(m-1)}(V_{i_1}|\dots|V_{i_m}).$$

Hence

$$G\circ\Pi_n=\bigoplus_{m=1}^n\bigoplus_{i_1<\dots<i_m}G^{(m-1)}\circ\pi^n_{\{i_1,\dots,i_m\}}.$$

Moreover, using that $\widehat{\iota_I^n}$ is left adjoint to $\widehat{\pi_I^n}$, in case $m\neq n$ we get

$$\begin{array}{rl} Hom_{n-\mathcal{A}_R}((R\otimes\boxtimes^n),G^{(m-1)}\circ\pi^n_{\{i_1,\dots,i_m\}}) = & \\ = & Hom_{n-\mathcal{A}_R}((R\otimes\boxtimes^n)\circ\iota^n_{\{i_1,\dots,i_m\}},G^{(m-1)})=0 \end{array}$$

since at least one of the factors in the tensor product is 0. Therefore

$$Hom_{\mathcal{A}_R}((R\otimes id^{\otimes n},G)=Hom_{n-\mathcal{A}_R}((R\otimes\boxtimes^n),G^{(n-1)})$$

Since G is of degree at most n, the functor $G^{(n-1)}$ is additive in each variable. Hence its value is given by the R-module $G^{(n-1)}(\mathbb{Z}|\dots|\mathbb{Z})$. However, $(R\otimes id^{\boxtimes^n})(\mathbb{Z}|\dots|\mathbb{Z})=R$ and $Hom_R(R,G^{(n-1)}(\mathbb{Z}|\dots|\mathbb{Z}))=G^{(n-1)}(\mathbb{Z}|\dots|\mathbb{Z})$. Hence a natural transformation in $Hom_{n-\mathcal{A}_R}((R\otimes\boxtimes^n),G^{(n-1)})$ induces an element in $G^{(n-1)}(\mathbb{Z}|\dots|\mathbb{Z})$. On the other hand, any element in $G^{(n-1)}(\mathbb{Z}|\dots|\mathbb{Z})$ induces a natural transformation in $Hom_{n-\mathcal{A}_R}((R\otimes\boxtimes^n),G^{(n-1)})$. Therefore

$$Hom_{\mathcal{A}_R}(R\otimes id^{\otimes n},G)=G^{(n-1)}$$

as claimed and we obtain the statement. ∎

Proposition 3.8.

$$\begin{array}{ccccc} & \leftarrow & & \leftarrow & \\ \mathcal{A}^{n-1}_{\mathbb{F}_p} & \rightarrow & \mathcal{A}^{n}_{\mathbb{F}_p} & \overset{e}{\rightarrow} & \mathbb{F}_p\mathfrak{S}_n - mod \\ & \leftarrow & & \leftarrow & \end{array}$$

is a recollement diagram with

$$e := Hom_{\mathcal{A}_{\mathbb{F}_p}}(\mathbb{F}_p \otimes_{\mathbb{Z}} \underbrace{id \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} id}_{n\ factors}, -) : \mathcal{A}^n_{\mathbb{F}_p} \longrightarrow \mathbb{F}_p\mathfrak{S}_n - mod.$$

Proof. We shall show the following auxiliary lemma needed for the proof of the proposition.

Lemma 3.9. $\mathbb{F}_p \otimes id^{\otimes n}$ *is projective and injective in* $\mathcal{A}^n_{\mathbb{F}_p}$.

Proof. Taking cross effects is exact. By Lemma 3.7 we see that $\mathbb{F}_p \otimes id^{\otimes n}$ is projective. Using that the duality D on functors $F : \mathbb{Z} - free \longrightarrow \mathbb{F}_p - Mod$ given by

$$(DF)(V) = Hom_{\mathbb{F}_p}(F(Hom_{\mathbb{Z}}(V, \mathbb{Z})), \mathbb{F}_p),$$

and observing that $\mathbb{F}_p \otimes id^{\otimes n}$ is self-dual, we obtain that $\mathbb{F}_p \otimes id^{\otimes n}$ is injective as well. ∎

We need to prove that

$$\mathcal{A}^{n-1}_{\mathbb{F}_p} = \{F \in \mathcal{A}^n_{\mathbb{F}_p} |\ Hom_{\mathcal{A}_{\mathbb{F}_p}}(\mathbb{F}_p \otimes id^{\otimes n}, F) = 0\}.$$

But this is clear by Lemma 3.7

Since $\mathbb{F}_p \otimes id^{\otimes n}$ is projective there is a right adjoint and a left adjoint to $Hom_{\mathcal{A}_{\mathbb{F}_p}}(\mathbb{F}_p \otimes id^{\otimes n}, -)$, namely the functor $M \mapsto (\mathbb{F}_p \otimes id^{\otimes n} \otimes M)^{\mathfrak{S}_n}$ is the right adjoint, and the functor $M \mapsto (\mathbb{F}_p \otimes id^{\otimes n} \otimes M)_{\mathfrak{S}_n}$ is the left adjoint.

Moreover, the unit and the counit of the adjunctions induce the identity on $\mathbb{F}_p\mathfrak{S}_n$-mod. This can be done literally as in Piriou [30, Proposition 2.2.2].

This shows Proposition 3.8. ∎

As a consequence we show the following lemma.

Lemma 3.10. *Any simple object in* $\mathcal{A}^n_{\mathbb{F}_p}$ *is in the image of* $\circ(\mathbb{F}_p \otimes -) : \mathcal{F}^n_{\mathbb{F}_p} \hookrightarrow \mathcal{A}^n_{\mathbb{F}_p}$ *and any simple object of* $\mathcal{A}^n_{\mathbb{F}_p}$ *gives a simple object in* $\mathcal{F}^n_{\mathbb{F}_p}$ *this way.*

Proof. We shall use induction on n. There is a morphism of recollement diagrams as follows, where the vertical functors are fully faithful embeddings of categories by Lemma 1.2.

$$\begin{array}{ccccc} & \leftarrow & & \leftarrow & \\ \mathcal{F}^{n-1}_{\mathbb{F}_p} & \rightarrow & \mathcal{F}^{n}_{\mathbb{F}_p} & \overset{e_{\mathcal{F}^n}}{\rightarrow} & \mathbb{F}_p\mathfrak{S}_n - mod \\ & \leftarrow & & \leftarrow & \\ \downarrow i_{n-1} & & \downarrow i_n & & \| \\ & \leftarrow & & \leftarrow & \\ \mathcal{A}^{n-1}_{\mathbb{F}_p} & \rightarrow & \mathcal{A}^{n}_{\mathbb{F}_p} & \overset{e_{\mathcal{A}^n}}{\rightarrow} & \mathbb{F}_p\mathfrak{S}_n - mod \\ & \leftarrow & & \leftarrow & \end{array}$$

Therefore the number of simple objects in $\mathcal{A}^n_{\mathbb{F}_p}$ and in $\mathcal{F}^n_{\mathbb{F}_p}$ coincides.

The statement is clear for $n \leq p-1$ by Lemma 1.6. Let $n \geq p$ and let S be a simple object in $\mathcal{F}^n_{\mathbb{F}_p}$. We may suppose, using the induction hypothesis that $e_{\mathcal{F}^n}(S) \neq 0$. Then, suppose X is a simple and proper subobject of i_nS. Since $e_{\mathcal{A}^n}$ is exact, $e_{\mathcal{A}^n}(X)$ is a subobject of $e_{\mathcal{A}^n}(i_nS) = e_{\mathcal{F}^n}(S)$. Since $e_{\mathcal{F}^n}$ is exact, $e_{\mathcal{F}^n}(S)$ is simple. So, $e_{\mathcal{A}^n}(X)$ is either 0 or isomorphic to $e_{\mathcal{F}^n}(S)$. Since X is a proper subobject of i_nS we see that $e_{\mathcal{A}^n}(X) = 0$, and hence $X \in \mathcal{A}^{n-1}$. Since the simple objects of $\mathcal{F}^{n-1}$ and of $\mathcal{A}^{n-1}$ coincide by the induction hypothesis, X is a proper non zero subobject of S. This gives the contradiction. ∎

We have seen in Lemma 1.6 and Remark 3.2 that

$$\mathcal{F}_{\mathbb{F}_p}^{p-1} \simeq \mathcal{A}_{\mathbb{F}_p}^{p-1} \simeq \prod_{n<p} \prod_{\lambda \vdash n} \mathbb{F}_p - mod$$

4. Short Review of Brauer tree algebras

Proposition 3.8 shows that the representation theory of the symmetric group is closely related to $\mathcal{A}^n(p)$ we shall need some information from group representations. We give Benson [4, Section 4.18 and Section 6.5] as a general reference.

Let k be a field of characteristic $p > 0$ and let G be a finite group so that k is a splitting field for G, i.e. the endomorphism ring of each simple kG-module is k. Then it is well-known that there are only a finite number of isomorphism classes of indecomposable kG-modules if and only if the Sylow p subgroups of G are all cyclic. In particular the symmetric group $\mathfrak{S}_p$ is a group with every Sylow p subgroup being cyclic.

An indecomposable ring-direct factor of kG is called a p-block of G, and the representation theory of a block B is completely understood in case the Sylow p subgroups of G are cyclic. The algebras occurring in this case are so-called Brauer tree algebras.

We explain briefly the theory of Brauer tree algebras and refer to Benson [4, Section 4.18 and Section 6.5] or to Auslander-Reiten-Smalø [1, Section X.3] for more details.

A Brauer tree is a finite connected tree Γ, i.e. a connected graph without cycles and multiple edges, together with additional structure as we shall describe now: Let Γ_0 be the vertices of Γ and let Γ_1 be the edges of Γ. For each $e \in \Gamma_1$ let $v_1(e) \in \Gamma_0$ and $v_2(e) \in \Gamma_0$ be the vertices adjacent to e.

