DEGENERATION-LIKE ORDERS IN TRIANGULATED CATEGORIES

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Abstract. In an earlier paper we defined a relation $\leq_\Delta$ between objects of the derived category of bounded complexes of modules over a finite dimensional algebra over an algebraically closed field. This relation was shown to be equivalent to the topologically defined degeneration order in a certain space $comproj(A, \underline{d})$ for derived categories. This space was defined as a natural generalization of varieties for modules. We remark that this relation $\leq_\Delta$ is defined for any triangulated category and show that under some finiteness assumptions on the triangulated category $\leq_\Delta$ is always a partial order.

1. Introduction

For a finite dimensional k-algebra A over an algebraically closed field k one defines for any positive integer d a variety $mod(A, d)$ as the algebraic variety of d-dimensional A-modules. The general linear group $Gl_d(k)$ acts naturally by conjugation on $mod(A, d)$ and two points in $mod(A, d)$ correspond to isomorphic modules if and only if they belong to the same orbit under this group action. A module M degenerates to a module N in $mod(A, d)$ if N is in the topological closure of the orbit of M. Riedtmann and Zwara characterized in [7, 10] this degeneration by an algebraic relation namely M degenerates into N if and only if there is a module Z so that N can be embedded into $M \oplus Z$ with quotient being isomorphic to Z. Zwara proved in [9] algebraically that this so-defined relation on isomorphism classes of modules is transitive.

In [4] we defined a topological space $comproj(A, \underline{d})$ so that points in this space correspond to right bounded complexes of projective A-modules with fixed homogeneous components in each degree. A base change group acts on this space as well and orbits correspond to isomorphism classes in the derived category. We showed that degeneration in $mod(A, d)$ induces a degeneration in $comproj(A, \underline{d})$ for suitably defined $\underline{d}$. Moreover, we defined algebraically a partial order $\leq_\Delta$ by setting $X \leq_\Delta Y$ if there is a complex Z so that $Y \longrightarrow X \oplus Z \longrightarrow Z \longrightarrow Y[1]$ is a distinguished triangle in the derived category. Then, we showed that for two complexes X and Y in the bounded derived category, $X \leq_\Delta Y$ if and only if X degenerates to Y in $comproj(A, \underline{d})$ for sufficiently big $\underline{d}$. By consequence, $\leq_\Delta$ is a partial order on isomorphism classes of complexes in $D^b(A)$. We need that A is finite dimensional over an algebraically closed field in order to be able to apply methods from algebraic geometry.

We show in the present paper that for any commutative ring R the obvious generalization of $\leq_\Delta$ to an arbitrary triangulated category is a partial order for any triangulated R-linear category $\mathcal{T}$ satisfying the following three properties:
1) $Hom_{\mathcal{T}}(X, Y)$ is an R-module of finite length for any X, Y
2) for any $X, Y \in \mathcal{T}$ there is a non zero integer $n_{X,Y}$ so that $Hom_{\mathcal{T}}(X, Y[n_{X,Y}]) = 0$, and
3) idempotent morphisms split in $\mathcal{T}$.
These hypotheses are satisfied for the bounded derived category of finitely generated modules over a finite dimensional algebra over a field R. Moreover, they imply that $\mathcal{T}$ satisfies the Krull-Remak-Schmidt theorem.

The paper is organized as follows. In Section 2 we give the basic definitions and show by an example that without some finiteness hypothesis on the triangulated categories, it will not be reasonable to have a partial order defined by $\leq_\Delta$. In Section 3 we imitate Zwara's

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proof [9] in order to show transitivity of our relation $\leq_\Delta$. In Section 4 we show that if $M \leq_\Delta N$ then, $Hom_{\mathcal{T}}(X, M)$ has smaller length over R than $Hom_{\mathcal{T}}(X, N)$ for any object X. Moreover, as a consequence we show that $\leq_\Delta$ is anti-symmetric. We finish the proof of the main result Theorem 5 in Section 5.

