Derived Equivalences of Orders

Alexander Zimmermann*†

Abstract

Using the result of Roggenkamp and Scott [33] that the p-adic group ring of a p-group determines its group bases we prove that for a p–group G, the derived equivalence class of the group ring over the p-adic integers determines the group basis up to isomorphism. We furthermore prove that two Green orders [30] are isomorphic if they are 'of the same type'. These two results represent results that arose in collaboration with Klaus W. Roggenkamp. We give a survey on the classification of the set of tilting complexes over a hereditary order over a complete discrete rank 1 valuation domain. The section on hereditary orders reports on joint work with Steffen König [17]. The last section gives a definition and first properties of Picard groups of derived module categories of orders. This section represents joint work with Raphaël Rouquier.

1 Introduction

Derived categories arose first in algebraic geometry in the early 60's and proved there to be a powerful tool. Inspired by the work of D. Happel [8] and Cline, Parshall, Scott [3] J. Rickard proved in 1989 a Morita theorem for derived categories [23, 25]. He proved that the derived categories of any two rings are equivalent if and only if there is a so called tilting complex over one of the rings, the other one is the endomorphism ring of.

In Section 2 we explain some of the technicalities in the definitions concerning derived module categories and equivalences between them in more detail.

*1991 *Mathematics Subject Classification* Primary: 16G30, 18E30, 20C10, Secondary: 16S34, 20C05

†The author acknowledges financial support from the Deutsche Forschungsgemeinschaft.

We apply in the present paper the theory of J. Rickard to various sorts of problems. In Section 3 we discuss derived equivalences of Green orders. Green orders are defined by K. W. Roggenkamp to clarify the structure of blocks of p-adic group rings of cyclic defect [30]. They generalize in a sense Brauer tree algebras since for every ordinary Brauer tree algebra there is a Green order which reduces modulo the radical of the base ring to this Brauer tree algebra. Green orders are as well associated to trees, though they have additional datas associated to each vertex, not only to one exceptional vertex. We prove that two Green orders are derived equivalent if they share the same set of data.

In Section 4 we give a summary of joint work with Steffen König on derived equivalences of hereditary orders. Analogous to the situation in the representation theory of algebras we say that an order is a hereditary order if it is an order whose ideals are projective. Their structure is clarified by [2, 9] (see [22]) and we shall use this knowledge to describe over a complete discrete rank 1 valuation domain R for a given hereditary R–order Λ all derived equivalences, all tilting complexes and their endomorphism rings in a purely combinatorial manner. The result gives an example for two R–algebras Λ and Γ which are derived equivalent, but $R/rad\ R \otimes_R \Lambda$ is not derived equivalent to $R/rad\ R \otimes_R \Gamma$, though Λ is an R–order. If also Γ was an R–order, then the theorem of J. Rickard [25] would establish a derived equivalence between $R/rad\ R \otimes_R \Lambda$ and $R/rad\ R \otimes_R \Gamma$.

In Section 5 we discuss derived equivalences of local orders. We prove there that derived equivalences of local orders are nothing else than Morita equivalences. As a consequence we see by the theorem of K. W. Roggenkamp and L. L. Scott [33] that if two group rings of p-groups over the p-adic integers are derived equivalent, then the two groups are already isomorphic. The converse is obvious.

Section 6 is devoted to define Picard groups for derived module categories of orders. The derived equivalences of standard type between R–algebras Λ and Γ are introduced in [25], R being a commutative ring. We shall elaborate on derived equivalences of standard type with $\Lambda = \Gamma$ in case of an R–order Λ. We give in Section 6 the definition and some of the properties. We also look at examples mainly where this newly defined Picard group of derived module categories gives nothing new in addition to the theory of A. Fröhlich on Picard groups for module categories of orders.

Our interest in derived module categories of orders arose with the far reaching conjectures of M. Broué. M. Broué observed a very close connection between the modular character theory of blocks with abelian defect group and their Brauer correspondent. He suspected that this connection is just the observable surface of a much deeper and more structural connection between Brauer corresponding blocks. He conjectured in [1] that if G is a finite group and k an algebraically closed field of characteristic p, if, furthermore, B is a block of kG with abelian defect group D and if b is the Brauer correspondent of B in $kN_G(D)$, the group ring of the normalizer of the defect group, then B and b are derived equivalent.

J. Rickard proved in [24] that Brauer tree algebras over an algebraically closed field of characteristic $p > 0$ are derived equivalent to each other if and only if they share the same number of edges and the multiplicity of the exceptional vertex of their defining Brauer tree. In [18] M. Linckelmann generalized the result of J. Rickard to blocks of cyclic defect of group rings over a complete discrete valuation domain R of characteristic 0 and *algebraically closed residue field* of characteristic $p > 0$. The tilting complexes, M. Linckelmann used, reduce to those, J. Rickard introduced, when they are taken modulo the radical of R. R. Rouquier constructed the *twosided* tilting complexes out of a stable equivalence of Morita type [34]. This is an equivalence of stable module categories which is induced by tensoring with a bimodule. As is known, a derived equivalence induces a stable equivalence à la Morita [25, 16].

We are, hence, generalizing in Section 3 the results of M. Linckelmann [18] since cyclic blocks of p-adic group rings are Green orders and since for each generalized Brauer tree algebra there is a Green order reducing modulo the radical of the ground ring to that generalized Brauer tree algebra.

Our result on Green orders should be seen as an extension of M. Linckelmann's work on Broué's conjecture for integral cyclic blocks, especially without the assumption of the residue field being algebraically closed or large enough.

We finish the introduction by emphasising that the Sections 3 and 5 represent joint work with Klaus W. Roggenkamp, Section 4 reports on joint work with Steffen König and Section 6 reports on joint work with Raphaël Rouquier.

Acknowledgement: Part of the work in the last section was done while the author was guest at the École Normale Supérieure and the author wants to express his thanks to the members of the "Équipe des Groupes Finis" for their warm hospitality during his stay.

2 Preliminaries

For defining the derived category we use the notation of [37]. Let Λ be a ring. A complex is a pair (C, d) where C is a $\mathbb{Z}$–graded Λ–module and d is a $\mathbb{Z}$–graded endomorphism of C homogeneous of degree 1 with $d \cdot d = 0$. A complex of finitely generated modules is a complex such that the homogeneous components are finitely generated modules. Objects in the various categories below are always complexes satisfying additional constraints, as described below. Morphisms are complex morphisms modulo homotopy. A complex is called finitely generated if its underlying graded module is finitely generated.

- $D^b(\Lambda)$: The objects of the *derived category* are complexes of finitely generated projective Λ–modules, such that only finitely many homogeneous components of positive degree are not zero and the total homology is a finitely generated Λ–module.
- $K^b(\Lambda)$: The objects of the *homotopy category* are finitely generated complexes of finitely generated projective Λ–modules.

Obviously, $K^b(\Lambda)$ is a full subcategory of $D^b(\Lambda)$. The category of finitely generated Λ–modules embeds into $D^b(\Lambda)$ by choosing for each module a fixed projective resolution, which then can represent the module in $D^b(\Lambda)$. The definition of a projective resolution of a bounded complex is more technical and we refer for a definition to [10]. $D^b(\Lambda)$ as well as $K^b(\Lambda)$ are not necessarily exact categories. They, however, carry another structure, they are *triangulated*. For each X and Y in $D^b(\Lambda)$ and each mapping $\alpha \in Hom_{D^b(\Lambda)}(X,Y)$, we associate the *mapping cone*

$$C(\alpha) := (X[1] \oplus Y, \begin{pmatrix} -d_X & \alpha \\ 0 & d_Y \end{pmatrix}),$$

where $X[1]$ differs from X by a shift by $+1$ in the graduation. There is a natural mapping homogeneous of degree 0 from Y to $C(\alpha)$ and a natural mapping homogeneous of degree 1 from $C(\alpha)$ to X. The triangulated structure is preserved by a functor F if and only if $C(F(\alpha)) \simeq F(C(\alpha))$, with the isomorphism being natural in the obvious way.

For the tensor product there is the so called left derived tensor product functor extending the usual tensor product. This is quite technical and we refer to [10] for the definition and the most fundamental properties. However, we shall use this functor $- \otimes^L_\Lambda -$ frequently.

Following J. Rickard's fundamental paper [23] we call two rings Λ and Γ *derived equivalent* if there is a complex T, bounded above and bounded below, of finitely generated projective Λ–modules such that

1. each endomorphism of T homogeneous of degree n is homotopic to 0 if $n \neq 0$ and the endomorphism ring of T of complex morphisms homogeneous of degree 0 modulo homotopy is isomorphic to Γ,

2. and, furthermore, the smallest triangulated category generated by direct summands of finite sums of T inside the category of bounded complexes of projective Λ–modules modulo homotopy contains Λ as complex concentrated in degree 0.

T is then called a *tilting complex from* Λ *to* Γ.

J. Rickard proved in [23] that the categories $D^b(\Lambda)$ and $D^b(\Gamma)$ are equivalent as triangulated categories if and only if there is a tilting complex T from Λ to Γ.

In [25], J. Rickard proved that if Λ and Γ are derived equivalent R–orders, R being an integral domain, then there is a so called twosided tilting complex X of Λ–Γ–bimodules, such that X as complex of Λ–modules is a tilting complex from Λ to Γ and X as complex of Γ–modules is a tilting complex from Γ to Λ, and there is a complex Y of Γ–Λ–bimodules, such that

$$X \otimes_\Gamma Y \simeq \Lambda \text{ in } D^b(\Lambda \otimes_R \Lambda^{op}),$$

and

$$Y \otimes_\Lambda X \simeq \Gamma \text{ in } D^b(\Gamma \otimes_R \Gamma^{op}).$$

Here, for a ring A, the ring A^{op} is the opposite ring. The functor

$$X \otimes^L_\Gamma - : D^b(\Gamma) \longrightarrow D^b(\Lambda)$$

induces an equivalence with quasi-inverse $Y \otimes^L_\Lambda -$.