- Then for each vertex $v \in \Gamma_0$ let E_v be the edges adjacent to v, and suppose given a transitive permutation $\sigma_v \in \mathfrak{S}_{E_v}$.
- Moreover, we choose one vertex $v_0 \in \Gamma_v$ and an integer $\mu > 0$, called the exceptional multiplicity.

A Brauer tree algebra B is then a finite dimensional symmetric k-algebra so that

- the isomorphism classes of simple B-modules are parameterised by Γ_1, denoting by S_e a simple B-module whose isomorphism class corresponds to the edge $e \in \Gamma_1$,
- the projective cover P_e of S_e has the property that $rad(P_e)/soc(P_e) = U_{v_1(e)} \oplus U_{v_2(e)}$, where $U_{v_1(e)}$ and $U_{v_2(e)}$ are both uniserial modules.
- for each $j \in \{1, 2\}$,
  - if $v_j(e)$ is not the exceptional vertex, then the composition length of $U_{v_j(e)}$ is equal to $|E_{v_j(e)}| - 1$
  - if $v_j(e)$ is the exceptional vertex, then the composition length of $U_{v_j(e)}$ is equal to $\mu \cdot |E_{v_j(e)}| - 1$
- for each i, for which the i-th radical quotient is not 0,
  $$rad^i(U_{v_1(e)})/rad^{i+1}(U_{v_1(e)}) \simeq S_{\sigma_{v_1(e)}^{i+1}(e)}$$
  and
  $$rad^i(U_{v_2(e)})/rad^{i+1}(U_{v_2(e)}) \simeq S_{\sigma_{v_2(e)}^{i+1}(e)}$$

One observes that in case $\mu = 1$, then there is no difference between the exceptional vertex and a non exceptional vertex. Hence, in this case we do not need to fix an exceptional vertex and we say that the Brauer tree has no exceptional vertex if the exceptional multiplicity μ is 1.

Observe further that if a vertex v is a leaf, i.e. $E_v = \{e\}$, then the projective cover of S_e is uniserial and $U_v = 0$.

A particular case is when each vertex has at most 2 edges adjacent to it, or in other words if $|E_v| \leq 2$ for all $v \in \Gamma_0$. In this case we call the Brauer tree a stem. If moreover the Brauer tree has no exceptional vertex then the tree can be visualised as

$$\bullet - \bullet - \bullet - \cdots - \bullet - \bullet$$

The projective indecomposable modules P are then all of Loewy length 3, i.e. P has simple top, simple socle and $rad(P)/soc(P)$ is semisimple of composition length at most 2.

This situation occurs for $\mathbb{F}_p\mathfrak{S}_p$ which has several blocks for $p \geq 5$, one of which is a Brauer tree algebra associated to a stem without exceptional vertex and $p-1$ edges, and all the other blocks are simple algebras.

5. The structure of polynomial functors modulo p

The situation of polynomial functors of degree p is different from those of degree $n < p$. We are going to describe in this section their structure completely. From now on we assume that $p \geq 5$ since the representation theory of $\mathbb{F}_2\mathfrak{S}_2$ and of $\mathbb{F}_3\mathfrak{S}_3$ is slightly different from the case $p \geq 5$.

Remark 5.1. Let us recall the relation between the Schur algebra $S_{\mathbb{F}_p}(p,p)$ and the group algebra $\mathbb{F}_p\mathfrak{S}_p$. We shall use the structure of the Schur algebra and the corresponding Hecke algebra as it is proved in [20, Corollary 1.3]. The group algebra of the symmetric group is a special case of the Hecke algebra. For general informations on Schur algebras and Brauer tree algebras see [17] and [21]. The algebra $S_{\mathbb{F}_p}(p,p)$ admits p simple modules $S_1,\dots,S_{p-1},S_p$ whereas the group algebra $\mathbb{F}_p\mathfrak{S}_p$ admits $p-1$ simple modules $S'_1,\dots,S'_{p-1}$. The projective indecomposable $S_{\mathbb{F}_p}(p,p)$-modules have composition series

$$\begin{matrix} S_1 \\ S_2 \\ S_1 \end{matrix}\ ,\quad \begin{matrix} & S_2 & \\ S_1 & & S_3 \\ & S_2 & \end{matrix}\ ,\quad \begin{matrix} & S_3 & \\ S_2 & & S_4 \\ & S_3 & \end{matrix}\ ,\ \dots,\ \begin{matrix} & S_{p-1} & \\ S_{p-2} & & S_p \\ & S_{p-1} & \end{matrix}\ ,\quad \begin{matrix} S_p \\ S_{p-1} \end{matrix}$$

whereas the projective indecomposable $\mathbb{F}_p\mathfrak{S}_p$-modules have composition series

$$\begin{matrix} S'_1 \\ S'_2 \\ S'_1 \end{matrix}\ ,\quad \begin{matrix} & S'_2 & \\ S'_1 & & S'_3 \\ & S'_2 & \end{matrix}\ ,\quad \begin{matrix} & S'_3 & \\ S'_2 & & S'_4 \\ & S'_3 & \end{matrix}\ ,\ \dots,\ \begin{matrix} & S'_{p-2} & \\ S'_{p-3} & & S'_{p-1} \\ & S'_{p-2} & \end{matrix}\ ,\quad \begin{matrix} S'_{p-1} \\ S'_{p-2} \\ S'_{p-1} \end{matrix}$$

Lemma 5.2. *Let L be the simple polynomial functor in $\mathcal{F}^p_{\mathbb{F}_p}$ so that L is of degree p and so that L corresponds to the trivial representation of $\mathbb{F}_p\mathfrak{S}_p$. Then, $Ext^1_{\mathcal{F}_{\mathbb{F}_p}}(id,L) \neq 0 \neq Ext^1_{\mathcal{F}_{\mathbb{F}_p}}(L,id)$. Moreover, if $Ext^1_{\mathcal{F}_{\mathbb{F}_p}}(id,S) \neq 0$ or $0 \neq Ext^1_{\mathcal{F}_{\mathbb{F}_p}}(S,id)$ for a simple degree p-functor S, then $L \simeq S$.*

Proof. Let S_1 be the simple polynomial functor of degree 1. The identity functor id is trivially of degree 1 and simple, which implies $S_1 = id$. Then, $Ext^1_{\mathcal{F}^p}(L,id)$ is not necessarily zero for L being an irreducible polynomial functor of degree p. Now, since we are working over $\mathbb{F}_p$, we get that $id^{(1)} \simeq id$ as polynomial functor, but not as strict polynomial functor. As strict polynomial functor, $I^{(1)}$ is of degree p.

Theorem 2 implies that there is only one simple functor L of degree p with $Ext^1_{\mathcal{F}^p}(L,id) \neq 0$. Proposition 2.2 in connection with Theorem 1, imply that L is the simple functor corresponding to the trivial $\mathbb{F}_p\mathfrak{S}_p$-module, since this is the module which has an extension with the unique simple module of the Schur algebra $S_{\mathbb{F}_p}(p,p)$ which is not a simple $\mathbb{F}_p\mathfrak{S}_p$-module (cf Remark 5.1).

This implies that

$$Ext^1_{\mathcal{P}(p)}(L,id^{(1)}) \simeq Ext^1_{\mathcal{F}_{\mathbb{F}_p}}(L,id^{(1)}) \simeq Ext^1_{\mathcal{F}_{\mathbb{F}_p}}(L,id)\ .$$

Since the category of degree p strict polynomial functors is equivalent to the category of modules over the Schur algebra $S_{\mathbb{F}_p}(p,p)$, one sees that

$$Ext^1_{\mathcal{P}_{\mathbb{F}_p}}(L,id^{(1)}) \simeq Ext^1_{S_{\mathbb{F}_p}(p,p)}(V,I_0)$$

for I_0 being the simple $S_{\mathbb{F}_p}(p,p)$-module corresponding to the p singular partition of p and V being the simple $S_{\mathbb{F}_p}(p,p)$-module corresponding to L. Finally, it is a classical fact (cf e.g. [20]) that $Ext^1_{S_{\mathbb{F}_p}(p,p)}(I_0,V) \neq 0$ or $Ext^1_{S_{\mathbb{F}_p}(p,p)}(V,I_0) \neq 0$ if and only if V corresponds to the trivial module of the symmetric group, and in this case, the dimension of $Ext^1_{S_{\mathbb{F}_p}(p,p)}(I_0,V)$ and of $Ext^1_{S_{\mathbb{F}_p}(p,p)}(V,I_0)$ is 1. This proves the statement. ■

Using the embedding $\mathcal{F}^p_{\mathbb{F}_p} \hookrightarrow \mathcal{A}^p_{\mathbb{F}_p}$ from Lemma 1.2, the functor L of $\mathcal{F}^p_{\mathbb{F}_p}$ induces a functor $L(\mathbb{F}_p \otimes -)$ in $\mathcal{A}^p_{\mathbb{F}_p}$. In order to avoid additional notational burden we shall denote this functor $L(\mathbb{F}_p \otimes -)$ by L as well.

We get as a corollary the following statement.