2. Basic definitions and an auxiliary results

Definition 1.
- Let $\mathcal{T}$ be a triangulated category with shift functor [1]. Then, we say for two objects X and Y of $\mathcal{T}$ that $Y \leq_\Delta X$ if there is an object Z of $\mathcal{T}$ and a distinguished triangle
$$X \longrightarrow Y \oplus Z \longrightarrow Z \longrightarrow X[1]$$
- We say that idempotent morphisms split in $\mathcal{T}$ if for any object X and any $e^2 = e \in End_{\mathcal{T}}(X)$ there is an isomorphism $X \simeq X_e \oplus X_{e'}$ so that via this isomorphism the endomorphism e is mapped to the endomorphism $\begin{pmatrix} id_{X_e} & 0 \\ 0 & 0 \end{pmatrix}$ of $X_e \oplus X_{e'}$.

Remark 2.1. For module categories this partial order is studied by Riedtmann [7] and Zwara [10, 9]. In [4] we extend the main result of Zwara [10] to derived categories.

Example 2.2. (1) Let $\mathcal{A}$ be an abelian category admitting countable direct sums and M and N be two objects. Then,
$$0 \longrightarrow M \stackrel{id}{\longrightarrow} M \longrightarrow 0 \longrightarrow 0$$
is exact and hence also
$$0 \longrightarrow M \stackrel{(id,0)}{\longrightarrow} M \oplus \left(\bigoplus_{i=1}^{\infty}(N \oplus M)\right) \longrightarrow \bigoplus_{i=1}^{\infty}(N \oplus M) \longrightarrow 0$$
is exact. Now, since
$$M \oplus \left(\bigoplus_{i=1}^{\infty}(N \oplus M)\right) \simeq \bigoplus_{i=1}^{\infty}(M \oplus N) \simeq \bigoplus_{i=1}^{\infty}(N \oplus M) \simeq N \oplus \left(\bigoplus_{i=1}^{\infty}(M \oplus N)\right)$$
we get $N \leq_\Delta M$ for any two modules M and N in the derived category $D^b(\mathcal{A})$ of bounded complexes of objects in $\mathcal{A}$. Hence $M \leq_\Delta N \leq_\Delta M$ for any two objects M and N in $\mathcal{A}$ even though M and N may be not isomorphic.

This example shows that it does not make sense to try to prove that $\leq_\Delta$ is a partial order without some finiteness assumption on the category.

(2) Let $G = Q_{32}$ be the generalized quaternion group of order 32 and let $\mathbb{Z}G$ be its integral group ring. In [8] Swan gives a projective non-free ideal $\mathfrak{a}$ of $\mathbb{Z}G$ so that $\mathfrak{a} \oplus \mathbb{Z}G \simeq \mathbb{Z}G \oplus \mathbb{Z}G$. Since $\mathfrak{a}$ is not free, we have that $\mathbb{Z}G \not\simeq \mathfrak{a}$. The split exact sequence
$$0 \longrightarrow \mathfrak{a} \longrightarrow \mathbb{Z}G \oplus \mathfrak{a} \longrightarrow \mathbb{Z}G \longrightarrow 0$$
induces an exact sequence
$$0 \longrightarrow \mathfrak{a} \longrightarrow \mathbb{Z}G \oplus \mathbb{Z}G \longrightarrow \mathbb{Z}G \longrightarrow 0$$
which shows $\mathbb{Z}G \leq_\Delta \mathfrak{a}$. Likewise, $\mathfrak{a} \leq_\Delta \mathbb{Z}G$ and we get $\mathfrak{a} \leq_\Delta \mathbb{Z}G \leq_\Delta \mathfrak{a}$ in the derived category of $\mathbb{Z}G$-modules.

Hence it is clear that it will be necessary to have conditions to ensure the Krull Schmidt property in $\mathcal{T}$ in order to make $\leq_\Delta$ a partial order.

We recall a property of triangulated categories.