3 Green–orders

This section represents results of joint work with Klaus Roggenkamp.

In this section we give a sufficient criterion for Green–orders (cf. [30]) to be derived equivalent. The tilting complex is explicitly constructable and the method is entirely combinatorial.

In [30] a Green order is defined to clarify the structure of blocks of integral group rings with cyclic defect. For the reader's convenience we include the definition. Earlier, Wilhelm Plesken obtained results in this direction in his Habilitationsschrift [21].

Following [30] we define a Green–order Λ as the following.

Definition 1 Let R be a Dedekind domain with field of fractions K.

1. An R–order Λ is called an *isotypic order* provided there is a twosided Λ–ideal J with
   (a) J contains a K–basis of $K\Lambda$,
   (b) J is a projective left Λ–module,
   (c) Λ/J is a direct product of local R–algebras,
   (d) J is nilpotent modulo the Higman ideal[1] of Λ.
2. An R–order Λ is called a *Green order* if
   (a) there exists a finite connected tree T with vertices $\{v_i\}_{i=1,\dots n}$ and edges $e_{i,j}$ numbered in such a way that edges with numbers $e_{i,j}$ connect the vertices v_i and v_j.
   (b) A vertex corresponds to a central idempotent η_i of $K\Lambda$ with $\eta_i\cdot(\sum_{j=1}^n \eta_j) = \eta_i$ and $\sum_{i=1}^n \eta_i = 1$.
   (c) The edges correspond to indecomposable projective Λ–lattices $P_{i,j}$.
   (d) There is an embedding of the tree in the plane such that the projective resolution of $\eta_{i_0} P_{i_0,j_0}$ is given by Green's walk around the Brauer tree T [6].

Here the walk around the Brauer tree is defined formally as follows. To each vertex of the graph there is an ordering of the edges incident to the vertex by saying that the orientation is counterclockwise, say, which means that the edge e which comes before another edge f, both incident to the vertex v in the clockwise orientation of the embedding in the plane, the edge e is larger than f at v. Take now $M := \eta_{i_0} P_{i_0,j_0}$ as in the definition. A projective resolution is then a complex with homology concentrated in degree 0 and projective entries all of which are zero in positive degrees. The first projective in the projective resolution is P_{i_0,j_0}. The second is given by the ordering at the vertex j_0. Take the largest edge which is smaller and not equal to j_0. This is associated to the projective P_{j_0,j_1} and is obtained by going counterclockwise around the vertex j_0. In degree -1 we hence

[1] The Higman ideal of an R–order Γ is the R–annihilator of $Ext^1_{\Lambda\otimes_R\Lambda^{op}}(\Lambda, -)$ [32].

write P_{j_0,j_1}. The mapping between P_{i_0,j_0} and P_{j_0,j_1} is the maximal one. As next projective we take the edge which is the largest one incident to j_1 which is smaller than P_{j_0,j_1} but not equal to it, take the projective associated to it, write it in degree -2 and proceed this way. Proceeding this way we get a complex of projective modules. The resulting complex is required to have homology M in degree 0 and homology 0 elsewhere.

It is shown in [30] that an indecomposable isotypic order is Morita equivalent to an order with the following structure:

Assume first that Λ is a basic isotypic order. Let Ω be the endomorphism ring of an irreducible Λ–module. Let m be the cardinality of a complete set of orthogonal idempotents in $K\Lambda$. Then there is an element $\omega \in \Omega$ such that J is a principal ideal, generated by γ and $\gamma^m = \omega \cdot 1_{m\times m}$. Then,

$$\Lambda = (\omega \cdot \Omega)_{m\times m} + UT(\Omega, m),$$

where we denote for any ring Γ by $UT(\Gamma, m)$ the upper triangular $m \times m$ matrix with entries in Γ. The integer m is called the *size* of the isotypic order. The pair (Ω, ω) is called the *type* of the isotypic order.

A Green order is built in the following way. We fix orders Ω_i in skewfields D_i, regular elements $\omega_i \in \Omega_i$, one for each vertex and isomorphisms $f_{i,j} : \Omega_i/\omega_i \longrightarrow \Omega_j/\omega_j$, one for each edge. We abbreviate

$$\overline{\Omega} := \Omega_1/(\omega_1\Omega_1).$$

Then, we put at each vertex an isotypic order Λ_i of type (Ω_i, ω_i) and size 'number of vertices adjacent to the vertex v_i'. We identify the diagonal entries of Λ_i and Λ_j according to the following rule: We fix a walk around the Brauer tree. If a vertex v_i is passed for the k_i^{th} time, and the walk then turns to the vertex v_j going one step further, the order Λ_j is hit for the k_j^{th} time, then the k_i^{th} entry of Λ_i is identified modulo ω_i with the k_j^{th} entry of Λ_j modulo ω_j via $f_{i,j}$. In [30] Klaus W. Roggenkamp proved that this is a Green order and every Green order arises this way.

This section is devoted to prove the

Theorem 1 *Two Green orders are derived equivalent if the following data coincide:*

1. *The number of vertices,*
2. *the set*[2] *of pairs* (Ω_i, ω_i) *and*
3. *the mappings* $\Omega_i \longrightarrow \overline{\Omega}$.

3.1 The tools

We begin with an almost trivial lemmata.

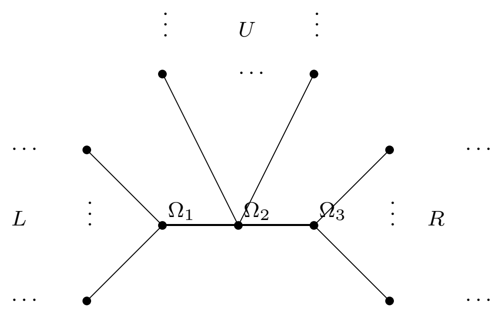
Let P and Q be non isomorphic indecomposable projective lattices. Let $P \stackrel{\phi}{\longrightarrow} Q$ be a homomorphism. An endomorphism of complexes (α, β) of $P \stackrel{\phi}{\longrightarrow} Q$ induces morphisms $(\rho, \sigma, \tau) \in End(ker\phi) \oplus End(im\phi) \oplus End(coker\phi)$.

[2] we *count* multiply occuring elements

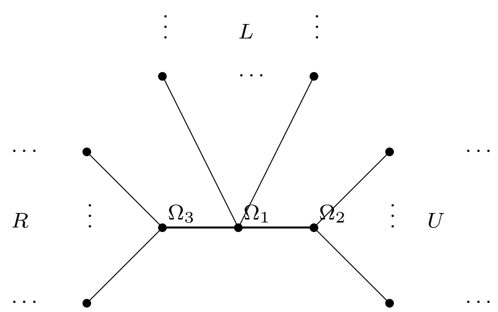
Lemma 1 *Let Λ be an order and let P and Q be indecomposable projective modules. With the above notation, if $\tau = 0$, then (α, β) is homotopic to $(\alpha', 0)$. If $\rho = 0$ and $\tau = 0$ then (α, β) is homotopic to 0.*

Proof: Since $\tau = 0$, the morphism β factors through $im\phi$, which is an irreducible sublattice of Q. But, $P \longrightarrow im\phi$ is an epimorphism, and hence, β even factors through P. This gives a homotopy annihilating β. We even showed more: The factoring annihilates σ, leaving ρ and τ untouched. This proves the lemma. ∎

Proposition 1 *Let Λ be a Green order, with data $(\Omega_i, \omega_i)_{i=1,..,n}$, $n \in I\!N$ and fixed isomorphisms $\Omega_i/(\omega_i\Omega_i) \simeq \Omega_1/(\omega_1\Omega_1)$ for $i = 1, ..., n$, associated with the following tree. (We walk around the tree counterclockwise.)*



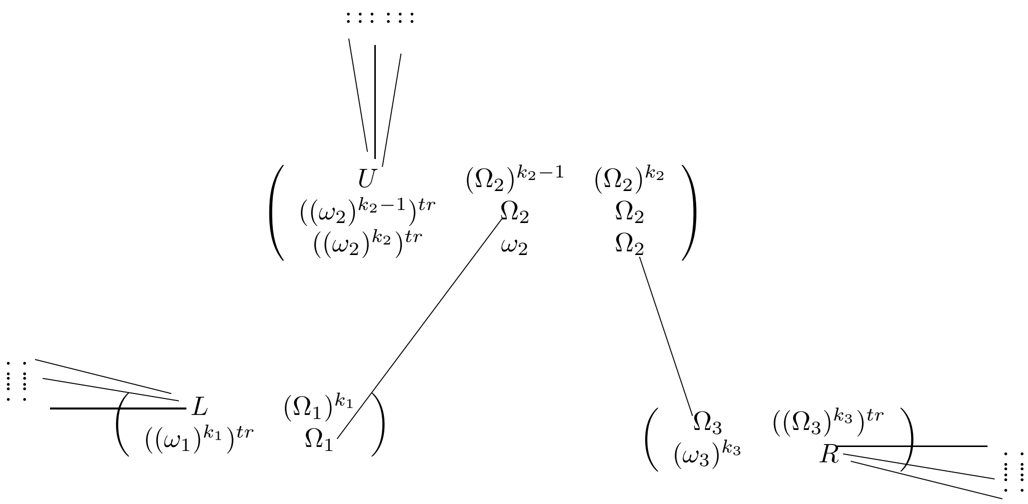
Then, the Green order Λ is derived equivalent to the Green order Γ which shares the data $(\Omega_i, \omega_i)_{i=1,..,n}$, $n \in I\!N$ and the fixed isomorphisms $\Omega_i/(\omega_i\Omega_i) \simeq \Omega_1/(\omega_1\Omega_1)$ for $i = 1, ..., n$ with Λ and which has the Brauer tree



Proof of the Proposition 1.

