Degeneration in triangulated categories

ALEXANDER ZIMMERMANN
(joint work with Bernt Tore Jensen, Xiuping Su; Manuel Saorín)

1. The classical situation

Let k be an algebraically closed field and let A be a finite dimensional k-algebra. Then an A-module structure on k^d is just a k-algebra homomorphism $A \to Mat_{d\times d}(k)$. Two such maps give isomorphic module structures if and only if they are conjugate by a matrix in $GL_d(k)$. Hence the set of A-module structures on k^d form an algebraic affine variety $mod(A,d)$ on which $GL_d(k)$ acts, and orbits of this action correspond to isomorphism classes. The module M degenerates to N (denoted $M \leq_{\rm deg} N$) if N belongs to the Zariski closure of the orbit of M. How to characterise this algebraically? This is solved in the following result.

Theorem 1. *(Riedtmann* [3]*, Zwara* [10]*) Let k be an algebraically closed field and let A be a finite dimensional k-algebra, then for any two A-modules M and N we get $M \leq_{deg} N$ if and only if there is an A-module Z and a short exact sequence $0 \to Z \to Z \oplus M \to N \to 0$. We denote this second condition by $M \leq_{Zwara} N$.*

The geometric version $\leq_{\rm deg}$ is a partial order on the set of isomorphism classes of finite dimensional A-modules, as is easily seen.

2. Carrying the algebraic degeneration to the triangulated world

The goal of our research is to carry these constructions to the setting of triangulated categories. The first easy step is to generalise $\leq_{\rm Zwara}$ to triangulated categories. k denotes from now on a commutative ring, and occasionally a field.

Definition 2. (Jensen-Su-Zimmermann [1, 2], Yoshino [8]) Let $\mathcal{T}$ be a triangulated category. Then for any two objects M and N we denote $M \leq_\Delta N$ if there is an object Z and a distinguished triangle $Z \to M \oplus Z \to N \to Z[1]$.

In [2] we proved partial order properties of $\leq_\Delta$. In particular we show

Theorem 3. [2] *Let $\mathcal{T}$ be a k-linear triangulated category with split idempotents.*

- *If all endomorphism algebras of objects of $\mathcal{T}$ are artinian, then $\leq_\Delta$ is reflexive and transitive on isomorphism classes of objects of $\mathcal{T}$.*
- *If $Hom_{\mathcal{T}}(X,Y)$ is of finite k-length for all objects X and Y, and if there is $n \in \mathbb{Z}\setminus\{0\}$ such that $Hom_{\mathcal{T}}(M,N[n]) = 0$, then*

$$M \leq_\Delta N \leq_\Delta M \Rightarrow M \simeq N.$$

We see that the pre-order property is relatively general, whereas the partial order property needs strong hypotheses. We should mention that Peter Webb showed in [7] the antisymmetry by completely different methods for triangulated categories which admit almost split triangles. Singular categories are one of our main intended application, and there the hypotheses are not satisfied. Zhengfang

Wang proved in [6] independently that $\leq_\Delta$ is a partial order on isomorphism classes of the singular category $D_{sg}(A)$ of a finite dimensional algebra A.

Remark 4. We should mention that this concept may allow to compare modules of different dimension. Indeed, it may happen that M is an indecomposable module of infinite projective dimension and $M \leq_{\mathrm{Zwara}} N_1 \oplus N_2$ where N_1 is a non-zero module of finite projective dimension, and N_2 is a module of infinite projective dimension. Then $M \leq_\Delta N_2$ in $D_{sg}(A)$. A more explicit example is the following: If A is a self-injective algebra, and P is an indecomposable projective A-module with non zero submodule S, then $P \leq_{\mathrm{deg}} S \oplus P/S$ and therefore $P \leq_{\Delta+\mathrm{nil}} S \oplus P/S$ in the stable category of A-modules. But $P \simeq 0$ in the stable category, whereas the pieces S and P/S are not. Nevertheless, by definition, degeneration preserves the class in the Grothendieck group.

3. Geometric degeneration in triangulated categories

In joint work with Saorín we concentrated on a geometric definition for degeneration in triangulated categories. We model our geometric version on Yoshino's concept of a degeneration along a dvr.

Definition 5. (Yoshino) Let k be a field and let A be a k-algebra. Then for any two A-modules M and N we say that $M \leq_{\mathrm{dvr}} N$ if there is a discrete valuation k-algebra V with maximal ideal tV and $k = V/tV$ and a V-flat $V \otimes_k A$-module Q such that $Q/tQ \simeq N$ as A-modules, and $Q[\frac{1}{t}] \simeq M \otimes_k V[\frac{1}{t}]$.