Corollary 5.3. *$Ext^1_{\mathcal{A}^p_{\mathbb{F}_p}}(\mathbb{F}_p \otimes_{\mathbb{Z}} id, L) \neq 0 \neq Ext^1_{\mathcal{A}^p_{\mathbb{F}_p}}(L, \mathbb{F}_p \otimes_{\mathbb{Z}} id)$.*

Proof. We know that $Ext^1_{\mathcal{F}^p_{\mathbb{F}_p}}(id, L) \neq 0$. So, there is a non split exact sequence

$$0 \longrightarrow L \longrightarrow X \longrightarrow id \longrightarrow 0$$

for some functor X in $\mathcal{F}^p_{\mathbb{F}_p}$. Since $\mathcal{F}^p_{\mathbb{F}_p} \hookrightarrow \mathcal{A}^p_{\mathbb{F}_p}$ by Lemma 1.2. This induces an exact sequence

$$0 \longrightarrow L(\mathbb{F}_p \otimes_{\mathbb{Z}} id) \longrightarrow X(\mathbb{F}_p \otimes_{\mathbb{Z}} id) \longrightarrow \mathbb{F}_p \otimes_{\mathbb{Z}} id \longrightarrow 0$$

in $\mathcal{A}^p_{\mathbb{F}_p}$ where $L(\mathbb{F}_p \otimes_{\mathbb{Z}} id)$ is simple by Lemma 3.10. This sequence is non split since the functor pre-composing with $\mathbb{F}_p \otimes_{\mathbb{Z}} id$ is a fully faithful embedding. Hence $Ext^1_{\mathcal{A}^p_{\mathbb{F}_p}}(\mathbb{F}_p \otimes_{\mathbb{Z}} id, L) \neq 0$. Therefore, L is a direct factor of the top of the radical of the projective cover of $\mathbb{F}_p \otimes_{\mathbb{Z}} id$. Similarly, $Ext^1_{\mathcal{A}^p_{\mathbb{F}_p}}(L, \mathbb{F}_p \otimes_{\mathbb{Z}} id) \neq 0$. ■

Actually, the argument of Lemma 5.2 gives another slightly different statement.

Lemma 5.4. *Let X and Y be two simple functors of degree at most p. Then,*

$$Ext^1_{\mathcal{F}^p_{\mathbb{F}_p}}(X,Y) \neq 0 \Rightarrow deg(X) - deg(Y) \in \{0, p-1\}.$$

Proof. By Lemma 3.10 we know that the simple functors X and Y can be considered to lie in $\mathcal{F}^p_{\mathbb{F}_p}$. Proposition 2.2 in connection with Theorem 1 imply this result. ■

Remark 5.5. At the present stage it might happen that $Ext^1_{\mathcal{A}^p_{\mathbb{F}_p}}(X,Y) \neq 0$ even though $Ext^1_{\mathcal{F}^p_{\mathbb{F}_p}}(X,Y) = 0$.

Denoting by $\rho_p(i)$ the number of p-regular partitions of i, the algebra $\Gamma^p_{\mathbb{F}_p}$ is Morita equivalent to a direct product of $\left(\sum_{i=1}^{p-1} \rho_p(i)\right) - 1$ copies of $\mathbb{F}_p$ and of an indecomposable ring $\Gamma^p_{\mathbb{F}_p,0}$. By the recollement diagram preceding Lemma 3.3 this ring $\Gamma^p_{\mathbb{F}_p,0}$ has a projective module $P = \Gamma^p_{\mathbb{F}_p,0} \cdot e$ so that the endomorphism ring of P is Morita equivalent to the Brauer tree algebra corresponding to $\mathbb{F}_p\mathfrak{S}_p$. There is a projective indecomposable $\Gamma^p_{\mathbb{F}_p,0}$-module P_0 so that $P_0 \oplus P$ is a progenerator of $\Gamma^p_{\mathbb{F}_p,0}$ and the endomorphism ring of $P_0 \oplus P$ is basic and Morita equivalent to $\Gamma^p_{\mathbb{F}_p,0}$. Moreover, $\Gamma^p_{\mathbb{F}_p,0}/(\Gamma^p_{\mathbb{F}_p,0} \cdot e \cdot \Gamma^p_{\mathbb{F}_p,0})$ is Morita equivalent to $\mathbb{F}_p$.

The next remark constructs appropriate projective objects.

Let R and S be commutative rings. The functor $R[-]/(I^{n+1})$ which assigns to an S-module V the quotient of the semi-group ring $R[V]$ on V by the $n+1$-st power of the augmentation ideal. This functor $S - free \longrightarrow R - mod$ is polynomial of degree n. Define $proj^m_n := \mathbb{F}_p[Hom_{\mathbb{Z}}(\mathbb{Z}^m, -)]/(I^{n+1})$ and $proj^m_\infty := \mathbb{F}_p[Hom_{\mathbb{Z}}(\mathbb{Z}^m, -)]$.

Lemma 5.6. *The functor $proj^m_n$ in $\mathcal{A}^n_{\mathbb{F}_p}$ is projective and contains a projective cover of the reduction modulo p functor $\mathbb{F}_p \otimes_{\mathbb{Z}} id$.*

Proof. For any degree n polynomial functor F one gets $Hom_{\mathcal{A}_{\mathbb{F}_p}}(proj^m_n, F) \simeq F(\mathbb{Z}^m)$. This is true if one does not factorizes the power of the augmentation ideal, and since all functors are of degree at most n, each natural transformation from $proj^m_\infty$ to F is zero on I^{n+1} (see [12, Section 1]).

So, $Hom_{\mathcal{A}}(proj_n^m, -)$ is exact, as evaluation on exact sequences of functors is exact. Hence, $proj_n^m$ is a projective object in $\mathcal{A}^n_{\mathbb{F}_p}$ and since $Hom_{\mathcal{A}}(proj_n^m, id) = id(\mathbb{Z}^m) = \mathbb{Z}^m \neq 0$, the projective cover of the reduction modulo p functor is a direct summand in $proj_n^m$. ∎

Remark 5.7. The situation is different for $\mathcal{F}_{\mathbb{F}_p}$. Indeed, the functor $\mathbb{F}_p[Hom_{\mathbb{F}_p}(\mathbb{F}_p^m, -)]/I^{n+1}$ is projective in $\mathcal{F}^n_{\mathbb{F}_p}$ again since

$$Hom_{\mathcal{F}^n_{\mathbb{F}_p}}(\mathbb{F}_p[Hom_{\mathbb{F}_p}(\mathbb{F}_p^m, -)]/I^{n+1}, F) \simeq F(\mathbb{F}_p^m).$$

But, automatically $I^p = 0$ for $m = 1$ and the evaluation at $\mathbb{F}_p$ in this case, so that the endomorphism ring of $\mathbb{F}_p[Hom_{\mathbb{F}_p}(\mathbb{F}_p, -)]/I^{n+1}$ is an ℓ-dimensional vector space where $\ell = min(p, n)$.

Recall the embedding $\mathcal{F}^n_{\mathbb{F}_p} \longrightarrow \mathcal{A}^n_{\mathbb{F}_p}$ given by pre-composing with $\mathbb{F} \otimes_{\mathbb{Z}} id$. The image of $\mathbb{F}_p[Hom_{\mathbb{F}_p}(\mathbb{F}_p, -)]/I^{n+1}$ under this embedding is $\mathbb{F}_p[Hom_{\mathbb{F}_p}(\mathbb{F}_p, \mathbb{F}_p \otimes_{\mathbb{Z}} -)]/I^{n+1}$ which is different from $\mathbb{F}_p[Hom_{\mathbb{Z}}(\mathbb{Z}, -)]/I^{n+1}$. As we will see, the projective indecomposable cover of $\mathbb{F}_p \otimes_{\mathbb{Z}} id$ in $\mathcal{A}^p_{\mathbb{F}_p}$ is a direct factor of the functor $\mathbb{F}_p[Hom_{\mathbb{Z}}(\mathbb{Z}, -)]/I^{p+1}$ and this is the only indecomposable functor which is not in the image of the embedding $\mathcal{F}^p_{\mathbb{F}_p} \longrightarrow \mathcal{A}^p_{\mathbb{F}_p}$.

Lemma 5.8. *The projective functor $proj_n^1$ has one composition factor of degree d for each $0 \leq d \leq n$ for all $n \leq p-1$. In particular $proj_n^1$ contains a simple constant functor as a direct summand.*

Proof. First, $proj_n^1(0) = \mathbb{F}_p$, and so the simple functor of degree 0 is a direct factor of $proj_n^1$.

Furthermore, $proj_n^1 \twoheadrightarrow proj_{n-1}^1$ for trivial reasons. Moreover, $proj_n^1$ is of degree n and not of degree $n-1$. We compute that $End_{\mathcal{A}^n_{\mathbb{F}_p}}(proj_n^1) = proj_n^1(\mathbb{Z}) = \mathbb{F}_p[\mathbb{Z}]/I^{n+1}$ is an $n+1$-dimensional vector space. Moreover, by Lemma 1.6 and Remark 3.2 the projective module $proj_{p-1}^1$ is semisimple since $\mathcal{A}^{p-1}_{\mathbb{F}_p}$ is a semisimple category. Observe that $proj_n^1$ has exactly one composition factor more than $proj_{n-1}^1$. This composition factor is of degree n since

$$Hom_{\mathcal{A}^n_{\mathbb{F}_p}}(proj_n^1, \mathbb{F}_p \otimes_{\mathbb{Z}} \underbrace{id \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} id}_{n \text{ factors}}) = \mathbb{F}_p \otimes_{\mathbb{Z}} \mathbb{Z} \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} \mathbb{Z} = \mathbb{F}_p$$

and since $\mathbb{F}_p \otimes_{\mathbb{Z}} \underbrace{id \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} id}_{n \text{ factors}}$ is the projective object corresponding to degree n polynomial functors in the recollement diagram. ∎

Recall that L denotes the simple functor in $\mathcal{A}^p_{\mathbb{F}_p}$ mapping to the trivial $\mathbb{F}_p\mathfrak{S}_p$-module in the recollement diagram.