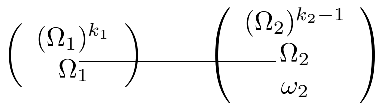
We denote the indecomposable projective Λ module linking Ω_1 with Ω_2 by P and the indecomposable projective Λ module linking Ω_2 with Ω_3 by Q. For the notation of a path in a graph we understand that as usually there is only one path between two edges.

We denote by $k_i + 1$ the number of vertices adjacent to the vertex with label Ω_i with $i = 1, 2, 3$. The order Λ then is Morita equivalent to the following:

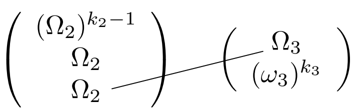


Here, the dots indicate that there are a number of other matrix rings following always linked with these lines which indicate the pullback of the orders Ω_i and Ω_j along the fixed isomorphism $f_{i,j}$ of $\Omega_i/(\omega_i\Omega_i)$ to $\Omega_j/(\omega_j\Omega_j)$.

We see that

$P =$ 

and

$Q =$ 

We form

$$\mathcal{L} := \{ \text{ vertices } i | \text{ the path from } i \text{ to } \Omega_2 \text{ passes } \Omega_1\} \setminus \{\Omega_1\},$$

$$\mathcal{R} := \{ \text{ vertices } i | \text{ the path from } i \text{ to } \Omega_2 \text{ passes } \Omega_3\} \setminus \{\Omega_3\}$$

and

$$\mathcal{U} := \{ \text{ vertices } i | \text{ the path from } i \text{ to } \Omega_2 \text{ passes neither } \Omega_1 \text{ nor } \Omega_3\} \setminus \{\Omega_2\}.$$

$$R := \bigoplus_{\binom{\Pi \text{ indecomposable projective}}{\text{involving vertices in } \mathcal{R}}} \Pi,$$

$$L := \bigoplus_{\binom{\Pi \text{ indecomposable projective}}{\text{involving vertices in } \mathcal{L}}} \Pi$$

and

$$U := \bigoplus_{\binom{\Pi \text{ indecomposable projective}}{\text{involving vertices in } \mathcal{U}}} \Pi$$

Definition of the tilting complex: Let

$$X := (0 \longrightarrow U \oplus L \oplus P \oplus P \longrightarrow Q \oplus R \longrightarrow 0)$$

with $U \oplus P \oplus P \oplus L$ in degree 0, $Q \oplus R$ in degree 1 and the mapping is the maximal one between the second copy of P and Q and zero between the other indecomposable projectives.

Proof that this defines a tilting complex: Each homomorphism from a summand of U to Q factorizes through P as is seen from the matrix representation of Λ. Each homomorphism from Q to $U \oplus P$ which has to be zero, if composed with the maximal mapping $P \longrightarrow Q$, has to be zero since all homomorphisms are settled rationally in the same simple algebra. There are no non zero mappings between L and $Q \oplus R$, since they do not share a rational component. By the same reason there is no non zero mapping between U and $L \oplus R$ as well as between P and R. Each homomorphism from P to Q has to factor via the maximal mapping $P \longrightarrow Q$, by definition.

Forming the mapping cone of the mapping

$$(0 \longrightarrow P \longrightarrow Q \longrightarrow 0) \longrightarrow (0 \longrightarrow P \longrightarrow 0 \longrightarrow 0),$$

which consists of the identity mapping in degree 0 and zero elsewhere, we get as mapping cone Q, such that decomposing the complex X into direct summands, we get that

$$P \oplus Q \oplus L \oplus R \oplus U \simeq \Lambda$$

belongs to the smallest triangulated category which contains all summands of X.

Hence, X forms a tilting complex over Λ.

The endomorphism ring of the tilting complex: We have to compute the endomorphism ring of X. We abbreviate by $K := (0 \longrightarrow P \longrightarrow Q \longrightarrow 0)$ with the maximal mapping in between and $Hom_{K^b(P_\Lambda)}(-,-) = (-,-)$.

$$End(X) = \begin{pmatrix} (U,U) & (U,P) & (U,L) & (U,K) & (U,R[1]) \\ (P,U) & (P,P) & (P,L) & (P,K) & (P,R[1]) \\ (L,U) & (L,P) & (L,L) & (L,K) & (L,R[1]) \\ (K,U) & (K,P) & (K,L) & (K,K) & (K,R[1]) \\ (R[1],U) & (R[1],P) & (R[1],L) & (R[1],K) & (R[1],R[1]) \end{pmatrix}$$

Let us examine the various sets of homomorphisms.
By different degrees

$$(U,R[1]) = (L,R[1]) = (P,R[1]) = (R[1],P) = (R[1],L) = (R[1],U) = 0.$$

By the matrix representation one sees that

$$(U,L) = (L,U) = 0.$$

Since $(P,R) = 0$, we see that $(K,R[1]) = (Q,R)$. Similarly, $(R,P) = 0$ implies $(R[1],K) = (R,Q)$. Analogously,

$$(L,K) = (L,P) \text{ and } (K,L) = (P,L),$$

Obviously,

$$(P,K) = (P, ker(P \longrightarrow Q)) \text{ and } (K,P) = (P,P)/((P,Q) \circ (Q,P)).$$

Now, $(U,K) = (U, ker(P \longrightarrow Q)) = 0$, since the modules belong to different rational components, and $(K,U) = 0$, since every morphism from P to U factorizes through Q, as one sees by the matrix representation.

We abbreviate

$$ker(P \longrightarrow Q) =: \kappa_{P,Q} \text{ and } (P,Q) \circ (Q,P) =: \Phi_{P,Q}.$$

Recall that we denote by $k_1 + 1$ the number of vertices adjacent to Ω_1 and by $k_3 + 1$ the number of vertices adjacent to Ω_3. The endomorphism ring now is

$$End(X) \simeq \begin{pmatrix} (U,U) & (U,P) & 0 & 0 & 0 \\ (P,U) & (P,P) & (P,L) & (P,\kappa_{P,Q}) & 0 \\ 0 & (L,P) & (L,L) & (L,P) & 0 \\ 0 & (P,P)/(\Phi_{P,Q}) & (P,L) & (K,K) & (Q,R) \\ 0 & 0 & 0 & (R,Q) & (R,R) \end{pmatrix}$$

We now use the special structure of a Green order and see that

$$(P,P)/\Phi_{P,Q} = \Omega_1 \text{ as well as } (P,\kappa_{P,Q}) = \omega_1 \cdot \Omega_1.$$

Furthermore, denoting by $k_2 + 2$ the number of vertices adjacent to Ω_2,

$$(P,U) = (\Omega_2^{k_2})^{tr} \text{ and } (U,P) = (\omega_2 \cdot \Omega_2)^{k_2}.$$

The endomorphism ring of P is a pullback

$$\begin{array}{ccc} (P,P) & \longrightarrow & \Omega_1 \\ \downarrow & & \downarrow \\ \Omega_2 & \longrightarrow & \overline{\Omega} \end{array}$$

which we abbreviate as usual as $\Omega_1 - \Omega_2$. We look at the endomorphism ring of K more closely. The endomorphism ring as complex is just $\Omega_1 - \Omega_2 - \Omega_3$ and we factor the middle term by Lemma 1. Thus,

$$(K,K) = \Omega_1 - \Omega_3.$$

The equations

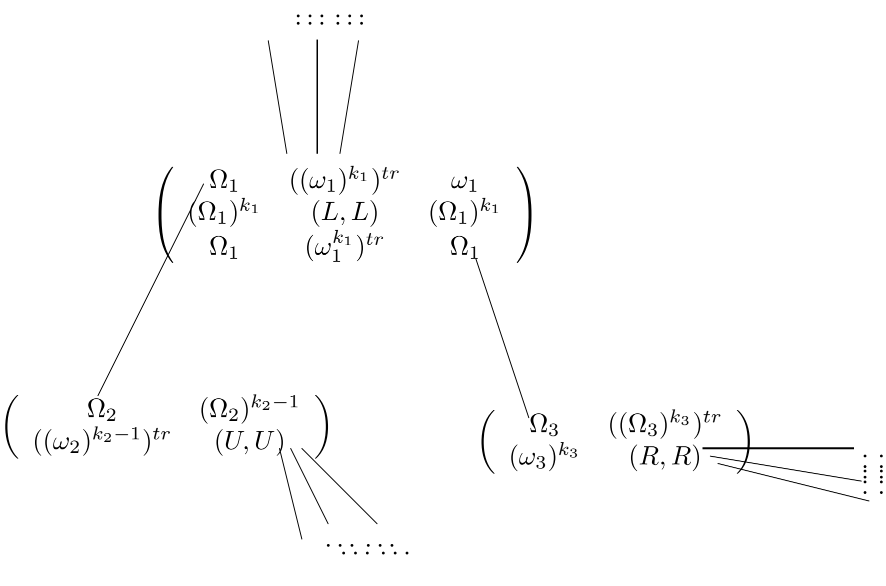
$$(L,P) = \Omega_1^{k_1},\ ((P,L) = (\omega_1 \cdot \Omega_1)^{k_1})^{tr},\ (R,Q) = (\omega_3 \cdot \Omega_3)^{k_3},\ (Q,R) = (\Omega_3^{k_3})^{tr}$$

are also immediate from the description of Λ as above.

