Invariance of generalized Reynolds ideals under derived equivalence

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Let k be an algebraically closed field of characteristic $p > 0$ and let A be a symmetric finite dimensional k-algebra with symmetrising bilinear form $(\ ,\ )$. Let KA be the k-vector space generated by the subset $\{xy - yx|\ x, y \in A\}$ of A. This space was defined and used by R. Brauer in [1].

Külshammer defined in [3] and [4] for any integer n the spaces

$$T_n(A) := \{x \in A \mid x^{p^n} \in KA\}$$

and $T_n(A)^\perp$ the orthogonal space to $T_n(A)$ with respect to the symmetrising form $(\ ,\ )$. Then, $T_n(A)^\perp$ is an ideal of the centre $Z(A)$ of A.

$$Z(A) = T_0(A)^\perp \supseteq T_1(A)^\perp \supseteq T_2(A)^\perp \supseteq \cdots \supseteq \bigcap_{n\in\mathbb{N}} T_n(A)^\perp = soc(A) \cap Z(A)$$

Héthelyi et al. show in [2] that $Z_0A \subseteq (T_1A^\perp)^2 \subseteq HA$, where HA is the Higman ideal, that is the image of the trace map of A, and where Z_0A is the sum of the centres of those blocks of A which are simple algebras. They show that for odd p the left inclusion is an equality, whereas for $p = 2$ one gets $Z_0A = (T_1A^\perp)^3 = (T_1A^\perp) \cdot (T_2A^\perp)$. Many more interesting properties of these ideals are given there.

The authors of [2] show that the ideals are invariant under Morita equivalence in the obvious sense and they ask if for derived equivalent symmetric algebras A and B there is an isomorphism $\varphi : Z(A) \longrightarrow Z(B)$ with $\varphi(T_n(A)^\perp) = T_n(B)^\perp$ for all $n \in \mathbb{N}$.

Let B be a k-algebra. By Rickard's theory [5] given an equivalence $D^b(A) \simeq D^b(B)$ as triangulated categories there is a complex X in $D^b(B \otimes_k A^{op})$ so that

$$X \otimes^{\mathbb{L}}_A - : D^b(A) \longrightarrow D^b(B)$$

is an equivalence, called "of standard type". Now, for a symmetric algebra A, given such an equivalence the algebra B is symmetric as well again by [5], or in a more general context by [6]. Then, replacing X by a suitable isomorphic copy consisting of a complex formed of left and right projective A-modules,

$$X \otimes_A - \otimes_A Hom_k(X, k) : D^b(A \otimes_k A^{op}) \longrightarrow D^b(B \otimes_k B^{op})$$

is an equivalence. Moreover, this equivalence maps the bimodule A to B and therefore induces an isomorphism

$$HH^n(A) = Hom_{D^b(A\otimes_k A^{op})}(A, A[n]) \simeq Hom_{D^b(B\otimes_k B^{op})}(B, B[n]) = HH^n(B)$$

between the degree n Hochschild cohomology of A and B. Now, observe that $HH^0(A) = Z(A)$ and $HH^0(B) = Z(B)$.

Theorem. [7] *Let k be an algebraically closed field of characteristic $p > 0$ and let A and B be finite dimensional k-algebras. Then, the isomorphism $\varphi : ZA \longrightarrow ZB$ between the centres ZA of A and ZB of B induced by an equivalence $D^b(A) \simeq D^b(B)$ of standard type has the property $\varphi(T_nA^\perp) = T_nB^\perp$ for all positive integers $n \in \mathbb{N}$.*

Remark: As an application of the theorem it is possible to distinguish the derived categories of certain algebras arising as blocks of group rings of tame representation type, which could not be distinguished otherwise. This will be subsequent joint work with Thorsten Holm.

The proof of the theorem uses first that the ideals $T_n(A)^\perp$ are images of mappings ζ_n defined by the property

$$(\zeta_n(x), y)^{p^n} = (x, y^{p^n}) \; \forall x \in Z(A) \forall y \in A/KA \; .$$

Then, we study in detail the mapping ζ_n as a composition of mappings

$$\begin{array}{rcl} Hom_{A\otimes_k A^{op}}(A,A) & \longrightarrow & Hom_{A\otimes_k A^{op}}(A, Hom_k(A,k)) \\ & \stackrel{\psi}{\longrightarrow} & Hom_{A\otimes_k A^{op}}(A, Hom_k(A,k)) \\ & \longrightarrow & Hom_{A\otimes_k A^{op}}(A,A). \end{array}$$

Here ψ is the composition of the n fold p-power mapping and the inverse of the Frobenius mapping. It is then possible to study the behaviour under a derived equivalence and this discussion gives the statement.

References

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