Proposition 5.9. *Suppose $p \geq 5$. The projective cover $P_{\mathbb{F}_p \otimes id}$ of $\mathbb{F}_p \otimes_{\mathbb{Z}} id$ in $\mathcal{A}^p_{\mathbb{F}_p}$ is uniserial with top and socle being $\mathbb{F}_p \otimes_{\mathbb{Z}} id$ and with $rad(P_{\mathbb{F}_p \otimes_{\mathbb{Z}} id})/soc(P_{\mathbb{F}_p \otimes_{\mathbb{Z}} id}) \simeq L$. Moreover,*

$$proj_p^1 \simeq S_0 \oplus P_{\mathbb{F}_p \otimes id} \oplus S_2 \oplus \cdots \oplus S_{p-1}$$

for simple functors S_i of degree i.

Proof. We shall divide the proof into various claims.

Claim 5.10. *No direct summand of the top of $proj_p^1$ is of degree p.*

Proof: We have $Hom_{\mathcal{A}^p_{\mathbb{F}_p}}(proj_p^1, (\mathbb{F}_p \otimes_{\mathbb{Z}} id)^{\otimes p}) = \mathbb{F}_p$. On the other hand, we know from the recollement diagram that $(\mathbb{F}_p \otimes_{\mathbb{Z}} id)^{\otimes p}$ is the projective cover of the simple modules coming from $\mathbb{F}_p\mathfrak{S}_p$, that is those of degree p. Now, each projective indecomposable module of $\mathbb{F}_p\mathfrak{S}_p$ has the property that the top of this module is isomorphic to the socle of this module and that the top and the socle of this projective indecomposable module are different. Hence, suppose a simple polynomial functor of degree p would be in the top of $proj_p^1$, then let Q be its projective cover in $\mathcal{A}^p_{\mathbb{F}_p}$. Further, Q is a direct summand of $(\mathbb{F}_p \otimes_{\mathbb{Z}} id)^{\otimes p}$. Since the top of $(\mathbb{F}_p \otimes_{\mathbb{Z}} id)^{\otimes p}$ is isomorphic to the socle of $(\mathbb{F}_p \otimes_{\mathbb{Z}} id)^{\otimes p}$, the above homomorphism space would be at least 2-dimensional, corresponding to the mapping of $proj_p^1$ on the top and on the socle of $(\mathbb{F}_p \otimes_{\mathbb{Z}} id)^{\otimes p}$. ∎

Lemma 5.11. *If $Ext^1_{\mathcal{A}^p_{\mathbb{F}_p}}(S,T) \neq 0$ for two simple functors S and T, then $deg(S) - deg(T) \in \{0, p-1\}$ and if $deg(S) = deg(T)$, then $deg(S) = p$.*

Proof. We know by Lemma 1.6 that $\mathcal{A}^{p-1}_{\mathbb{F}_p}$ is semisimple. Moreover, the category of constant functors is a direct factor in the category of polynomial functors. Using Lemma 3.6 this shows the statement. ∎

We denote by P_V the projective cover of the functor V in $\mathcal{A}^p_{\mathbb{F}_p}$.

Claim 5.12. *$proj_p^1 \simeq S_0 \oplus S_2 \oplus S_3 \oplus \cdots \oplus S_{p-1} \oplus M$ for simple projective functors S_i of degree i and the projective cover M of the functor $\mathbb{F}_p \otimes_{\mathbb{Z}} id$.*

Proof. $End_{\mathcal{A}^p_{\mathbb{F}_p}}(proj_p^1) = proj_p^1(\mathbb{Z}) = \mathbb{F}_p[\mathbb{Z}]/I^{p+1}$ is a $p+1$-dimensional $\mathbb{F}_p$-vector space. We know already that $proj^1_{p-1}$ is a quotient of $proj_p^1$ and that this is a semisimple functor with $p-1$ direct factors. So, every direct summand of the semisimple functor $proj^1_{p-1}$ is a direct factor of the head of $proj_p^1$. Denote by $S_0, S_1, S_2 \cdots S_{p-1}$ the simple direct factors of $proj^1_{p-1}$ and let S_i be of degree i. Then, since the degree 0 functors split off in any case, $P_{S_0} = S_0$. Moreover, $S_0 \oplus P_{S_0} \oplus P_{S_1} \oplus \cdots \oplus P_{S_{p-1}}$ is a direct factor of $proj_p^1$.

We need to study the functors P_{S_i}. If all the composition factors of P_{S_i} for an $i \leq p-1$ are of degree $p-1$ at most, then by Lemma 3.6 we get that P_{S_i} is of degree at most $p-1$ as well. Since the category $\mathcal{A}^{p-1}_{\mathbb{F}_p}$ is semisimple, we get $P_{S_i} = S_i$.

Suppose that a degree p simple polynomial functor S is a composition factor of $P_{S_{i_0}}$. Then, since $\mathcal{A}^{p-1}_{\mathbb{F}_p}$ is semisimple, again by Lemma 3.6, $Ext^1_{\mathcal{A}^p_{\mathbb{F}_p}}(S_{i_0}, S) \neq 0$. Now, simple functors are self-dual under the duality $(DF)(V) := F(V^*)^*$ (cf [23] for functors in $\mathcal{F}_{\mathbb{F}_p}$ and by Lemma 3.10 for simple functors in $\mathcal{A}_{\mathbb{F}_p}$). So, $Ext^1_{\mathcal{A}^p_{\mathbb{F}_p}}(S, S_{i_0}) \neq 0$. Since $Hom_{\mathcal{A}^p_{\mathbb{F}_p}}(proj_p^1, \mathbb{F}_p \otimes_{\mathbb{Z}} -)$ is a one-dimensional $\mathbb{F}_p$-vector space, this happens for precisely one $i_0 \in \{1, 2, \ldots, p-1\}$. We already know by Claim 5.3 that $i_0 = 1$ and S is the simple module corresponding to the trivial representation of $\mathbb{F}_p\mathfrak{S}_p$. This proves the claim. ∎

Claim 5.13. *For the projective cover P_L of L we get $rad(P_L)/rad^2(P_L) \simeq (\mathbb{F}_p \otimes id) \oplus L_2$, where L_2 is simple of degree p, $L_2 \not\simeq L$ and $rad^2(P_L) \simeq L$ as well as $rad^3(P_L) = 0$.*

Remark 5.14. We do not claim here that $soc(P_L)$ is simple. However, the radical layer structure of P_L can be described by

$$\begin{array}{ccc} & L & \\ (\mathbb{F}_p \otimes id) & & L_2 \\ & L & \end{array}$$

and where it is not clear if $soc(P_L)$ is simple and isomorphic to L or if the socle is isomorphic to $L \oplus L_2$ or to $L \oplus \mathbb{F}_p \otimes id$. But $Hom_{\mathcal{A}^p_{\mathbb{F}_p}}(\mathbb{F}_p \otimes id^{\otimes p}, P_L)$ is the projective cover of the trivial $\mathbb{F}_p\mathfrak{S}_p$-module. This projective cover is uniserial with composition series

$$\begin{array}{c} L \\ L_2 \\ L \end{array} .$$

Since L_2 is simple of degree p, its image in $\mathbb{F}_p\mathfrak{S}_p$ is given by the known module structure of $\mathbb{F}_p\mathfrak{S}_p$. In particular, $L \not\simeq L_2$. This shows that the only uniserial module of length 3 which is a quotient of P_L, if there is any, can have composition series

$$\begin{matrix} L \\ L_2 \\ L \end{matrix} .$$

In particular, $soc(P_L) \not\simeq L \oplus L_2$.

Proof of Claim 5.13. By Claim 5.3 we know that $(\mathbb{F}_p \otimes id)$ is composition factor of $top(rad(P_L))$ and since $Hom_{\mathcal{A}^p_{\mathbb{F}_p}}(proj^1_p, \mathbb{F}_p \otimes_{\mathbb{Z}} -)$ is a one-dimensional $\mathbb{F}_p$-vector space we know that it has multiplicity 1. Since the image of P_L in $\mathbb{F}_p\mathfrak{S}_p - mod$ is uniserial with top and socle L and simple $rad(P_L)/soc(P_L) \simeq L_2$, we have the above structure. ∎

Claim 5.15. *The projective cover $M = P_{\mathbb{F}_p \otimes id}$ of $\mathbb{F}_p \otimes_{\mathbb{Z}} id$ has $rad(M)/rad^2(M) \simeq L$.*

Proof. By Claim 5.3 we know that L is a direct factor of $top(rad(M))$. Since $\mathcal{A}^{p-1}_{\mathbb{F}_p}$ is semisimple, using Lemma 3.6 we see that no simple functor of degree $p-1$ at most can be a direct factor of $top(rad(M))$. Suppose $top(rad(M))$ has a second simple direct factor T of degree p. Then, $Ext^1_{\mathcal{A}^p_{\mathbb{F}_p}}(\mathbb{F}_p \otimes_{\mathbb{Z}} id, T) \neq 0$. Simple functors are self-dual (cf as above [23] for functors in $\mathcal{F}_{\mathbb{F}_p}$ and by Lemma 3.10 for simple functors in $\mathcal{A}_{\mathbb{F}_p}$). So, $Ext^1_{\mathcal{A}^p_{\mathbb{F}_p}}(T, \mathbb{F}_p \otimes_{\mathbb{Z}} id) \neq 0$. But, we have seen in Corollary 5.3 that there is one simple functor of degree p with a non trivial extension group with $\mathbb{F}_p \otimes_{\mathbb{Z}} id$, namely L. Moreover, since $proj^1_p$ contains the projective cover of the simple degree 1-functor as a direct factor (see Claim 5.12), and since by Lemma 5.11 this is the only degree where non trivial first extension groups can occur, we see that $T \simeq L$.