Now,

$$End(X) \quad \simeq \quad \begin{pmatrix} (U,U) & (\omega_2 \cdot \Omega_2)^{k_2} & 0 & 0 & 0 \\ (\Omega_2^{k_2})^{tr} & \Omega_2 - \Omega_1 & ((\Omega_1)^{k_1})^{tr} & (\omega_1 \cdot \Omega_1) & 0 \\ 0 & \Omega_1^{k_1} & (L,L) & (\Omega_1)^{k_1} & 0 \\ 0 & \Omega_1 & (\omega_1 \cdot \Omega_1^{k_1})^{tr} & \Omega_1 - \Omega_3 & (\Omega_3^{k_3})^{tr} \\ 0 & 0 & 0 & (\omega_3 \cdot \Omega_3)^{k_3} & (R,R) \end{pmatrix}$$

This results then is a Green order which looks like the following:



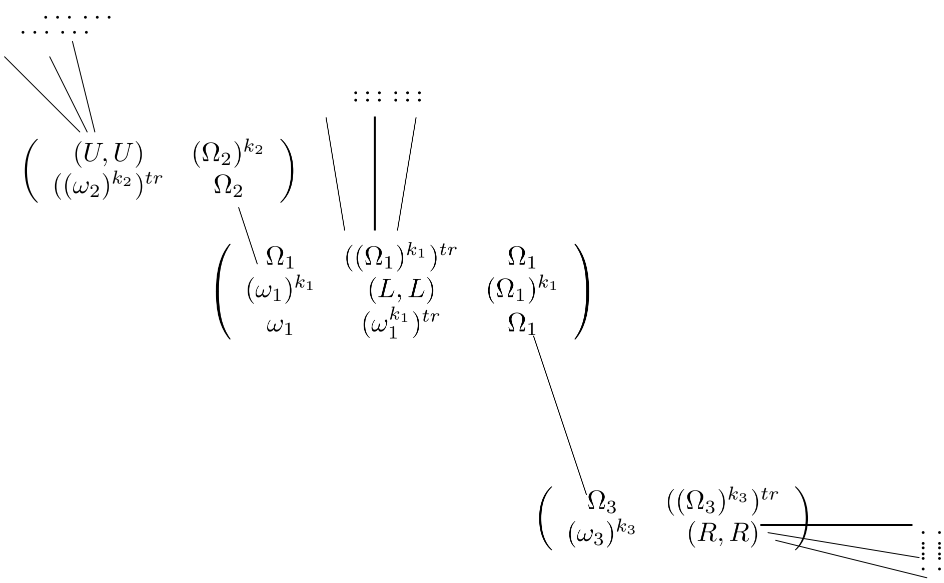
In the three by three matrix ring we change second and the last row and column and then the first and the last row and column. In the matrix ring corresponding to the vertex where Ω_2 is attached to, we conjugate by a suitable power of

$$\begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & 1 \\ \omega_2 & 0 & \dots & \dots & 0 \end{pmatrix}$$

and in the matrix where Ω_1 is attached to we conjugate by

$$\begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & 1 \\ \omega_1 & 0 & \dots & \dots & 0 \end{pmatrix}$$

and we get the following Greenorder:



which is Morita equivalent to the one we claimed it to be. ■

3.2 The theorem and its proof

Recall the theorem.

Theorem 1 *Two Green orders are derived equivalent if the following data coincide:*

1. *The number of vertices,*
2. *the pairs* (Ω_i, ω_i) *and*

3. the mappings $\Omega_i \longrightarrow \overline{\Omega}$.

This result generalizes the result [18] of Linckelmann considerably since we cover all Green orders and we do not need the ground ring to have an algebraically closed residue field.

The proof illuminates the derived equivalence given in [18] and shows how to get a tilting complex since it is entirely constructive and combinatorical. This is the case since if Λ_1 is a Green order and T_1 is a tilting complex over Λ_1 with endomorphism ring a Green order Λ_2 and if T_2 is a tilting complex over Λ_2 with endomorphism ring a Green order Λ_3, it is not so difficult, using certain mapping cone constructions, to obtain a tilting complex T_3 over Λ_1 with endomorphism ring Λ_3.

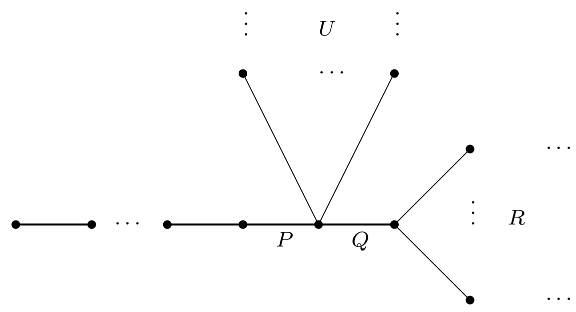
The proposition hence gives an inductive procedure to obtain a tilting complex over a Green order Λ which has endomorphism ring Γ if Λ and Γ share the same data, in other words fit into the framework of the theorem.

The theorem also implies part of the result of F. Membrillo–Hernandez [20], since the ramification index of Ω_i over R gives the multiplicity of the corresponding exceptional vertex, if one reduces modulo the radical of the ground ring.

Proof of the theorem:

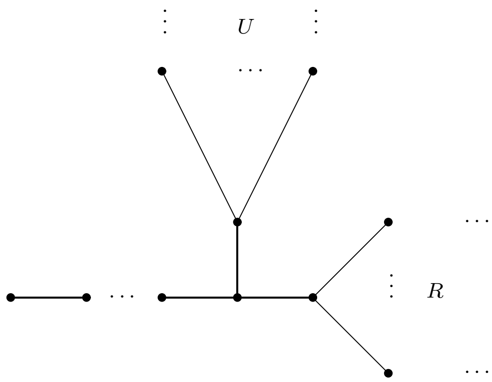
We reduce the number of vertices of valencity[3] greater than two in the tree.

1) Choose any leaf in the tree. From that leaf walk along the Brauer tree walk until one encounters a vertex with multiplicity greater than two. The indecomposable projective just before is P, the indecomposable projective attached to the vertex immediately after P is Q. L, R and U are now defined, using the notation in our tilting complex X in the proof of the proposition. We are in the following situation:



2) The endomorphism ring of this tilting complex is a Green order with one edge less on the line which starts from the leaf we started from. Its graph looks as the following:

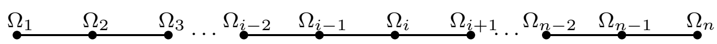
[3]The valencity of a vertex in a Green order is the number of edges which are adjacent to the vertex



Since the number of edges is an invariant under derived equivalences, we obtain by induction a stem using the inverse of that equivalence.

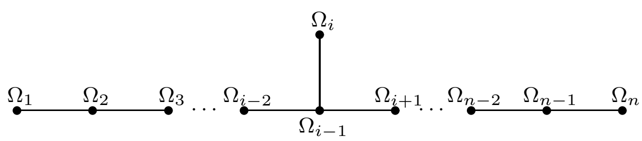
3) We are left to show that for a stem we can permute the various (Ω_i, ω_i). In fact this can be done by the tilting complex above.

We are given the following stem:



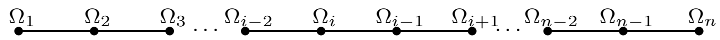
In our procedure we may choose for P the indecomposable projective linking Ω_1 and Ω_2 and for Q the indecomposable projective linking Ω_2 and Ω_3. The result is a stem with Ω_1 and Ω_2 interchanged, hence, the permutation $(1, 2)$ as element of the symmetric group S_n of degree n acting in the obvious way on our stem. By symmetry, we also get $(n-1, n)$.

Let P be the indecomposable projective linking the vertices $i-1$ and i and let Q be the indecomposable projective linking i and $i+1$. The procedure tilts then the above order to the order $\tilde{\Lambda}$ associated to the tree



We choose now, in $\tilde{\Lambda}$, as P the projective indecomposable linking Ω_i with Ω_{i-1} and as Q the projective indecomposable linking Ω_{i-2} and Ω_{i-1}.

The result is a Green order, again denoted by $\tilde{\Lambda}$ to the tree



This enables us to interchange Ω_{i-1} and Ω_i. Hence, we get the involutions $(i-1, i)$ of S_n for all $i \in \{3, ..., n-1\}$.

Since we can obtain now the involutions $(i, i+1)$ for all $i \in \{1, .., n-1\}$, the involutions obtained so far generate the symmetric group of degree n. ■

4 Hereditary orders

This section is a summary on joint work with Steffen König.

The paper [17] describes all tilting complexes over hereditary R–orders Λ combinatorially with R being a complete discrete rank 1 valuation domain. There we then compute their endomorphism rings using that knowledge. Let us begin with some observations on hereditary orders and hereditary rings.

Lemma 2 *Let Λ be a hereditary ring. Then, each bounded, indecomposable complex of finitely generated projective Λ–modules is isomorphic in $D^b(\Lambda)$ to a module.*

Proof.

Let X be a bounded, indecomposable complex of finitely generated projective modules. The largest degree with non zero homology is n, say, and let P be an indecomposable summand of X_n, the degree n homogeneous part of X. Then, $d_n : X_{n-1} \longrightarrow X_n$ is the differential. $d_n(d_n^{-1}(P)) =: U$, may be assumed to be a proper submodule of P. U is projective. Since X_{n-1} is projective, U is isomorphic by a mapping ϕ to a direct summand V of X_{n-1}, and d_n restricted to V is an isomorphism (with inverse ϕ) to U. We claim that $0 \longrightarrow U \stackrel{\iota}{\longrightarrow} P \longrightarrow 0$ with P being in degree n, and with ι being the natural embedding, is a direct summand of X. We map P to X_n by its natural mapping σ and U is isomorphic to V, a summand of X_{n-1}. By Definition,

$$\iota\sigma = \phi d_n$$

and, denoting by π_σ the splitting of σ,

$$\phi^{-1}\iota = d_n \pi_\sigma.$$

Furthermore, $d_{n-1}\phi^{-1} = 0$, since $\ker d_n \cap V = 0$.