Proposition 1. [5, Lemma 1.4.3, page 55] [6, Lemma 1.1] *Let*

$$A \xrightarrow{a} C \xrightarrow{c} E \xrightarrow{e_a} A[1]$$

and

$$B \xrightarrow{b} D \xrightarrow{d} E \xrightarrow{e_b} B[1]$$

be distinguished triangles, and let $f : A \longrightarrow B$ *be given so that* $e_a f[1] = e_b$*. Then, there is a mapping* $g : C \longrightarrow D$ *so that the following two properties are fulfilled:*

(1) (f, g, id_E) *is a mapping of triangles*

$$\begin{array}{ccccccc} A & \xrightarrow{a} & C & \longrightarrow & E & \longrightarrow & A[1] \\ \downarrow f & & \downarrow g & & \| & & \downarrow f[1] \\ B & \xrightarrow{b} & D & \longrightarrow & E & \longrightarrow & B[1] \end{array}$$

(2) *and*

$$A \xrightarrow{(f,a)} B \oplus C \xrightarrow{\binom{b}{-g}} D \longrightarrow A[1]$$

is a distinguished triangle for some mapping $D \longrightarrow A[1]$.

3. Transitivity

In [9] Zwara showed transitivity of the relation defined algebraically on isomorphism classes of A-modules. In this section we shall adapt his proof to our situation of triangulated categories.

Proposition 2. *Let* $\mathcal{T}$ *be a triangulated category in which idempotent morphisms split. Suppose that for any object* X *of* $\mathcal{T}$ *the endomorphism ring* $End_{\mathcal{T}}(X)$ *is an artinian ring. Then, the relation* $\leq_\Delta$ *is a reflexive and transitive relation on isomorphism classes in* $\mathcal{T}$*.*

Proof. It is clear that $\leq_\Delta$ is reflexive, since

$$X \xrightarrow{id} X \oplus 0 \longrightarrow 0 \longrightarrow X[1]$$

is a distinguished triangle.

We want to show that $\leq_\Delta$ is transitive. Let M, W, N so that $M \leq_\Delta W$ and $W \leq_\Delta N$. Then, there are distinguished triangles

$$W \xrightarrow{\omega} M \oplus Z \longrightarrow Z \longrightarrow W[1] \quad \text{and} \quad N \longrightarrow W \oplus Z_1 \xrightarrow{\binom{g}{f}} Z_1 \longrightarrow N[1]$$

for some objects Z and Z_1 in $\mathcal{T}$.

First, we shall show that we may assume that f is nilpotent.

Let $A = End(Z_1)$. By the Wedderburn-Artin theorem we can write

$$A/rad(A) \simeq \oplus_{i=1}^{L} Mat_{n_i}(\Delta_i),$$

where Δ_i are division rings for all $i \in \{1, \ldots, L\}$.

We consider the image $\overline{f} \in A/rad(A)$ of f. Suppose that $\overline{f} \neq 0$. We write $\overline{f} = (\overline{f}_i)_i$, where $\overline{f}_i \in Mat_{n_i}(\Delta_i)$.

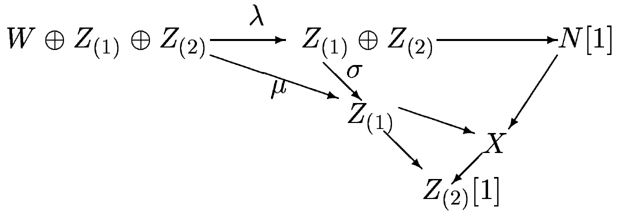
By Gauss' algorithm, for each i there exist invertible elements $\overline{g}_i$ and $\overline{h}_i$ of $Mat_{n_i}(\Delta_i)$ such that $\overline{g}_i \overline{f}_i \overline{h}_i$ is idempotent with units or zeroes on the diagonal and zero elsewhere. Since $rad(A)$ is nilpotent, we can find an idempotent $e \in A$ so that e maps to the idempotent $\overline{e} = (\overline{g}_i \overline{f}_i \overline{h}_i)_i$ of $A/rad(A)$ (cf e.g. [2, Theorem 1.7.3]). Let G and H be elements of A such that the images $\overline{G}$ and $\overline{H}$ in $A/rad(A)$ are $\overline{G} = (\overline{g}_i)_i$ and $\overline{H} = (\overline{h}_i)_i$, respectively. Then G and H are invertible and $GfH - e \in rad(A)$.