Suppose $L \oplus L | rad(M)$. Then, $Ext^1_{\mathcal{A}^p_{\mathbb{F}_p}}(\mathbb{F}_p \otimes_{\mathbb{Z}} id, L)$ is two-dimensional at least, and again by the self-duality of the simple functors, $Ext^1_{\mathcal{A}^p_{\mathbb{F}_p}}(L, \mathbb{F}_p \otimes_{\mathbb{Z}} id)$ is at least two-dimensional. Therefore, $\mathbb{F}_p \otimes_{\mathbb{Z}} id$ occurs twice in $top(rad(P_L))$. Since M is a direct factor of $proj^1_p$, the space $Hom_{\mathcal{A}^p_{\mathbb{F}_p}}(proj^1_p, (\mathbb{F}_p \otimes_{\mathbb{Z}} id)^{\otimes p})$ would be two-dimensional at least. This contradiction shows that $rad(M)$ has simple top L. Hence, $top(rad(M)) \simeq L$. ∎

Remark 5.16. The radical layer structure of M is therefore given by

$$\begin{matrix} (\mathbb{F}_p \otimes id) \\ L \\ rad^2(M) \end{matrix}$$

and by Claim 5.13 we get $rad^3(P_L) = 0$, and therefore we obtain $rad^4(M) = 0$.

Claim 5.17. *For the projective cover M of $\mathbb{F}_p \otimes_{\mathbb{Z}} id$ we get that $top(rad^2(M)) \simeq \mathbb{F}_p \otimes_{\mathbb{Z}} id$.*

Proof. Degree p functors can only have extensions with degree p-functors or degree 1-functors by Lemma 5.11. Moreover, the structure of P_L implies that we get that $top(rad^2(M))$ is a direct summand of $top(rad(P_L))$, whence is isomorphic to either 0, or to L_2 (which is defined in Claim 5.13), or to $\mathbb{F}_p \otimes_{\mathbb{Z}} id$, or to $L_2 \oplus (\mathbb{F}_p \otimes_{\mathbb{Z}} id)$.

Suppose $L_2 \oplus (\mathbb{F}_p \otimes_{\mathbb{Z}} id) \simeq top(rad^2(M))$. We shall use the fact that by Claim 5.13 we know the structure of P_L.

We get two possibilities for the projective resolution of $\mathbb{F}_p \otimes_{\mathbb{Z}} id$. Either

$$P_L \hookrightarrow M \longrightarrow\!\!\!\!\!\rightarrow (\mathbb{F}_p \otimes_{\mathbb{Z}} id)$$

is exact, or

$$L \hookrightarrow P_L \longrightarrow M \longrightarrow\!\!\!\!\!\rightarrow (\mathbb{F}_p \otimes_{\mathbb{Z}} id)$$

is exact.

In the second case, $Ext^2_{\mathcal{A}^p_{\mathbb{F}_p}}((\mathbb{F}_p \otimes_{\mathbb{Z}} id), L) \neq 0$. By the self-duality of the simple functors, $Ext^2_{\mathcal{A}^p_{\mathbb{F}_p}}(L, (\mathbb{F}_p \otimes_{\mathbb{Z}} id)) \neq 0$. Our information is sufficient for being able to write down the first terms of the projective resolution of L,

$$0 \longleftarrow L \longleftarrow P_L \longleftarrow M \oplus P_{L_2} \longleftarrow P_L \oplus P_{L_3} \longleftarrow \dots$$

for some projective P_{L_3}, for some simple object L_3 of degree p, given by the known projective resolution of the trivial $\mathbb{F}_p\mathfrak{S}_p$-module. Since $p \neq 2$, we get $L_3 \not\simeq L$. In an case $Hom(P_L \oplus P_{L_3}, (\mathbb{F}_p \otimes_{\mathbb{Z}} id)) = 0$, and therefore $Ext^2_{\mathcal{A}^p_{\mathbb{F}_p}}((\mathbb{F}_p \otimes_{\mathbb{Z}} id), L) = 0$. This contradiction excludes the case

$$L \hookrightarrow P_L \longrightarrow M \longrightarrow\!\!\!\!\!\rightarrow (\mathbb{F}_p \otimes_{\mathbb{Z}} id)$$

is exact.

If $P_L \hookrightarrow M \longrightarrow\!\!\!\!\!\rightarrow (\mathbb{F}_p \otimes_{\mathbb{Z}} id)$ is exact, the projective dimension of $(\mathbb{F}_p \otimes_{\mathbb{Z}} id)$ is 1. But, we know by Theorem 3 and the example following it, that $Ext^2_{\mathcal{F}^p_{\mathbb{F}_p}}(id, id) \neq 0$. By consequence, also $Ext^2_{\mathcal{A}^p_{\mathbb{F}_p}}((\mathbb{F}_p \otimes_{\mathbb{Z}} id), (\mathbb{F}_p \otimes_{\mathbb{Z}} id)) \neq 0$ and therefore the projective dimension of $(\mathbb{F}_p \otimes_{\mathbb{Z}} id)$ is at least 2.

These two observations exclude $L_2 \oplus (\mathbb{F}_p \otimes_{\mathbb{Z}} id) \simeq top(rad^2(M))$.

Suppose $L_2 \simeq top(rad^2(M))$**.** Then, using the structure of P_L, we get either $rad^3(M) = 0$ or $rad^3(M) = L$, and then $rad^4(M) = 0$.

If $rad^3(M) = 0$, then we get a non split exact sequence

$$0 \longleftarrow (\mathbb{F}_p \otimes_{\mathbb{Z}} id) \longleftarrow M \longleftarrow P_L \longleftarrow M \longleftarrow L_2 \longleftarrow 0$$

and therefore $Ext^3_{\mathcal{F}^p_{\mathbb{F}_p}}(L_2, (\mathbb{F}_p \otimes_{\mathbb{Z}} id)) \neq 0$. Dualizing, $Ext^3_{\mathcal{F}^p_{\mathbb{F}_p}}((\mathbb{F}_p \otimes_{\mathbb{Z}} id), L_2) \neq 0$. Our knowledge of the various projective covers of simples is sufficient to write down the first terms of the projective resolution of L_2. We get

$$0 \longleftarrow L_2 \longleftarrow P_{L_2} \longleftarrow P_L \oplus P_{L_3} \longleftarrow M \oplus P_{L_2} \oplus P_{L_4} \longleftarrow P_L \oplus P_{L_3} \oplus P_{L_5} \longleftarrow \dots$$

for projective objects P_{L_4} and P_{L_5} corresponding to degree p simple functors L_4 and L_5, given by the known projective resolution of the trivial $\mathbb{F}_p\mathfrak{S}_p$-module. Moreover, L_3 and L_5 are both different from L_2, since $p \geq 5$. This implies $Ext^3_{\mathcal{F}^p_{\mathbb{F}_p}}((\mathbb{F}_p \otimes_{\mathbb{Z}} id), L_2) = 0$. This contradiction excludes this case as well.

So, assume $rad^3(M) = L$ and $rad^4(M) = 0$. This is impossible since then the endomorphism ring of M would be one-dimensional. This contradicts the fact that $End_{\mathcal{A}^p_{\mathbb{F}_p}}(proj^1_p)$ is $p+1$-dimensional.

Hence, $L_2 \not\simeq top(rad^2(M))$.

We still have the possibility that $rad^2(M) = 0$**.** But again, this would imply that $End_{\mathcal{A}^p_{\mathbb{F}_p}}(M)$ would be one-dimensional and therefore $End_{\mathcal{A}^p_{\mathbb{F}_p}}(proj^1_p)$ is p-dimensional. Contradiction.

This proves the claim. ∎

Claim 5.18. $rad^3(M) = 0$.

Proof. We know by Claim 5.13 that $rad^3(P_L) = 0$. Since by Claim 5.15 we have $top(rad(M)) \simeq L$, one sees that $rad^4(M) = 0$. Moreover, $rad^3(M)$ is either 0 or L, since $top(rad^2(M)) \simeq (\mathbb{F}_p \otimes_{\mathbb{Z}} id)$ by Claim 5.17 and $top(rad(M)) \simeq L$ by Claim 5.15.

Suppose $rad^3(M) \simeq L$. Then M is uniserial with composition length 4, and $top(rad(M)) = L$. Therefore P_L maps onto $rad(M)$ with image being a uniserial module N of length 3 with $rad(N)/soc(N) = \mathbb{F}_p \otimes id$.

But this contradicts the structure of P_L as described in Claim 5.13 and in particular Remark 5.14.

This proves the claim. ∎

Examining what we showed implies that $proj_p^1$ is as stated in Proposition 5.9. ∎

We now come to our first main result in describing the structure of $\Gamma_{\mathbb{F}_p,0}$. For the relevant definitions on Brauer tree algebras we refer to Section 4.

Theorem 4. *$\Gamma_{\mathbb{F}_p,0}$ is a Brauer tree algebra over $\mathbb{F}_p$ without exceptional vertex and associated to a stem with p edges.*

$$\bullet_1 - \bullet_2 - \bullet_3 - \cdots - \bullet_{p+1}$$

Proof. The case $p \leq 3$ is a consequence of Drozd's results. By Proposition 5.9 we know that the projective cover of the functor $\mathbb{F}_p \otimes_{\mathbb{Z}} -$ is uniserial with top and socle $\mathbb{F}_p \otimes_{\mathbb{Z}} -$ and with second layer V, where V is the simple functor corresponding to the trivial $\mathbb{F}_p\mathfrak{S}_p$-module.

We know furthermore that except the projective cover of $\mathbb{F}_p \otimes_{\mathbb{Z}} -$ only the projective indecomposable functor P_L has a composition factor $\mathbb{F}_p \otimes_{\mathbb{Z}} -$ and that this composition factor is a direct summand of $top(rad(P_L))$.