Throughout this section let R be a complete discrete rank 1 valuation domain with field of fractions K, residue field k and prime ideal p. It is well known (cf. [2, 9, 22]) that any indecomposable hereditary R–order Λ is Morita equivalent to one described below. There is a division ring D, finite dimensional over K, such that for the maximal order Δ in D and a prime element π_D of Δ the hereditary order is Morita equivalent to one of the following form.

$$\Lambda = \begin{pmatrix} \Delta & \dots & \dots & \dots & \Delta \\ (\pi_D) & \ddots & & & \vdots \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ (\pi_D) & \dots & \dots & (\pi_D) & \Delta \end{pmatrix}_{n\times n} .$$

Here we denote by (π_D) the Δ–ideal generated by π_D.

Taking left modules, the isomorphism classes of indecomposable projective Λ–lattices are numbered $P(i)$ (where $i \in \mathbb{Z}/n\mathbb{Z}$) in such a way that $rad(P(i+1)) \simeq P(i)$. Thus, $P(i)$ may and will be chosen as the i–th column in the above matrix.

Let $T : 0 \longrightarrow P(k) \longrightarrow P(m) \longrightarrow 0$ be an indecomposable direct summand of a tilting complex, with homology in degree i, say. Let

$$\begin{array}{ccccccc} 0 & \longrightarrow & P(k) & \longrightarrow & P(m) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & 0 & \longrightarrow & H^i(M) & \longrightarrow & 0 \end{array}$$

be the natural projection, which is by Lemma 2 an isomorphism in $D^b(\Lambda)$.

Let T be as above. The differential $P(k) \longrightarrow P(m)$ is realized by an element $a \in \Delta$ by the structure of Λ as described above. We denote the valuation on D again by v. We claim that the valuation of a is the minimum of $\{v(b)|P(k)\cdot b \leq P(m)\}$. In fact, if $v(a) \neq min\{v(b)|\ \exists\, b \in D : P(k)\cdot b \leq P(m)\}$, then let b_0 be an element with minimal valuation. We get a non trivial mapping

$$\begin{array}{ccccccccc} 0 & \longrightarrow & 0 & \longrightarrow & P(k) & \longrightarrow & P(m) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \cdot b_0 & & \downarrow & & \\ 0 & \longrightarrow & P(k) & \longrightarrow & P(m) & \longrightarrow & 0 & \longrightarrow & 0 \end{array}$$

which is not homotopic to zero.

Corollary 1 *An indecomposable summand of a tilting complex T is determined up to isomorphism by a triple (P, Q, i) of projective indecomposable modules (P might be 0 there) and a degree i where the homology is concentrated on.*

Definition 2 An interval $(i + n\mathbb{Z}, j + n\mathbb{Z})$ with $i, j \in \mathbb{Z}$ is the smallest non empty image of intervals $(i + nz_1, j + nz_2)$ with $z_1, z_2 \in \mathbb{Z}$ under the natural projection $\mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z}$. That means

$$(i + n\mathbb{Z}, j + n\mathbb{Z}) := \{k|\exists d \in \mathbb{Z} : i < k < j + dn, \frac{(j-i)}{n} < d \leq 1 + \frac{(j-i)}{n}\}.$$

With the natural identification we use that notation for intervals of projective modules: For $i, j \in \mathbb{Z}/n\mathbb{Z}$, $(P(i), P(j)) := (i, j)$. We, furthermore, denote by $[i, j] := (i, j) \cup \{i, j\}$ and $[P(i), P(j)] := [i, j]$.

According to this definition an interval $(P(i), P(j))$ is never empty.

To formulate our results we need to define two more objects:

Definition 3 The set $\mathfrak{TC}$ contains the equivalence classes of the multiplicity free tilting complexes over Λ modulo isomorphisms and modulo shift in the derived category $D^b(\Lambda - mod)$.

Fix a circle in the plane and n different vertices on the circle, numbered clockwise by $1, \ldots, n$, identified in the natural way with elements of $\mathbb{Z}/n\mathbb{Z}$.

Definition 4 The elements of the set $\mathfrak{C}$ are quadruples (n_1, n_2, M_1, M_2) satisfying the following conditions:

(a) Both n_1 and n_2 are integers, n_1 is positive, n_2 is non–negative and the sum $n_1 + n_2$ equals n (the size of the hereditary order),

(b) the set M_1 contains n_1 of the fixed vertices (which in the following will be called 'stars'),

(c) the set M_2 contains n_2 ordered pairs (i, j) of fixed vertices (which in the following will be seen as 'arrows' going from i to j and which will be written $i \longrightarrow j$),

(d) **(0)** there is a star at the vertex 1; if i is the starting vertex of an arrow ending at 1 and j is the ending vertex of an arrow starting at 1, then the intervals $(1, j]$ and $[i, 1)$ have empty intersection,

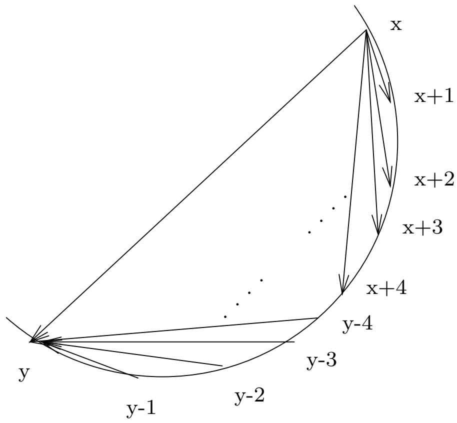
- **(I)** two different arrows (drawn as straight lines) do not intersect in interior points of the circle,
- **(II)** viewing the union of the arrows as an (non oriented) graph, it is the disjoint union of n_1 trees, each of them containing exactly one vertex contained in M_1,
- **(III)** if there is an arrow leading from i to j, then there is no star strictly between i and j.

Theorem 2 *[17] There is a bijection between the sets $\mathfrak{TC}$ and $\mathfrak{C}$.*

We shall elaborate more on this bijection in a moment. It turns out that this bijection is very explicit.

Using the rules (0) to (III) it turns out to be convenient to introduce the notation of a *cascade of fans.*

A fan is attached to an arrow $a = (v_\alpha \longrightarrow v_\omega)$, called the basic arrow, with either $v_\omega = 1$ or else $v_\alpha < v_\omega$, and consists of a set S_e of arrows all ending in v_ω and a *disjoint* set S_b of arrows all starting in v_α. Furthermore, the arrows in S_e have beginning vertex larger than v_α and the arrows in S_b have ending vertex smaller than v_ω. The smallest vertex that occurs as a beginning vertex of an arrow in S_e is strictly larger than the largest vertex that occurs as an ending vertex of an arrow in S_b, i.e. there is no crossing. So, a fan looks like the following:



where $x = v_\alpha$ and $y = v_\omega$. Since the elements of a fan are arrows, each of these arrows itself can serve as a basic arrow a of another fan. The resulting figure is called cascade of fans if for each basic arrow a there is no directed path from the beginning vertex of a to the ending vertex of a besides the basic arrow itself, i.e. if the underlying graph is a tree. The largest interval $[i_l, i_r]$ such that there is a path from i_l to i_r only using arrows belonging to the cascade of fans is called the interval the cascade of fans is based on. A cascade of fans is called complete if for all basic arrows a of a fan in the cascade between the beginning vertex b and the ending vertex e all the vertices in the interval (b, e) are ending or beginning vertices of an arrow.

Lemma 3 *A connected component of a combinatorial object in $\mathcal{C}$ is a complete cascade of fans. Conversely, given a complete cascade of fans, one can choose a star at a vertex belonging to the interval, the cascade of fans is based on, and this then gives rise to a connected component of an object of $\mathcal{C}$. A set of cascades of fans in pairwise disjoint intervals give rise to an element of $\mathcal{C}$.*

The procedure to get the tilting complexes is now very instructive.

A vertex with number k represents $P(k)$. An arrow from l to k represents an indecomposable summand $P(l) \longrightarrow P(k)$. A star at a vertex k represents an indecomposable summand $P(k)$. The graduation is then prescribed by the following procedure.

We take the circle and divide the circle into segments such that each complete cascade of fans belongs to exactly one segment. We cut the circle line at the boundaries of the segments and pull the circle segments to a straight line. Stars are adjusted to degree 0. The arrows all point in direction left to right. Replacing stars by projectives and arrows by torsion modules such that the degree is determined 'by the place the arrow is standing'. That means, that inductively if $P(k)$ is adjusted to degree d_k, say, then, for an arrow $l \longrightarrow k$ we put $P(l)$ at degree $d_k - 1$, for an arrow $k \longrightarrow l$, we put $P(l)$ at degree $d_k + 1$. Inductively, we place each indecomposable summand of the tilting complex at a certain degree.

For a given cascade of fans we are able to give the endomorphism ring of the corresponding tilting complex explicitly.

We form for each vertex v a subring of a matrix ring of size m where m is the number of arrows ending at v plus the number of arrows starting at v plus the number of stars attached to v (=0 or 1). We form a upper triangular matrix with entries in $\overline{\Delta} := \Delta/(\pi_D)$. We shall define three integers b_v, s_v, e_v associated to v in the following. The $m = b_v + s_v + e_v$ diagonal entries are numbered consecutively with

1. firstly the b_v arrows emanating from v with increasing ending vertex,

2. secondly the number s_v of stars, (0 or 1), if present,

3. thirdly the e_v arrows ending at v, with increasing beginning vertex.