Let $F := GfH$. We have the following morphism of triangles.

$$\begin{array}{ccccccc} N & \longrightarrow & W \oplus Z_1 & \xrightarrow{\binom{g}{f}} & Z_1 & \longrightarrow & N[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ N & \longrightarrow & W \oplus Z_1 & \xrightarrow{\binom{gH}{F}} & Z_1 & \longrightarrow & N[1], \end{array}$$

where in the diagram the endomorphism of $W \oplus Z_1$ is $\begin{pmatrix} 1 & 0 \\ 0 & G^{-1} \end{pmatrix}$ and the endomorphism of Z_1 is H. So we get a triangle $N \longrightarrow W \oplus Z_1 \xrightarrow{\binom{gH}{F}} Z_1 \longrightarrow N[1]$. We will show that eFe has a right inverse.

Since e is idempotent and since idempotents split in $\mathcal{T}$, there is a decomposition $Z_1 = Z_{(1)} \oplus Z_{(2)}$ such that e preserves the decomposition, is identity on $Z_{(1)}$ and 0 on $Z_{(2)}$. Moreover, we have $eFe \in End(Z_{(1)})$ and $eFe - e \in rad(End(Z_{(1)}))$. By [2, Lemma 1.2.2] we know that eFe has a right inverse in $End(Z_{(1)})$, which we denote by F'. Applying the octahedral axiom on the following diagram of triangles,



where $\lambda = \begin{pmatrix} gH \\ F \end{pmatrix} = \begin{pmatrix} g' & g'' \\ eFe & eFe' \\ e'Fe & e'Fe' \end{pmatrix}$, $\sigma = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mu = \begin{pmatrix} g' \\ eFe \\ e'Fe \end{pmatrix}$, we get a triangle

$$N \longrightarrow X[-1] \longrightarrow Z_{(2)} \longrightarrow N[1].$$

We have $\begin{pmatrix} 1 & g'F' & 0 \\ 0 & 1 & 0 \\ 0 & e'FeF' & 1 \end{pmatrix} \in Aut_{\mathcal{T}}(W \oplus Z_{(1)} \oplus Z_{(2)})$ and $F' \in Aut_{\mathcal{T}}(Z_{(1)})$ making the left most square of the following diagram commutative.

$$\begin{array}{ccccccc} W \oplus Z_{(1)} \oplus Z_{(2)} & \xrightarrow{\mu} & Z_{(1)} & \longrightarrow & X & \longrightarrow & (W \oplus Z_{(1)} \oplus Z_{(2)})[1] \\ \downarrow & & \downarrow & & & & \downarrow \\ W \oplus Z_{(1)} \oplus Z_{(2)} & \xrightarrow{(0,1,0)^t} & Z_{(1)} & \longrightarrow & (W \oplus Z_{(2)})[1] & \longrightarrow & (W \oplus Z_{(1)} \oplus Z_{(2)})[1], \end{array}$$

Since the vertical mappings are isomorphisms, they induce an isomorphism $X \cong (W \oplus Z_{(2)})[1]$.

So the above triangle becomes a triangle

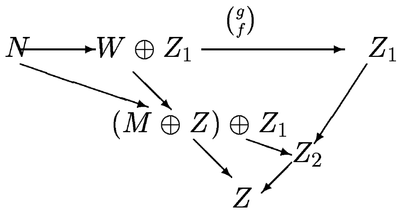
$$N \longrightarrow W \oplus Z_{(2)} \longrightarrow Z_{(2)} \longrightarrow N[1].$$

Since $Z_{(2)}$ is a direct summand of Z_1, by induction on the length of $\mathrm{End}(Z_1)$ we can assume that f is nilpotent.