Since we know that the principal block of $\mathbb{F}_p\mathfrak{S}_p$ is a Brauer tree algebra without exceptional vertex associated to a stem with p vertices, this means that the we only need to show that $\mathbb{F}_p \otimes_{\mathbb{Z}} -$ is not in the socle of P_L, since the only basic algebra with the composition series as a Brauer tree algebra associated to a stem is actually a Brauer tree algebra associated to a stem.

For this we use the duality D on the category of polynomial functors. The projective indecomposable functor P_L is a direct factor of $\mathbb{F}_p \otimes_{\mathbb{Z}} \underbrace{id \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} id}_{p \text{ factors}}$, since this is the projective cover of all the degree p simple functors.

It is clear that $\mathbb{F}_p \otimes_{\mathbb{Z}} \underbrace{id \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} id}_{p \text{ factors}}$ is self dual. Since all the simple functors are self-dual, also $DP_L \simeq P_L$. If $\mathbb{F}_p \otimes_{\mathbb{Z}} -$ is in the socle of P_L, the simple functor $D(\mathbb{F}_p \otimes_{\mathbb{Z}} -) \simeq \mathbb{F}_p \otimes_{\mathbb{Z}} -$ is in the top of $DP_L \simeq P_L$, but the top of P_L is L by definition.

This proves the Theorem. ∎

6. Lifting to characteristic 0

6.1. Lifting Brauer tree algebras to orders.

Proposition 6.1. *Let R be a complete discrete valuation ring with residue field k and field of fractions K. Let B be a Brauer tree algebra over k associated to a Brauer tree which is a stem without exceptional vertex. Let Λ be an R-order. Then, for any proper two-sided ideal $I \neq 0$ of B we get that*

$$\Lambda \otimes_R k \simeq B/I \Longrightarrow rank_{\mathbb{Z}}(K_0(K \otimes_R \Lambda)) \leq rank_{\mathbb{Z}}(K_0(B/I))$$

Proof. We shall first suppose that Λ is indecomposable and that $I \leq rad(B)$.

Let $S_1, S_2, \ldots, S_n$ be representatives of the simple B-modules. The projective cover P_i of S_i has then a composition series where $soc(P_i) \simeq S_i$ and

$$rad(P_i)/soc(P_i) \simeq S_{i-1} \oplus S_{i+1}$$

for all $i \in \{2, 3 \ldots, n-1\}$,

$$rad(P_1)/soc(P_1) \simeq S_2$$

and

$$rad(P_n)/soc(P_n) \simeq S_{n-1}.$$

Denote $\overline{B} := B/I$. Since $I \leq rad(B)$, we get $\overline{B}$ has the same number of simple modules, and moreover, the simple $\overline{B}$-modules and the simple B-modules coincide by the epimorphism $B \longrightarrow \overline{B}$. Therefore the projective indecomposable $\overline{B}$-modules are $\overline{P}_i := \overline{B} \otimes_B P_i$ for $i \in \{1, 2, \ldots, n\}$. Moreover, $\overline{P}_i$ is the projective cover of S_i as $\overline{B}$-module.

Extending k if necessary, we may assume that the field of fractions K of R is a splitting field for Λ, since extending K does not decrease the rank of the Grothendieck group, using the Noether-Deuring theorem. Since k is a splitting field for B and for B/I, and since $k \otimes_R \Lambda \simeq \overline{B}$, the Cartan matrix of $\overline{B}$ is symmetric (cf e.g. [21, Proposition 4.2.11]). Since the Cartan matrix of B equals

$$C := \begin{pmatrix} 2 & 1 & 0 & \dots & \dots & 0 \\ 1 & 2 & 1 & \ddots & & \vdots \\ 0 & 1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 1 & 0 \\ \vdots & & \ddots & 1 & 2 & 1 \\ 0 & \dots & \dots & 0 & 1 & 2 \end{pmatrix},$$

we see that the composition length of $\overline{P}_i$ differs from the composition length of P_i by at most 1. Indeed, if this would not be the case, then the composition matrix of $\overline{B}$ would be decomposable into at least two blocks and $\overline{B}$ would be decomposable as algebra. But, since Λ is indecomposable, so is $k \otimes_R \Lambda \simeq \overline{B}$. So, $I \leq soc(B)$.

Since R is complete, we may assume that B and Λ are both basic algebras.

Let Q_i be the projective cover of $\overline{P}_i$ as Λ-module. Hence, $k \otimes_R Q_i \simeq \overline{P}_i$. Since

$$dim_K Hom_{K\Lambda}(KQ_i, KQ_j) = dim_R Hom_\Lambda(Q_i, Q_j) = dim_k Hom_{\overline{B}}(\overline{P}_i, \overline{P}_j)$$

we know that KQ_i and KQ_j do not have a character in common if $|i - j| > 1$ and do have one character in common if $|i - j| = 1$. Since $dim_k Hom_{\overline{B}}(\overline{P}_i, \overline{P}_i) \in \{1, 2\}$, the character of KQ_i for all $i \in \{1, \dots, n\}$ is a sum of at most two irreducible characters, and in case of two characters these are non isomorphic. Now, since $Hom_{K\Lambda}(KQ_i, KQ_j) = 0$ if $|i - j| > 1$, it follows that if

$$dim_K Hom_{K\Lambda}(KQ_{i_0}, KQ_{i_0}) = 1,$$

then $i_0 \in \{1, n\}$. Otherwise, the character of KQ_{i_0} would be a constituent of KQ_{i_0+1}, of KQ_{i_0} and of KQ_{i_0-1}, which implies then $Hom_{K\Lambda}(KQ_{i_0+1}, KQ_{i_0-1}) \neq 0$. This would give a contradiction. This gives that I equals either S_1 or S_n or $S_1 \oplus S_n$.

Suppose now that S_1 is a direct factor of I (as left module) and suppose

$$rank_{\mathbb{Z}}(K_0(K \otimes_R \Lambda)) > rank_{\mathbb{Z}}(K_0(B)).$$

We shall prove that S_1 is not a direct factor of I. By symmetry, then neither S_n is a direct factor of I, and therefore, $I = 0$.

Under these hypotheses, $dim(\overline{P}_1) = 2$ and as a consequence also $dim_R(Q_1) = 2$. So, for the Wedderburn components corresponding to $K \otimes_R Q_1$ in $K \otimes_R \Lambda$ we have two possibilities. Either $K \otimes_R Q_1$ is a sum of two one-dimensional characters or $K \otimes_R Q_1$ is isomorphic to one two-dimensional character. Since $K \otimes_R \Lambda$ admits at least $n+1$ irreducible characters, $K \otimes_R Q_1$ must have two constituents. So, $K \otimes_R Q_1$ is a sum of two one-dimensional characters.

But now, let $\{e_i | i \in \{1, \dots, n\}\}$ be an orthogonal set of primitive idempotents with $\Lambda e_i \simeq Q_i$. Then, since e_1 and e_2 must be non zero on the common Wedderburn component of KQ_1 and KQ_2, we get $e_2 e_1 \neq 0$. This is a contradiction to the fact that e_1 and e_2 are orthogonal.

We have to deal with the case Λ being decomposable. The structure of B implies that in this case, B/I is a direct product of algebras we have dealt with in the earlier case, and copies of k. By induction on the number of simple modules of each indecomposable factor the result holds for each of the pieces as well. Summing up for all of these pieces, we get the desired result.

Finally, we have to deal with the case that I is not contained in $rad(B)$. The same argument as for Λ decomposable applies here as well.

This proves the Proposition. ■

6.2. **Proving that the Baues-Dreckmann-Franjou-Pirashvili ring is an order.** Since $\hat{\mathbb{Z}}_p$ is a complete discrete valuation ring, we may lift idempotents from $\Gamma^p_{\mathbb{F}_p}$ to $\Gamma^p_{\hat{\mathbb{Z}}_p}$. Hence, there is an indecomposable direct factor $\Gamma^p_{\hat{\mathbb{Z}}_p,0}$ of the rank one free module $\Gamma^p_{\hat{\mathbb{Z}}_p}$ which maps surjectively to $\Gamma^p_{\mathbb{F}_p,0}$. Let $T^p_0 := t(\Gamma^p_{\hat{\mathbb{Z}}_p,0})$ be the torsion ideal in $\Gamma^p_{\hat{\mathbb{Z}}_p,0}$ and define $\Lambda^p_{\hat{\mathbb{Z}}_p,0} := \Gamma^p_{\hat{\mathbb{Z}}_p,0}/T^p_0$.

Proposition 6.2. *$\Lambda^p_{\hat{\mathbb{Z}}_p,0}$ is an order. Moreover, $\hat{\mathbb{Q}}_p \otimes_{\hat{\mathbb{Z}}_p} \Lambda^p_{\hat{\mathbb{Z}}_p,0}$ is a direct product of $p+1$ matrix rings over $\hat{\mathbb{Q}}_p$ and up to isomorphism there are at most p simple $\Lambda^p_{\hat{\mathbb{Z}}_p,0}$-modules.*

Proof. In fact, $\hat{\mathbb{Q}}_p \otimes_{\hat{\mathbb{Z}}_p} \Lambda^p_{\hat{\mathbb{Z}}_p} = \hat{\mathbb{Q}}_p \otimes_{\hat{\mathbb{Z}}_p} \Gamma^p_{\hat{\mathbb{Z}}_p}$ and their common module categories are equivalent to the category of polynomial functors $\hat{Q}_p - mod \longrightarrow \hat{Q}_p - mod$ of degree at most p by Lemma 1.3.