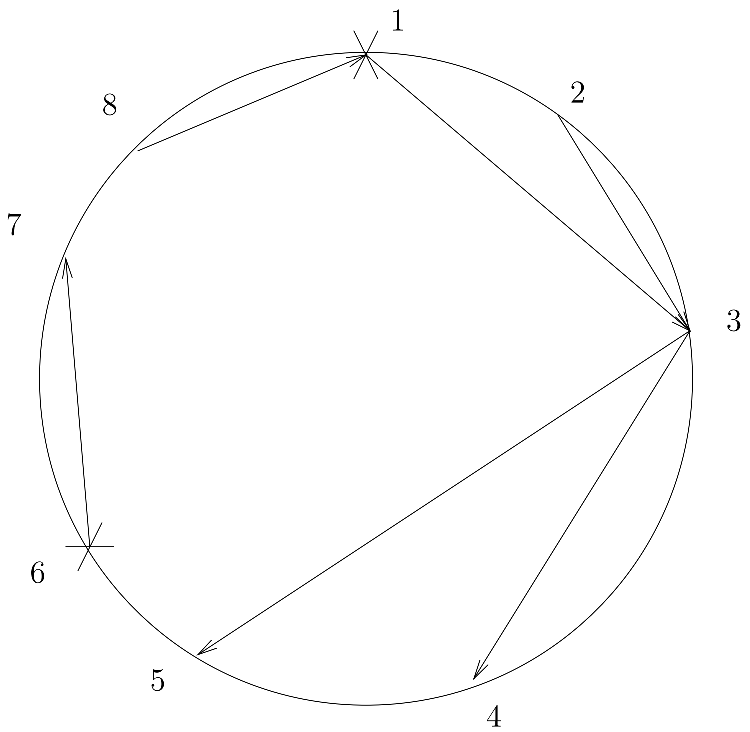
This way we associate to each arrow and each star a diagonal element in a matrix, in fact to each arrow we associate two matrix diagonal elements in two matrices and to each star we associate one matrix diagonal element.

We add a hereditary order (in its presentation as we described at the beginning of the section as we introduced the structure theorem) with entries in Δ of Δ–dimension k^2 with k being the number of stars.

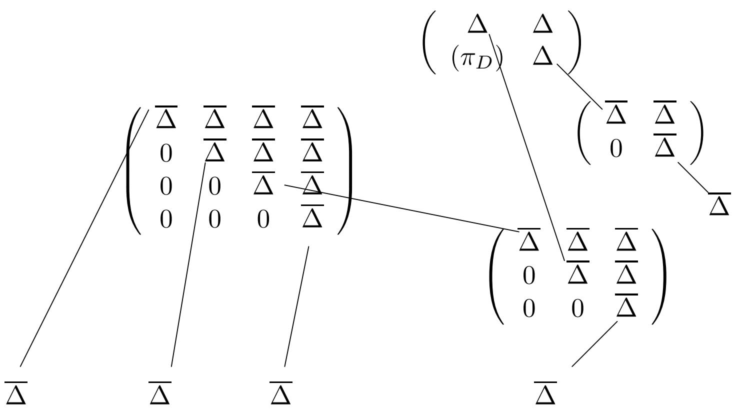
Therefore, also to each star there is attached two diagonal elements in two matrices.

We identify the diagonal entries of the hereditary order with the vertices stars are attached to, in increasing index from left upper to right lower. We then form the subring of the sum of the matrix rings defined by identifying modulo (π) the diagonal entries of the various matrices, corresponding to equal arrows or stars. (Observe that each arrow is counted twice, in the matrix corresponding to its beginning vertex and in the matrix corresponding to its ending vertex. Observe furthermore, that each star is also counted twice, in the hereditary order and in the matrix of the vertex it is attached to.)

Example 1 We have for example the following cascade of fans for a hereditary order with 8 indecomposable projectives.



The endomorphism ring here is as follows. The line between two rings indicates the image of the diagonal embedding into the two copies of the ring.

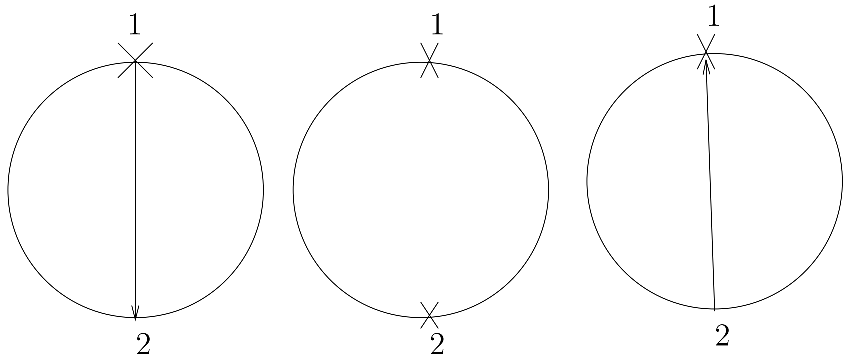


We give a corollary.

Corollary 2 *Let Λ be as above with more than one indecomposable projective module and let Γ be derived equivalent to Λ. Then, the ring $\Lambda/((rad\ R)\Lambda)$ is not derived equivalent to $\Gamma/((rad\ R)\Gamma)$ if the corresponding tilting complex corresponds to a cascade of fans with only one star.*

In fact, $\Lambda/((rad\ R)\Lambda)$ has infinite global dimension, but $\Gamma/((rad\ R)\Gamma)$ then has not. ∎

Example 2 Let $n = 2$. Then, we have the three possible cascade of fans.



The endomorphism rings are defined to be Γ_1, Γ_2 and Γ_3. We state without proof that all of the tilting complexes attached to them have a twosided tilting complex in the sense of Rickard [25]. This means, there is a complex X of $\Lambda - \Gamma_i$–bimodules and a complex Y of $\Gamma_i - \Lambda$-bimodules such that $X \otimes^L_{\Gamma_i} Y \simeq \Lambda$ and $Y \otimes^L_\Lambda X \simeq \Gamma_i$ as complexes of bimodules.

In the first case the complex

$$X^{[1]} := \left(\ldots \longrightarrow 0 \longrightarrow \begin{pmatrix} k & 0 \\ \Delta & \Delta \end{pmatrix} \longrightarrow 0 \longrightarrow \ldots\right)$$

with homology concentrated in degree 0 is a tilting complex of bimodules, even a tilting module.

$$X^{[1]} \otimes^L_{\Gamma_1} - : D^b(\Lambda) \longrightarrow D^b(\Gamma_1)$$

induces a derived equivalence between Γ_1 and the hereditary order Λ. The complex

$$X^{[-1]} := \left(\ldots \longrightarrow 0 \longrightarrow \begin{pmatrix} 0 & \Delta \\ 0 & \Delta \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 0 \\ k & k \end{pmatrix} \longrightarrow 0 \longrightarrow \ldots\right)$$

with homology in degree 0 and 1 is a tilting complex from either side, and the functor

$$X^{[1]} \otimes^L_{\Lambda} - : D^b(\Gamma_1) \longrightarrow D^b(\Lambda)$$

is a quasi-inverse to tensoring with $X^{[-1]}$.

In the second case, we have a Morita equivalence.

In the third case, we get a complex

$$X^{[3]} := \left(\ldots \longrightarrow 0 \longrightarrow \begin{pmatrix} \Delta & 0 \\ \Delta & k \end{pmatrix} \longrightarrow 0 \longrightarrow \ldots\right)$$

concentrated in degree 0, such that

$$X^{[3]} \otimes^L_{\Gamma_3} - : D^b(\Gamma_3) \longrightarrow D^b(\Lambda)$$

induces a derived equivalence. The complex

$$X^{[-3]} := \left(\ldots \longrightarrow 0 \longrightarrow \begin{pmatrix} \Delta & \Delta \\ 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} k & 0 \\ k & 0 \end{pmatrix} \longrightarrow 0 \longrightarrow \ldots\right)$$

with homology in degree 0 and 1 is a tilting complex with

$$X^{[-3]} \otimes_\Lambda^L - : D^b(\Lambda) \longrightarrow D^b(\Gamma_3)$$

inducing a quasi-inverse to tensoring with $X^{[3]}$.

Acknowledgement: After the author reported to Bernhard Keller about the twosided tilting complex in the direction that fits in the framework of Keller's theorem, that is $X^{[3]}$ exists by this theorem [12] in the last of the above cases, Bernhard Keller suspected that this might have an inverse. He, independently from our attempt, discovered the inverse complex in this case. We are grateful to Bernhard Keller for persuading us to try to prove that the complexes are in fact invertible.

5 Local Orders

This section represents result of joint work with Klaus Roggenkamp.

The next theorem states that a twosided tilting complex between local, symmetric orders has homology concentrated in a single degree.

Theorem 3 *Let R be a Dedekind domain of characteristic 0 and let Λ and Γ be local R–orders. If Λ is derived equivalent to Γ, then Λ is even Morita equivalent to Γ. Moreover, any tilting complex over Λ has homology concentrated in a single degree.*

Corollary 3 *The derived category of a group ring of a p-group G over any complete Dedekind domain, which is a finite extension of the p-adic integers, determines G up to isomorphism.*

Remark 1

1. Theorem 3 does not use [33, 31], while Corollary 3 heavily does.
2. Using G.Thompson's result [36] there is an immediate generalization to the following situation. Let S be a complete discrete valuation domain of characteristic 0 in which a rational prime p is not invertible. Let A be a local S–algebra which is finitely generated as S–module and let G be a finite p–group. Then, the derived equivalence class of $A \otimes_S SG$ determines G up to isomorphism as group basis. In other words, given groups G and H as above and S and A as required above, then the derived categories of $A \otimes_S SG$ and of $A \otimes_S SH$ are equivalent if and only if G and H are isomorphic.
3. J. Rickard gave independently a different proof of Theorem 3, which indeed uses only that Λ is local ([29]).

4. M. Linckelmann proves Theorem 3 independently in case of a p–adic group ring of a p–group [19]. More precisely: Let R be a complete discrete rank 1 valuation ring of characteristic 0 with residue field of characteristic p. Let P be a p–group. Then, for any group H, if RP and RH are stable equivalent à la Morita (see e.g.[25]), then $P \simeq H$. Since by J.Rickard's result in [25] derived equivalent self injective rings are stably equivalent à la Morita, Theorem 3 follows.