We continue the proof, showing that the relation is transitive. We get a factorisation

$$N \longrightarrow W \oplus Z_1 \longrightarrow M \oplus Z \oplus Z_1.$$

This gives



where the triangle

$$Z_1 \xrightarrow{\iota_1} Z_2 \longrightarrow Z \longrightarrow Z_1[1]$$

is given by the octahedral axiom. It follows that for the mapping

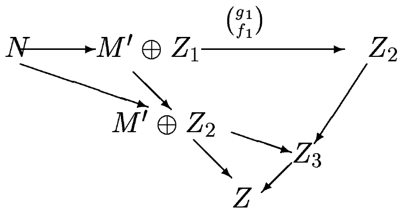
$$(M \oplus Z) \oplus Z_1 \xrightarrow{\binom{g_1}{f_1}} Z_2$$

one has $f\iota_1 = f_1$.

We abbreviate $M' := M \oplus Z$ and have a new factorisation

$$N \longrightarrow M' \oplus Z_1 \xrightarrow{\begin{pmatrix} id_{M'} & 0 \\ 0 & \iota_1 \end{pmatrix}} M' \oplus Z_2 \ .$$

This gives again a diagram



where the triangle

$$Z_2 \xrightarrow{\iota_2} Z_3 \longrightarrow Z \longrightarrow Z_2[1]$$

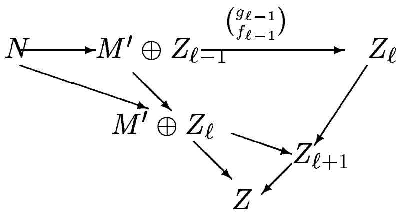
is given by the octahedral axiom. More generally we define inductively complexes Z_ℓ, distinguished triangles

$$Z_\ell \xrightarrow{\iota_\ell} Z_{\ell+1} \longrightarrow Z \longrightarrow Z_\ell[1]$$

and mappings $\binom{g_\ell}{f_\ell} : M' \oplus Z_\ell \longrightarrow Z_{\ell+1}$ given by the factorisation

$$N \longrightarrow M' \oplus Z_{\ell-1} \xrightarrow{\begin{pmatrix} id_{M'} & 0 \\ 0 & \iota_{\ell-1} \end{pmatrix}} M' \oplus Z_\ell$$

which induce the diagram



Hence, we get a sequence of mappings $(f_\ell, f_{\ell+1}, id_Z)$ of distinguished triangles

$$\begin{array}{ccccc} Z_\ell & \stackrel{f_\ell}{\longrightarrow} & Z_{\ell+1} & \stackrel{f_{\ell+1}}{\longrightarrow} & Z_{\ell+2} \\ \downarrow \iota_\ell & & \downarrow \iota_{\ell+1} & & \downarrow \iota_{\ell+2} \\ Z_{\ell+1} & \stackrel{f_{\ell+1}}{\longrightarrow} & Z_{\ell+2} & \stackrel{f_{\ell+2}}{\longrightarrow} & Z_{\ell+3} \\ \downarrow & & \downarrow & & \downarrow \\ Z & = & Z & = & Z \\ \downarrow & & \downarrow & & \downarrow \\ Z_\ell[1] & \stackrel{f_\ell[1]}{\longrightarrow} & Z_{\ell+1}[1] & \stackrel{f_{\ell+1}[1]}{\longrightarrow} & Z_{\ell+2}[1] \end{array}$$

Since the composition of morphism of triangles is a morphism of triangles, we get a morphism of distinguished triangles

$$\begin{array}{ccc} Z_1 & \stackrel{f_1 f_2 \dots f_k}{\longrightarrow} & Z_{k+1} \\ \downarrow \iota_1 & & \downarrow \iota_{k+1} \\ Z_2 & \stackrel{f_2 f_3 \dots f_{k+1}}{\longrightarrow} & Z_{k+2} \\ \downarrow & & \downarrow \\ Z & = & Z \\ \downarrow & & \downarrow \\ Z_1[1] & \stackrel{f_1 f_2 \dots f_k[1]}{\longrightarrow} & Z_{k+1}[1] \end{array}$$