By Friedlander-Suslin [16], the category of exact degree n polynomial functors $\hat{Q}_p - mod \longrightarrow \hat{Q}_p - mod$ is equivalent to the category of strict polynomial functors $\hat{Q}_p - mod \longrightarrow \hat{Q}_p - mod$ and this category is equivalent to the category of modules over the Schur algebra $S_{\hat{\mathbb{Q}}_p}(n,n)$. Moreover, the category of strict polynomial functors of degree at most n is equivalent to the direct sum of the category of strict polynomial functors of exact degree m for each $m \in \{0,1,\dots,n\}$. The Schur algebra $S_{\hat{\mathbb{Q}}_p}(p,p)$ is split semisimple (cf Green [17]) with exactly $p+1$ simple modules. This shows that $\hat{\mathbb{Q}}_p \otimes_{\hat{\mathbb{Z}}_p} \Lambda^p_{\hat{\mathbb{Z}}_p,0}$ is a direct product of $p+1$ full matrix rings over $\hat{\mathbb{Q}}_p$. Moreover, this shows also that $\Lambda^p_{\hat{\mathbb{Z}}_p}$ is an order since it is by definition torsion free and contains a basis of the semisimple algebra $\hat{\mathbb{Q}}_p \otimes_{\hat{\mathbb{Z}}_p} \Lambda^p_{\hat{\mathbb{Z}}_p}$ (which is Morita equivalent to $\prod_{i=0}^p S_{\hat{\mathbb{Q}}_p}(i,i)$).

In order to prove the second statement we just observe that the number of simple objects in $\mathcal{A}^p_{\mathbb{F}_p}$ equals the number of simple objects in $\mathcal{F}^p_{\mathbb{F}_p}$ by Lemma 3.10. Moreover, since $\Lambda^p_{\hat{\mathbb{Z}}_p,0}$ is a quotient of $\Gamma^p_{\hat{\mathbb{Z}}_p,0}$, every simple $\Lambda^p_{\hat{\mathbb{Z}}_p,0}$-module induces a simple $\Gamma^p_{\hat{\mathbb{Z}}_p,0}$-module. We know that $\Gamma^p_{\mathbb{F}_p,0}$ is a Brauer tree algebra with p simple modules. Moreover, $\mathbb{F}_p \otimes_{\hat{\mathbb{Z}}_p} \Gamma^p_{\hat{\mathbb{Z}}_p,0} \simeq \Gamma^p_{\mathbb{F}_p,0}$ and so, $\Gamma^p_{\hat{\mathbb{Z}}_p,0}$ admits p simple modules. As a consequence $\Lambda^p_{\hat{\mathbb{Z}}_p,0}$ admits at most p simple modules. This proves the proposition. ∎

Proposition 6.3. *$t(\Gamma^p_{\mathbb{F}_p,0}) = 0$ and therefore $\Lambda^p_{\hat{\mathbb{Z}}_p,0} = \Gamma^p_{\hat{\mathbb{Z}}_p,0}$.*

Proof. This is a consequence of Proposition 6.2, Theorem 4 and Proposition 6.1.

Indeed, since $\mathbb{F}_p \otimes_{\hat{\mathbb{Z}}_p} -$ is right exact, the epimorphism

$$\Gamma^p_{\hat{\mathbb{Z}}_p,0} \longrightarrow \Lambda^p_{\hat{\mathbb{Z}}_p,0}$$

induces an epimorphism

$$B = \Gamma^p_{\mathbb{F}_p,0} \longrightarrow \Lambda^p_{\hat{\mathbb{Z}}_p,0} \otimes_{\hat{\mathbb{Z}}_p} \mathbb{F}_p$$

with kernel I, for B being a Brauer tree algebra associated to a stem with p edges and without exceptional vertex (Theorem 4). Since $t(\Gamma^p_{\hat{\mathbb{Z}}_p,0}) \subseteq rad(\Gamma^p_{\mathbb{F}_p,0})$ by Proposition 6.2, $I \leq rad(B)$. Since $B/I \simeq \Lambda^p_{\hat{\mathbb{Z}}_p,0} \otimes_{\hat{\mathbb{Z}}_p} \mathbb{F}_p$ for an order $\Lambda^p_{\hat{\mathbb{Z}}_p,0}$, Proposition 6.1 implies that in this case $I = 0$. Hence, $\mathbb{F}_p \otimes_{\hat{\mathbb{Z}}_p} t(\Gamma^p_{\hat{\mathbb{Z}}_p,0}) = 0$ and therefore, $t(\Gamma^p_{\hat{\mathbb{Z}}_p,0}) = 0$. This proves the proposition. ∎

6.3. **Describing the order; the main result.** We shall describe $\Lambda^p_{\hat{\mathbb{Z}}_p,0}$ and prove our main result. For this purpose we introduce some notation (cf [21, Section 4.4]). Let

$$\hat{\mathbb{Z}}_p \overset{p^i}{\text{———}} \hat{\mathbb{Z}}_p \quad := \quad \{(a,b) \in \hat{\mathbb{Z}}_p \times \hat{\mathbb{Z}}_p |\ a - b \in p^i \hat{\mathbb{Z}}_p\}$$

and

$$\hat{\mathbb{Z}}_p \frac{\qquad}{} \hat{\mathbb{Z}}_p \quad := \quad \hat{\mathbb{Z}}_p \overset{p}{\frac{\qquad}{}} \hat{\mathbb{Z}}_p \ .$$

The following is the main result of our paper.

Theorem 5. *Let $\mathcal{A}^p_{\hat{\mathbb{Z}}_p}$ be the category of at polynomial functors from free abelian groups to $\hat{\mathbb{Z}}_p$-modules and of degree at most p. Then, $\mathcal{A}^p_{\hat{\mathbb{Z}}_p}$ is equivalent to $\Gamma^p_{\hat{\mathbb{Z}}_p}$-mod, where*

$$\Gamma^p_{\hat{\mathbb{Z}}_p} := (\prod_{1<n<p} \hat{\mathbb{Z}}_p) \times (\prod_{\lambda \vdash p \text{ and } \lambda \text{ not a hook}} \hat{\mathbb{Z}}_p) \times \Lambda^p_{\hat{\mathbb{Z}}_p,0}$$

and where

$$\Lambda^p_{\hat{\mathbb{Z}}_p,0} \simeq \hat{\mathbb{Z}}_p \oplus \begin{pmatrix} \hat{\mathbb{Z}}_p & \hat{\mathbb{Z}}_p \\ (p) & \hat{\mathbb{Z}}_p \end{pmatrix} \oplus \begin{pmatrix} \hat{\mathbb{Z}}_p & \hat{\mathbb{Z}}_p \\ (p) & \hat{\mathbb{Z}}_p \end{pmatrix} \oplus \dots \oplus \begin{pmatrix} \hat{\mathbb{Z}}_p & \hat{\mathbb{Z}}_p \\ (p) & \hat{\mathbb{Z}}_p \end{pmatrix} \oplus \hat{\mathbb{Z}}_p$$

$$= \left\{ (d_0) \times \left(\prod_{j=1}^{p-1} \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix} \right) \times (a_p) \;\middle|\; \forall j : a_j, b_j, c_j, d_j \in \hat{\mathbb{Z}}_p; p|c_j; p|(d_j - a_{j-1}) \right\}$$

is a Green order with p isomorphism classes of indecomposable projective modules.

Remark 6.4. Roggenkamp described the orders Λ which admit a set of lattices with periodic projective resolutions encoded by a Brauer tree ([34], see also [21]). Roggenkamp called these orders Green-orders and he described their structure in great detail.

Proof of the theorem. The case $p \leq 3$ was done by Drozd. Hence we may suppose that $p \geq 5$. Since $\mathbb{F}_p \otimes_{\hat{\mathbb{Z}}_p} \Lambda^p_{\hat{\mathbb{Z}}_p,0}$ is a Brauer tree algebra, there is a set of $\mathbb{F}_p \otimes_{\hat{\mathbb{Z}}_p} \Lambda^p_{\hat{\mathbb{Z}}_p,0}$-modules having a periodic projective resolution given by the Brauer tree of $\Lambda^p_{\mathbb{F}_p,0}$. Lifting these projective resolutions to the order $\Lambda^p_{\hat{\mathbb{Z}}_p,0}$ gives a periodic projective resolution of certain $\Lambda^p_{\hat{\mathbb{Z}}_p,0}$-modules M_i. These periodic resolutions are encoded by the same Brauer tree. It remains to show that the modules M_i are lattices. Actually, this is automatic. Indeed, since the resolution is periodic, each module M_i is also a kernel of a differential, after a complete period of the periodic projective resolution. Hence, $\Lambda^p_{\hat{\mathbb{Z}}_p,0}$ is a Green order with Brauer tree being a stem with p edges and without exceptional vertex.

We have to show that the maximal overorder of the Green order $\Lambda^p_{\hat{\mathbb{Z}}_p,0}$ is a direct product of matrix rings over $\hat{\mathbb{Z}}_p$ and that the image of $\Lambda^p_{\hat{\mathbb{Z}}_p,0}$ in each of the matrix rings is a hereditary order.

The first part is clear since $\hat{\mathbb{Q}}_p$ is a splitting field of $\Lambda^p_{\hat{\mathbb{Z}}_p,0}$, and $\Lambda^p_{\hat{\mathbb{Z}}_p,0}$ can be embedded into a direct product of matrix rings over the ring of integers in $\hat{\mathbb{Q}}_p$ (see e.g. [33]). Let $e_1, e_2, \dots, e_{p+1}$ be a complete set of primitive pairwise orthogonal idempotents of the center of $\hat{\mathbb{Q}}_p \otimes_{\hat{\mathbb{Z}}_p} \Lambda^p_{\hat{\mathbb{Z}}_p,0}$. Then,

$$\prod_{j=1}^{p-1} (\Lambda^p_{\hat{\mathbb{Z}}_p,0} \cdot e_i) \simeq \hat{\mathbb{Z}}_p \times \prod_{j=1}^{p-1} \begin{pmatrix} \hat{\mathbb{Z}}_p & \hat{\mathbb{Z}}_p \\ (p^{x_j}) & \hat{\mathbb{Z}}_p \end{pmatrix} \times \hat{\mathbb{Z}}_p$$

for some $x_j \in \mathbb{N} \setminus \{0\}$. Moreover, since $\Lambda^p_{\hat{\mathbb{Z}}_p,0} \otimes_{\hat{\mathbb{Z}}_p} \mathbb{F}_p$ is a Brauer tree algebra without exceptional vertex, $x_1 = x_2 = \dots = x_{p-1}$ and as a consequence, if one of the matrix rings is hereditary, all of them are hereditary. The structure theory of Green orders (cf Roggenkamp [34]; see also [21, Section 4.4]) and of hereditary orders, (cf e.g. Reiner [33]) then gives the statement.