Proof of Corollary 3: By Theorem 3 the group rings are Morita equivalent. Using the result [33, 31] the groups are isomorphic. ■

Proof of Theorem 3: We abbreviate $-^* := Hom_\Lambda(-,\Lambda)$. Let Γ and Λ be derived equivalent. Then, by Rickard's theorem [25] there is a twosided tilting complex

$$\mathcal{C} : ... \longrightarrow 0 \longrightarrow X_0 \stackrel{\partial_0}{\longrightarrow} X_1 \longrightarrow ... \longrightarrow X_{n-1} \stackrel{\partial_{n-1}}{\longrightarrow} X_n \longrightarrow 0 \longrightarrow ...$$

of $\Lambda - \Gamma$–bimodules such that

$$\mathcal{C}^* \otimes_\Lambda \mathcal{C} \simeq \Gamma$$

as complex of $\Gamma - \Gamma$–bimodules and

$$\mathcal{C} \otimes_\Gamma \mathcal{C}^* \simeq \Lambda$$

as complex of $\Lambda - \Lambda$–bimodules. Here, we already used that by shifting to an appropriate degree, which does not harm the property of being a twosided tilting complex between Λ and Γ, we may assume that the smallest degree with non zero entry is 0.

We assumed to have orders. Therefore, we have to deal with a tilting complex X whose homogeneous components X_i are projective on either side by J.Rickard's theorem [25]. Since now, all of the X_i are projective on either side, we may assume that ∂_0^* is not surjective and ∂_{n-1} is not surjective. In fact, X_{n-1} may be replaced by $ker\partial_{n-1}$ and X_n by 0, analogously for the dual complex.

We form the complex

$$\mathcal{C} \otimes_\Gamma \mathcal{C}^* : ... \longrightarrow 0 \longrightarrow X_0 \otimes X_n^* \stackrel{\partial_0 \otimes 1 \oplus 1 \otimes \partial_{n-1}^*}{\longrightarrow} X_1 \otimes_\Gamma X_n^* \oplus X_0 \otimes_\Gamma X_{n-1}^* \longrightarrow ...$$

$$... \longrightarrow X_{n-1} \otimes_\Gamma X_0^* \oplus X_n \otimes_\Gamma X_1^* \stackrel{\partial_{n-1} \otimes 1 + 1 \otimes \partial_0^*}{\longrightarrow} X_n \otimes_\Gamma X_0^* \longrightarrow 0 \longrightarrow ...$$

We look at the right end of the complex. Since ∂_{n-1} and ∂_0^* both are not surjective, their images have a non trivial Γ direct summand S and T^* respectively that belongs to the radical of X_n and the radical of X_0^* respectively.

We forget the Λ structure for the moment and concentrate only on the Γ structure of X_n and X_0^*.

We use [7, (6.2)], varying ∂_0^* and ∂_{n-1} *as* Γ*–module homomorphism*, to assume that

$$X_n = X_n^0 \oplus X_n^1$$

with $X_n^0 \leq im(\partial_{n-1})$ and $im(\partial_{n-1})/X_n^0 \leq rad(X_n^1)$ as Γ-modules. Similar notations are used for $X_0^* = (X_0^*)^0 \oplus (X_0^*)^1$. With this notation,

$$X_n \otimes_\Gamma X_0^* = X_n^0 \otimes_\Gamma (X_0^*)^0 \oplus X_n^0 \otimes_\Gamma (X_0^*)^1 \oplus X_n^1 \otimes_\Gamma (X_0^*)^0 \oplus X_n^1 \otimes_\Gamma (X_0^*)^1.$$

Then, we realize that

$$X_n^0 \otimes_\Gamma (X_0^*)^0 \leq im(\partial_{n-1} \otimes 1 + 1 \otimes \partial_0^*)$$

and

$$im(\partial_{n-1} \otimes 1 + 1 \otimes \partial_0^*)/(X_n^0 \otimes_\Gamma (X_0^*)^0) \neq (X_n \otimes_\Gamma X_0^*)/(X_n^0 \otimes_\Gamma (X_0^*)^0)$$

Therefore, $\partial_{n-1} \otimes 1 + 1 \otimes \partial_0^*$ is not surjective and the homology at the end position of $\mathcal{C} \otimes_\Gamma \mathcal{C}^*$ is not zero.

But the complex $\mathcal{C} \otimes_\Gamma \mathcal{C}^*$ is quasi-isomorphic to the complex centered at 0 with homology Λ. But then $n = 0$.

This proves that $\mathcal{C}$ is isomorphic to a progenerator, hence, $\mathcal{C}$ induces a Morita equivalence.

This proves the theorem. ∎

6 Picard groups of derived categories

This section reports on joint work with Raphaël Rouquier.

We define an extension of the Picard group of an order to derived categories and we prove some elementary properties.

Definition 5 Let R be a commutative ring and let A be an R–algebra, which is projective as an R–module. Then, we call a twosided tilting complex ${}_AX_A$ with

$$End_A({}_A|X) \simeq End_A(X|_A) \simeq A$$

a <u>twosided autotilting complex</u> of A. The set of isomorphism classes of twosided autotilting complexes of A forms a group under $- \otimes_A^L -$. This group is called $TrPic(A)$.

Remark 2 The neutral element is A as complex concentrated in degree 0, by definition of a twosided tilting complex, for each twosided autotilting complex X there is an inverse complex Y with $X \otimes_A Y \simeq A$ and $Y \otimes_A X \simeq A$ as complexes of bimodules. We have associativity since under our assumptions each projective bimodule is projective on either side and the left derived tensor product then reduces to the ordinary tensor product. The condition on A may be weakened but we do not elaborate on this here.

Now, Λ is an R–order in a semisimple artinian algebra A, with R being an integral domain. The field of fractions of R is K. Then, if X is a twosided autotilting complex of Λ, also $K \otimes_R^L X$ is a twosided autotilting complex of $K \otimes_R \Lambda$. Moreover,

$$K \otimes_R^L - : TrPic(\Lambda) \longrightarrow TrPic(K \otimes_R \Lambda)$$

is a group homomorphism.

Definition 6 We define for $\Lambda,\ K,\ R$ as above

$$TrI(\Lambda) := ker(K \otimes_R^L -).$$

Remark 3 For any ring B, let X be the module B, viewed as complex with homology concentrated in degree 1, then X generates an infinite cyclic group C_∞ in $TrPicent(B)$. In the same manner, if a ring

$$B \simeq \prod_{i=1}^{n} B_i,$$

decomposes into non zero factors B_i and $n \in \mathbb{N}$, then, the complexes with B_k in degree 1 and B_j in degree 0 for a $k \in \{1, \ldots, n\}$ and all $j \in \{1, \ldots, n\} \setminus \{k\}$, generate a subgroup $Sh(B) :\simeq \prod_{i=1}^{n} C_\infty$ of $TrPic(B)$.

Note 1 For a semisimple artinian algebra A, since all modules are projective, hence also the homology of a twosided (auto-)tilting complex is projective, each twosided autotilting complex of A is isomorphic to its homology. Being derived equivalent to a semisimple artinian algebra A is the same as being Morita equivalent to A. Each twosided (auto-)tilting complex is, modulo $Sh(A)$ isomorphic to a bimodule inducing a Morita equivalence.

We now look again to our R-order Λ in the semisimple K-algebra A.

Multiplication by elements c in the centre C of Λ from the right, provides an endomorphism of X. The endomorphism ring of X is isomorphic to Λ itself, by left multiplication with Λ. Hence, for each twosided autotilting complex X there is a ring homomorphism

$$\phi_X : C \longrightarrow \Lambda$$

sending multiplication by an element c from the right to multiplication by $\phi_X(c)$ from the left. By Note 1, ϕ_X depends only on the isomorphism class of X. We have

$$\begin{aligned}(\lambda\phi_X(c)) \cdot - &= (\lambda(\phi_X(c) \cdot -)) \\ &= (\lambda \cdot (- \cdot c)) \\ &= ((\lambda \cdot -) \cdot c) \\ &= (\phi_X(c)\lambda) \cdot -,\end{aligned}$$

and therefore, ϕ_X defines an endomorphism of the centre C of Λ. Furthermore, ϕ_X is multiplicative with respect to the twosided autotilting complex using again 1. This means that for any two twosided autotilting complexes X and Y of Λ we have

$$\phi_{X \otimes_\Lambda^L Y} = \phi_X \circ \phi_Y.$$

Multiplication from the left by elements of C defines similarly an endomorphism ψ_X of C, which turns out to be the inverse of ϕ_X. ϕ defines therefore a mapping

$$\phi : TrPic(\Lambda) \longrightarrow Aut(\text{ centre } (\Lambda))$$

Definition 7 We define for Λ and ϕ as above

$$TrPicent(\Lambda) := ker\phi.$$

Remark 4 1. The Definitions 5, 6 and 7 generalize the analogous definitions for $Pic(-)$ and $Picent(-)$ to derived auto equivalences of module categories in the sense that the classical objects embed into the analogue for derived equivalences by just viewing the corresponding module as complex concentrated in degree 0. We have for any ring A that

$$Pic(A) \subseteq TrPic(A)$$

and

$$Picent(A) \subseteq TrPicent(A).$$

2. $I(-)$ distinguishes the various isomorphic modules, $TrI(-)$, however, does not.

A. Fröhlich developed in [5] a sophisticated and powerful theory of Picard groups. We first tried to generalize the results there to $TrPic$, $TrPicent$ and TrI. Most of the generalizations work, some even quite immediate. We present here only some of the achievements. Further results and the omitted proofs of the results presented here are given in [35].

In [5] for any R–order Λ there is defined a mapping $\Phi : Aut(\Lambda) \longrightarrow Pic(\Lambda)$ by just mapping $f \in Aut(\Lambda)$ to the twisted bimodule ${}_f\Lambda_1$, defined by the following condition. The R–module structure of ${}_f\Lambda_1$ is as that of Λ, however, $\lambda \in \Lambda$ acts on ${}_f\Lambda_1$ by multiplication by $f(\lambda)$ from the left and by multiplication by λ from the right.