Now, since $f_1 = \iota_1 f$ and since $\iota_\ell f_{\ell+1} = f_\ell \iota_{\ell+1}$, we get $f_1 f_2 \dots f_k = f^k \iota_1 \iota_2 \dots \iota_k$. Since f is nilpotent we get $f_1 f_2 \dots f_k = 0$ for some k. Using Proposition 1 we get a triangle

$$Z_1 \stackrel{\binom{0}{\iota_1}}{\longrightarrow} Z_{k+1} \oplus Z_2 \longrightarrow Z_{k+2} \longrightarrow Z_1[1] \ .$$

Now,

$$Z_{k+2} \simeq cone(\begin{pmatrix} 0 \\ \iota_1 \end{pmatrix}) \simeq Z_{k+1} \oplus cone(\iota_1) \simeq Z_{k+1} \oplus Z$$

Since

$$N \longrightarrow (M \oplus Z) \oplus Z_{k+1} \longrightarrow Z_{k+2} \longrightarrow N[1]$$

is a triangle, we get $M \leq_\Delta N$ as required. ∎

4. Anti-symmetry

Proposition 3. *Let R be a commutative ring and let $\mathcal{T}$ be an R-linear triangulated category. Suppose that X is an object in $\mathcal{T}$ so that $Hom_{\mathcal{T}}(X,Y)$ is an R-module of finite length* $\text{length}_R(Hom(X,Y))$ *for all Y in $\mathcal{T}$. Then,*

$$N \leq_\Delta M \Rightarrow \text{length}_R(Hom(X, N[j])) \leq \text{length}_R(Hom(X, M[j]))$$

for any integer j.

If X is an object in $\mathcal{T}$ so that $Hom_{\mathcal{T}}(Y,X)$ is an R-module of finite length for all Y in $\mathcal{T}$, then

$$N \leq_\Delta M \Rightarrow \text{length}_R(Hom(N[j], X)) \leq \text{length}_R(Hom(M[j], X))$$

for any integer j.

Proof. Let $N \leq_\Delta M$. Then, there is an object Z of $\mathcal{T}$ so that

$$M \longrightarrow N \oplus Z \longrightarrow Z \longrightarrow M[1]$$

is a distinguished triangle. Apply $Hom_{\mathcal{T}}(X,-)$ to this triangle. Abbreviating $(X,-) := Hom_{\mathcal{T}}(X,-)$ we get a long exact sequence

$$\dots \longrightarrow \ (X, M[j]) \ \longrightarrow \ (X, N[j]) \oplus (X, Z[j]) \ \longrightarrow \ (X, Z[j]) \ \longrightarrow \dots$$

For any j one gets

$$\begin{aligned}\text{length}_R(Hom_{\mathcal{T}}(X, M[j])) + \text{length}_R(Hom_{\mathcal{T}}(X, Z[j])) \\ \geq \text{length}_R(Hom_{\mathcal{T}}(X, N[j]) \oplus Hom_{\mathcal{T}}(X, Z[j]))\end{aligned}$$

and so,

$$\text{length}_R(Hom_{\mathcal{T}}(X, M[j])) \geq \text{length}_R(Hom_{\mathcal{T}}(X, N[j]))$$

as soon as $\text{length}_R(Hom_{\mathcal{T}}(X, Z[j]))$ and $\text{length}_R(Hom_{\mathcal{T}}(X, M[j])$ are finite.

The dual statement follows by the dual arguments, applying $Hom_{\mathcal{T}}(-, X)$ to the triangle.