For a triangulated category $\mathcal{C}_V^\circ$ and an element $t : id_{\mathcal{C}_V^\circ} \to id_{\mathcal{C}_V^\circ}$ (i.e. an element t in the centre of $\mathcal{C}_V^\circ$) we can form the Gabriel-Zisman localisation $\mathcal{C}_V^\circ \xrightarrow{p} \mathcal{C}_V^\circ[t^{-1}]$, which is again triangulated and is universal amongst all triangulated categories in which t_X becomes invertible for all objects X.

Definition 6. [4] Let $\mathcal{C}_k^\circ$ be a triangulated k-category with split idempotents.

- A degeneration data for $\mathcal{C}_k^\circ$ is given by triangulated k-categories $\mathcal{C}_V^\circ$, $\mathcal{C}_k$ and $\mathcal{C}_V$ with split idempotents, such that $\mathcal{C}_V^\circ$ is full triangulated subcategory of $\mathcal{C}_V$, and $\mathcal{C}_k^\circ$ is full triangulated subcategory of $\mathcal{C}_k$, a triangle functor $\uparrow_k^V\colon \mathcal{C}_k \to \mathcal{C}_V$ restricting to a triangle functor $\mathcal{C}_k^\circ \to \mathcal{C}_V^\circ$, and a triangle functor $\phi : \mathcal{C}_V^\circ \to \mathcal{C}_k$, as well as an element t in the centre of $\mathcal{C}_V^\circ$. We require that $\phi(t_{M\uparrow_k^V})$ is a split monomorphism with cone M for all objects M of $\mathcal{C}_k^\circ$.
- An object M of $\mathcal{C}_k^\circ$ degenerates to an object N of $\mathcal{C}_k^\circ$ if there is an object Q of $\mathcal{C}_V^\circ$ such that $\phi(cone(t_Q)) \simeq N$ and $p(Q) \simeq p(M \uparrow_k^V)$. We write $M \leq_{\mathrm{cdeg}} N$ in this case.

It is not hard to see that this generalises Yoshino's concept in case of stable categories for finite dimensional self-injective algebras. The main result is

Theorem 7. *Let $\mathcal{C}_k^\circ$ be a triangulated k-category with split idempotents. Then*

$$M \leq_{\Delta+nil} N \Rightarrow M \leq_{cdeg} N.$$

If $\mathcal{C}_k^\circ$ is the category of compact objects in an algebraic compactly generated triangulated k-category, then we get

$$M \leq_{\Delta+nil} N \Leftrightarrow M \leq_{cdeg} N.$$

4. Symmetry in the definition of the triangle degeneration

The definition of $\leq_{\text{Zwara}}$ and $\leq_\Delta$ bears some non-symmetry. Zwara proved in [9] for finite dimensional algebras A over a field and A-modules M and N that there is an A-module Z and a short exact sequence $0 \to Z \to Z \oplus M \to N \to 0$ if and only if there is an A-module Z' and a short exact sequence $0 \to N \to Z' \oplus M \to Z' \to 0$.

For the relation $\leq_\Delta$ we may pose the same question. Since the opposite category of a triangulated category is again triangulated, we denote by $\leq_{\Delta^{op}}$ the triangle relation between the corresponding objects, i.e. $M \leq_{\Delta^{op}} N$ if and only if there is Z' and a distinguished triangle $N \to M \oplus Z' \to Z' \to N[1]$.

Theorem 8. [5] *Let $\mathcal{T}$ be a triangulated k-category with split idempotents.*

- *If the endomorphism ring of each object in $\mathcal{T}$ is artinian, then $M \leq_{\Delta+nil} N \Leftrightarrow M \leq_{\Delta^{op}+nil} N$.*
- *If $\mathcal{T}_k$ is the category of compact objects in an algebraic compactly generated triangulated k-category, then $M \leq_{\Delta+nil} N \Leftrightarrow M \leq_{\Delta^{op}+nil} N$.*

We do not know if the statement is true for $\leq_\Delta$ instead of $\leq_{\Delta+\text{nil}}$. The second statement uses our explicit construction of a degeneration and the construction of an explicit Q as in the definition of $\leq_{\text{cdeg}}$ in the main theorem.

References

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