Define a functor

$$Hom_{\mathcal{A}^p_{\hat{\mathbb{Z}}_p}}\left(\hat{\mathbb{Z}}_p \otimes_{\mathbb{Z}} (\bigotimes_{j=1}^{p} id), -\right) : \mathcal{A}^p_{\hat{\mathbb{Z}}_p} \longrightarrow \hat{\mathbb{Z}}_p\mathfrak{S}_p - mod$$

where we use again that the functor $\hat{\mathbb{Z}}_p \otimes_{\mathbb{Z}} (\bigotimes_{j=1}^p id) : \mathbb{Z}-free \longrightarrow \hat{\mathbb{Z}}_p - mod$ carries a natural $\hat{\mathbb{Z}}_p$-linear $\mathfrak{S}_p$ action. Denote for notational simplicity $E := Hom_{\mathcal{A}^p_{\hat{\mathbb{Z}}_p}}\left(\hat{\mathbb{Z}}_p \otimes_{\mathbb{Z}} (\bigotimes_{j=1}^p id), -\right)$. By Lemma 3.7 this functor is just the $p-1$-th cross effect, and by Lemma 3.6 this functor E is exact.

Since E is exact, and since by definition E is represented by $P := \hat{\mathbb{Z}}_p \otimes_{\mathbb{Z}} (\bigotimes_{j=1}^p id)$, this object P is projective. Let e be an idempotent in $\Gamma^p_{\hat{\mathbb{Z}}_p}$ which corresponds to the projective indecomposable $\Gamma^p_{\hat{\mathbb{Z}}_p}$-modules which occur in P. Then, replacing $\mathcal{A}^p_{\hat{\mathbb{Z}}_p}$ by $\Gamma^p_{\hat{\mathbb{Z}}_p} - mod$ the functor $Hom_{\mathcal{A}^p_{\hat{\mathbb{Z}}_p}}\left(\hat{\mathbb{Z}}_p \otimes_{\mathbb{Z}} (\bigotimes_{j=1}^p id), -\right)$ becomes the functor $E : \Gamma^p_{\hat{\mathbb{Z}}_p} - mod \longrightarrow \hat{\mathbb{Z}}_p\mathfrak{S}_p - mod$ and E is just multiplication by e.

We need to show that $End_{\mathcal{A}_{\hat{\mathbb{Z}}_p}}(P) \simeq \hat{\mathbb{Z}}_p\mathfrak{S}_p$, where the action is given by permutation of components in the tensor product. Once this is done, we know that $End_{\mathcal{A}_{\hat{\mathbb{Z}}_p}}(P) \simeq e \cdot \Gamma^p_{\hat{\mathbb{Z}}_p} \cdot e \simeq \hat{\mathbb{Z}}_p\mathfrak{S}_p$ and we observe that for all idempotents in $\Gamma^p_{\hat{\mathbb{Z}}_p}$ we get that this product $e \cdot \Gamma^p_{\hat{\mathbb{Z}}_p} \cdot e$ is again a product of Green orders with the same order of congruences. Since $\hat{\mathbb{Z}}_p\mathfrak{S}_p$ is a Green order with congruences modulo p only, we get that $x = 1$.

Claim 6.5. $End_{\mathcal{A}_{\hat{\mathbb{Z}}_p}}(\hat{\mathbb{Z}}_p \otimes_{\mathbb{Z}} (\bigotimes_{j=1}^p id)) \simeq \hat{\mathbb{Z}}_p\mathfrak{S}_p$

Proof. The proof given by Piriou-Schwartz [31, Lemma 1.9] of the corresponding statement for $\mathcal{F}_{\mathbb{F}_p}$ carries over literally. For the reader's convenience we recall the (short) arguments.

Given an $x = \sum_{\sigma \in \mathfrak{S}_p} x_\sigma \sigma \in \hat{\mathbb{Z}}_p\mathfrak{S}_p$, then associate to this x the natural transformation η_x in $End_{\mathcal{A}_{\hat{\mathbb{Z}}_p}}\left(\bigotimes_{j=1}^p \left(\hat{\mathbb{Z}}_p \otimes_{\mathbb{Z}} id\right)\right)$ given by

$$v_1 \otimes v_2 \otimes \cdots \otimes v_n \mapsto \sum_{\sigma \in \mathfrak{S}_p} x_\sigma (v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}).$$

Inversely, a natural transformation η in $End_{\mathcal{A}_{\hat{\mathbb{Z}}_p}}\left(\bigotimes_{j=1}^p \left(\hat{\mathbb{Z}}_p \otimes_{\mathbb{Z}} id\right)\right)$ is determined by its value on $\mathbb{Z}^n$. Fix a basis $\{e_1, \ldots, e_n\}$ of $\mathbb{Z}^n$. The image of $e_1 \otimes e_2 \otimes \cdots \otimes e_n$ under $\eta_{\mathbb{Z}^n}$ can be uniquely written as $x_\eta \cdot (e_1 \otimes e_2 \otimes \cdots \otimes e_n)$ for an $x_\eta \in \hat{\mathbb{Z}}_p\mathfrak{S}_n$. The two mappings $x \mapsto \eta_x$ and $\eta \mapsto x_\eta$ are mutually inverse and obviously ring homomorphisms. ■

This proves the theorem. ■

Remark 6.6. The Schur algebra $S_{\hat{\mathbb{Z}}_p}(p,p)$ is a classical order which was completely described by König in [20].

$$S_{\hat{\mathbb{Z}}_p}(p,p)' \simeq \begin{pmatrix} \hat{\mathbb{Z}}_p & \hat{\mathbb{Z}}_p \\ (p) & \hat{\mathbb{Z}}_p \end{pmatrix} \oplus \begin{pmatrix} \hat{\mathbb{Z}}_p & \hat{\mathbb{Z}}_p \\ (p) & \hat{\mathbb{Z}}_p \end{pmatrix} \oplus \ \ldots \oplus \begin{pmatrix} \hat{\mathbb{Z}}_p & \hat{\mathbb{Z}}_p \\ (p) & \hat{\mathbb{Z}}_p \end{pmatrix} \oplus \hat{\mathbb{Z}}_p$$

where we denote by $S_{\hat{\mathbb{Z}}_p}(p,p)'$ the basic algebra of the Schur algebra $S_{\hat{\mathbb{Z}}_p}(p,p)$.

Now, any strict polynomial functor induces a polynomial functor. So, composing further to the Green order lifting the principal block of the group ring of the symmetric group, we get an induced functor

$$S_{\hat{\mathbb{Z}}_p}(p,p)' - mod \longrightarrow \mathcal{A}^p_{\hat{\mathbb{Z}}_p} \stackrel{\phi}{\longrightarrow} e \cdot \Lambda \cdot e - mod.$$

Since the functor ϕ is induced by the Schur functor, this composed map is induced by the natural embedding of $e \cdot \Lambda \cdot e \hookrightarrow S_{\hat{\mathbb{Z}}_p}(p,p)'$.

7. Identifying the lattices as functors

We shall identify the indecomposable functors of $\mathcal{A}^p_{\hat{\mathbb{Z}}_p}$ which correspond to indecomposable $\Gamma_{\hat{\mathbb{Z}}_p}$-lattices. We call such polynomial functors 'polynomial lattices'.

The structure of $\Gamma_{\hat{\mathbb{Z}}_p,0}$ implies that there are exactly $3p-2$ indecomposable $\Gamma_{\hat{\mathbb{Z}}_p,0}$ lattices. Indeed, the indecomposable lattices are the p projective indecomposable modules $P_1, P_2, \dots, P_p$, the $p-1$ kernels of any fixed non zero homomorphism $P_i \longrightarrow P_{i+1}$ for $i = 1, 2, \dots, p-1$, as well as the $p-1$ kernels of any fixed non zero homomorphism $P_i \longrightarrow P_{i-1}$ for $i = 2, 3, \dots, p$. Therefore, there are exactly $3p-2$ indecomposable 'lattices' of exactly degree p polynomial functors in $\mathcal{A}^p_{\hat{\mathbb{Z}}_p}$.

Moreover, the proof of Theorem 5 shows that exactly the projective indecomposable $\Gamma_{\hat{\mathbb{Z}}_p}$-modules of degree $d \in \{2, \dots, p-1\}$ will give rise to indecomposable lattices. Denote by $\rho(k)$ the number of partitions of k into non zero integers, we get the following corollary to Theorem 5.

Corollary 7.1. *Up to isomorphism there are exactly* $3p - 2 + \sum_{k=2}^{p-1} \rho(k)$ *indecomposable polynomial lattices in* $\mathcal{A}^p_{\hat{\mathbb{Z}}_p}$ *and* $p + \sum_{k=2}^{p-1} \rho(k)$ *of them are projective, while* $2(p-1)$ *of them are not projective. The non projective polynomial lattices are kernels of mappings between projective indecomposable polynomial functors.*

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Université de Picardie, Département de Mathématiques et CNRS UMR 7352, 33 rue St Leu, F-80039 Amiens Cedex 1, France
E-mail address: alexander.zimmermann@u-picardie.fr