This shows the proposition. ■

Suppose that $M \leq_\Delta N$ and $N \leq_\Delta M$. Then, we know by Proposition 3 that for any object X in $\mathcal{T}$ so that for all objects Y one has that $Hom_{\mathcal{T}}(X, Y)$ and $Hom_{\mathcal{T}}(Y, X)$ are of finite length over R, one has

$$\text{length}_R(Hom_{\mathcal{T}}(X, N[j])) = \text{length}_R(Hom_{\mathcal{T}}(X, M[j]))$$

and

$$\text{length}_R(Hom_{\mathcal{T}}(N[j], X)) = \text{length}_R(Hom_{\mathcal{T}}(M[j], X))$$

for any integer j.

Proposition 4. *Let R be a commutative ring and let $\mathcal{T}$ be an R-linear triangulated category in which idempotent morphisms split and so that for any two objects X and Y of $\mathcal{T}$ the set $Hom_{\mathcal{T}}(X, Y)$ is of finite length as an R-module. Suppose that M and N are two objects in $\mathcal{T}$ so that there is $n \in \mathbb{Z} \setminus \{0\}$ satisfying $Hom_{\mathcal{T}}(M, N[n]) = 0$.*

If for any object X of $\mathcal{T}$ one has that the length of $Hom_{\mathcal{T}}(M, X)$ as R-module equals the length of $Hom_{\mathcal{T}}(N, X)$ as R-module, then $M \simeq N$.

Remark 4.1.
- Bongartz showed a similar result for an abelian category [3]. We see that his proof can be modified so that it applies to our situation as well.
- If we assume that $Hom_{\mathcal{T}}(M, M[n])$ vanishes instead of $Hom_{\mathcal{T}}(M, N[n])$ the conclusion of Proposition 4 still holds. The proof is similar.

Proof. Suppose M and N are objects satisfying the above hypothesis. Since $Hom_{\mathcal{T}}(M, N)$ is of finite length over R, take generators $f_1, f, \dots, f_\ell$ of $Hom_{\mathcal{T}}(M, N)$ as an R-module. Let $f := (f_1, f_2, \dots, f_\ell)^{tr} : M^l \longrightarrow N$. We show that f is split.

So, f induces an epimorphism

$$f^* := Hom_{\mathcal{T}}(M, f) : Hom_{\mathcal{T}}(M, M^\ell) \longrightarrow Hom_{\mathcal{T}}(M, N).$$

Let K be the cone of f so that $M^\ell \xrightarrow{f} N \longrightarrow K \longrightarrow M^\ell[1]$ is a distinguished triangle. Then,

$$\begin{aligned}(M, M^\ell[-j]) \longrightarrow (M, N[-j]) \longrightarrow \dots \longrightarrow (M, K[-1]) \longrightarrow (M, M^\ell) \longrightarrow (M, N) \longrightarrow (M, K) \\ \longrightarrow \dots \longrightarrow (M, M^\ell[j]) \longrightarrow (M, N[j])\end{aligned}$$

and

$$\begin{aligned}(N, M^\ell[-j]) \longrightarrow (N, N[-j]) \longrightarrow \dots \longrightarrow (N, K[-1]) \longrightarrow (N, M^\ell) \longrightarrow (N, N) \longrightarrow (N, K) \\ \longrightarrow \dots \longrightarrow (N, M^\ell[j]) \longrightarrow (N, N[j])\end{aligned}$$

are both exact. By construction $(M, M^\ell) \longrightarrow (M, N)$ is surjective. Hence,

$$(M, M^\ell[-j]) \longrightarrow (M, N[-j]) \longrightarrow \dots \longrightarrow (M, K[-1]) \longrightarrow (M, M^\ell) \longrightarrow (M, N) \longrightarrow 0$$

and

$$0 \longrightarrow (M, K) \longrightarrow \dots \longrightarrow (M, M^\ell[j]) \longrightarrow (M, N[j])$$

are both exact.

Suppose $Hom_{\mathcal{T}}(M, N[n]) = 0$ and let $j = |n|$. If $n < 0$, then one gets that

$$0 \longrightarrow (M, K[n+1]) \longrightarrow \ldots \longrightarrow (M, K[-1]) \longrightarrow (M, M^\ell) \longrightarrow (M, N) \longrightarrow 0$$

is exact. Since the R-length of $Hom_{\mathcal{T}}(M, N[n])$ equals the R-length of $Hom_{\mathcal{T}}(N, N[n])$, also $Hom_{\mathcal{T}}(N, N[n]) = 0$ and

$$0 \longrightarrow (N, K[n+1]) \longrightarrow \ldots \longrightarrow (N, K[-1]) \longrightarrow (N, M^\ell) \longrightarrow (N, N)$$

is exact. Since now the length over R of the corresponding terms coincide for $(N, -)$ and for $(M, -)$, counting the alternate sum of R-lengths, we see that

$$0 \longrightarrow (N, K[n+1]) \longrightarrow \ldots \longrightarrow (N, K[-1]) \longrightarrow (N, M^\ell) \longrightarrow (N, N) \longrightarrow 0$$

is exact. If $n > 0$, then

$$0 \longrightarrow (M, K) \longrightarrow \ldots \longrightarrow (M, M^\ell[n]) \longrightarrow 0$$

is exact, and since the R-lengths of $(M, N[i])$ equals the R-lengths of $(N, N[i])$, also

$$(N, K) \longrightarrow \ldots \longrightarrow (N, M^\ell[n]) \longrightarrow 0$$

is exact. But then, counting again the alternate sum of the R-lengths, one gets that

$$0 \longrightarrow (N, K) \longrightarrow \ldots \longrightarrow (N, M^\ell[n]) \longrightarrow 0$$

is exact and consequently

$$(N, K[-n+1]) \longrightarrow \ldots \longrightarrow (N, K[-1]) \longrightarrow (N, M^\ell) \longrightarrow (N, N) \longrightarrow 0$$

is exact.

Hence, in any case, f is split and N is a direct factor of M^n. The hypothesis on $\mathcal{T}$ implies that the Krull-Schmidt theorem holds in $\mathcal{T}$ [1, Ch 1, Theorem 3.6]. We see that therefore N and M have an indecomposable non zero direct factor U in common. Let $N = U \oplus N'$ and $M = U \oplus M'$. By induction on the R-length of $Hom(M, M)$ we are finished. ∎

Corollary 5. *Let $\mathcal{T}$ be a triangulated category satisfying the assumptions in Proposition 4. Then the relation $\leq_\Delta$ is an anti-symmetric relation on the isomorphism classes of $\mathcal{T}$.*

Proof. Let $M \leq_\Delta N$ and $N \leq_\Delta M$. Then, Proposition 3 shows that the hypothesis of Proposition 4 are fulfilled and therefore $N \simeq M$. □

5. The main result

We are now ready to formulate our main result.

Theorem. *Let R be a commutative ring and let $\mathcal{T}$ be an R-linear triangulated category satisfying the following three conditions.*

- *For any two objects X and Y of $\mathcal{T}$ the set $Hom_{\mathcal{T}}(X, Y)$ is of finite length as an R-module.*
- *For any two objects X and Y in $\mathcal{T}$ there is an integer $n_{XY} \in \mathbb{Z} \setminus \{0\}$ so that $Hom_{\mathcal{T}}(X, Y[n_{X,Y}]) = 0$,*
- *Idempotent morphisms in $\mathcal{T}$ split.*

Then, the relation $\leq_\Delta$ defines a partial order relation on the set of isomorphism classes of objects in $\mathcal{T}$.

Proof. The fact that $\leq_\Delta$ is reflexive and transitive is Proposition 2. The anti-symmetry is Corollary 5. ∎

Remark 5.1. Examples for triangulated categories $\mathcal{T}$ satisfying the hypotheses of Theorem 5 are

- the bounded derived category $D^b(A)$ of finitely generated A-modules over an artinian R-algebra A,
- the bounded derived category of coherent sheaves over a projective variety.

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