We denote for any ring A by $TC(A)$ the set of isomorphism classes of tilting complexes over A with endomorphism ring isomorphic to A. There is another mapping $\chi_A : TrPic(A) \longrightarrow TC(A)$ by mapping a twosided autotilting complex X to a tilting complex T by just forgetting the left structure.

Theorem 4 *[35] Let Λ be an R–order over an integral domain R with field of fractions K in a semisimple algebra A. Then let $X, Y \in TrPic(\Lambda)$.*

$$\chi_\Lambda(X) = \chi_\Lambda(Y) \Longleftrightarrow \text{ there is an } f \in Aut(\Lambda) : \Phi(f) \otimes^{\mathbb{L}}_\Lambda X \simeq Y \text{ in } TrPic(\Lambda).$$

One of the most powerful results of Fröhlich is the localization sequence. We are able to generalize this sequence to the following:

Theorem 5 *[35] Let R be a Dedekind domain with field of fractions K and let Λ be an R–order in the semisimple K–algebra A. Then there is an exact sequence*

$$1 \longrightarrow Picent(\text{ centre }(\Lambda)) \longrightarrow TrPicent(\Lambda) \longrightarrow \prod_{\wp \in Spec(R)} TrPicent(\Lambda \otimes_R R_\wp).$$

The mapping $Picent(\text{ centre }(\Lambda)) \longrightarrow TrPicent(\Lambda)$ factors through $Picent(\Lambda)$ and is just $- \otimes_{\text{centre }(\Lambda)} \Lambda$. The mapping on the right is just $- \otimes_R \prod_{\wp \in Spec(R)} R_\wp$.

Remark 5 1. In the classical situation it has been proven by Fröhlich that

$$1 \to Picent(\text{ centre } (\Lambda)) \to Picent(\Lambda) \to \prod_{\wp \in Spec(R)} Picent(\Lambda \otimes_R R_\wp) \to 1.$$

is exact. We cannot expect to have surjectivity at the right since by localizing we get more central idempotents and this causes shifts in the new ring direct components.

For example for $\Lambda = \mathbb{Z}C_2$, the integral group ring of the cyclic group of order 2, we get

$$TrPicent(\mathbb{Z}C_2) = Picent(\mathbb{Z}C_2) = TrPicent(\mathbb{Z}_2C_2) = 1$$

and for all $p \in Spec(\mathbb{Z}) \setminus \{2\}$ we have

$$TrPicent(\mathbb{Z}_pC_2) = C_\infty \times C_\infty.$$

2. From the above it follows immediately that $TrPicent(\Lambda) = Picent(\Lambda) \times C_\infty$ for commutative indecomposable R-orders Λ. This can be proven more directly.

Theorem 6 *[35] Let R be a Dedekind domain with field of fractions K and let Λ be a commutative indecomposable R–order in the semisimple K–algebra A. Then,*

$$TrPicent(\Lambda) = Picent(\Lambda) \times C_\infty.$$

Proof. We first reduce to the case where R is complete local. So let us assume the statement is true for R being in addition complete local. Let

$$X := (0 \longrightarrow X_0 \stackrel{\partial_0}{\longrightarrow} X_1 \longrightarrow .... \longrightarrow X_{n-1} \stackrel{\partial_{n-1}}{\longrightarrow} X_n \longrightarrow 0)$$

be a twosided auto tilting complex of Λ. Then, let $R_\wp$ be the completion of R at a prime $\wp$. $R_\wp \otimes_R X$ is also a tilting complex for $\Lambda_\wp$ for all $\wp \in Spec(R)$. Since

$$TrPicent(\Lambda_\wp) = Picent(\Lambda_\wp) \times \prod_{\text{number of indecomposable factors of } \Lambda_\wp} C_\infty$$

by the assumption, the homology $R_\wp \otimes H^n(X) = H^n(R_\wp \otimes X)$ is torsion free, hence is either zero or an $R_\wp$-lattice. Therefore $H^n(X)$ is torsion free over R, and since $H^n(X) \otimes R_\wp$ has to be a direct factor of $\Lambda_\wp$, for all $\wp \in Spec(R)$, $H^n(X)$ has to be a direct factor of Λ (cf. [4, 31.32]).

Claim 1 *Let Λ be a commutative indecomposable R-order in a separable K-algebra A with R being a complete local Dedekind domain. Then*

$$TrPicent(\Lambda) = Picent(\Lambda) \times C_\infty.$$

Proof. We have the Krull-Schmidt theorem for projective Λ–modules. Projective covers exist. Since Λ is indecomposable commutative, one sided projectives are two sided direct summands, and therefore the decomposition into projectives is the block decomposition. Hence, there is only one simple module and Λ is local. The result now follows from Theorem 3. ∎

In the same spirit we state without proof the

Theorem 7 *[35] Let R be a Dedekind domain and let Λ be an indecomposable hereditary order. Then,*

$$TrPicent(\Lambda) = Picent(\Lambda) \times C_\infty.$$

Remark 6 Theorem 7 follows from Theorem 5 and the discussion in Section 4.

Remark 7 After finishing the manuscript [35] we were informed during the ICRA VII that H. Lenzing and H. Meltzer proved, without knowing of our attempt, and with completely different methods, that for a canonical algebra A, $TrPic(A)$ is finitely generated. H. Meltzer and H. Lenzing clarified the structure of $TrPic(A)$ for A being a canonical algebra over an algebraically closed field to a large extent.

References

[1] M. Broué, *Isométries parfaites, types de blocs, catégories dérivées*, Astérisque **181–182** (1990), 61–92.

[2] A. Brumer, *Structure of Hereditary orders*, Thesis, Princeton 1963.

[3] E. Cline, B. Parshall and L. Scott, *Derived categories and Morita theory,* J. Algebra **104** (1986), 397–409.

[4] C.W.Curtis and I.Reiner, *Methods of Representation Theory*, John Wiley & Sons, New York 1982 & 1987.

[5] A. Fröhlich, *The Picard group of noncommutative rings, in particular of orders*, Trans. Amer. Math. Soc. **180** (1973), 1–45.

[6] J. A. Green, *Walking around the Brauer tree*, J. Austral. Math. Soc. **17** (1974), 197–213.

[7] K. W. Gruenberg, K. W. Roggenkamp, *Extension categories of groups and modules: Essential covers,* J. Algebra **44**, (1977), 564–574.

[8] D. Happel, *Triangulated categories in the representation theory of finite-dimensional algebras,* London Math. Soc. Lect. Notes **119** University press, Cambridge 1987.

[9] M. Harada, *Hereditary Orders*, Trans. Amer. Math. Soc. **107** (1963), 273–290.

[10] R. Hartshorne, Residues and dualities, Springer Lecture Notes in Math. **20**, Berlin 1966.

[11] B. Keller, *Chain complexes and stable categories*, Manuscr. Math. **67** (1990), 379–417.

[12] B. Keller, *A remark on tilting theory and DG algebras*, Manuscr. Math. **79** (1993), 247–253.

[13] B. Keller, *Deriving DG categories,* Ann. Sc. de l'Éc. Norm. Sup. 4^e série, t.27 (1994), 63–102.

[14] B. Keller, *Tilting theory and differential graded algebras*, pp. 183–190 in: V. Dlab and L.L. Scott (eds.), Proceedings of the NATO ARW on Representations of Algebras and related topics, 1994

[15] B. Keller, *Derived categories and their uses*, to be published in: M. Hazewinkel (ed.), Handbook of Algebra, Elsevier.

[16] B. Keller and D. Vossieck, *Sous les catégories dérivées,* C. R. Acad. Sci. Paris, **305**, Série I, (1987), 225–228.

[17] S. König and A. Zimmermann, *Tilting hereditary orders*, to be published.

[18] M. Linckelmann, *Derived equivalences for cyclic blocks over a p-adic ring,* Math. Zeitschrift **207** (1991), 293–304.

[19] M. Linckelmann, *Stable equivalences of Morita type for self–injective algebras and p–groups*, preprint (1994).

[20] F. Membrillo–Hernandez, *Homological properties of finite dimensional algebras*, Thesis Univ. College Oxford, 1993.

[21] W. Plesken, *Group rings of finite groups over p-adic integers,* Lect. Notes in Math. **1026** Springer Verlag Berlin 1983.

[22] I. Reiner, *Maximal Orders*, Academic Press, London 1975.

[23] J. Rickard, *Morita theory for derived categories,* J. London Math. Soc. **43** (1989), 436–456.

[24] J. Rickard, *Derived categories and stable equivalence,* J. Pure and Appl. Alg. **61** (1989), 303–317.

[25] J. Rickard, *Derived equivalences as derived functors*, J. London Math. Soc. **43** (1991), 436–456.

[26] J. Rickard, *Lifting theorems for tilting complexes,* J. of Algebra **142** (1991), 383–393.

[27] J. Rickard, *Splendid equivalences: derived categories and permutation modules,* (May 1994) to be published by the London Mathematical Society.

[28] J. Rickard, *Translation functors and equivalences of derived categories for blocks of algebraic groups*, pp. 255–264 in: V. Dlab and L.L. Scott (eds.), Proceedings of the NATO ARW on Representations of Algebras and related topics, 1994.

[29] J. Rickard, letter from 21. October 1993 to the author.

[30] K.W. Roggenkamp, *Blocks of cyclic defect and Green orders,* Comm. in Alg. **20** (6) 1992, 1715–1734.

[31] K.W. Roggenkamp, *Subgroup rigidity of p-adic group rings (Weiss' arguments revisited),* J. London Math. Soc. (2) **46** (1992), 432–448.

[32] K. W. Roggenkamp and V. Huber-Dyson, *Lattices over Orders Vol. 1* Lect. Notes in Math. **115** Springer Verlag Berlin 1970.

[33] K. W. Roggenkamp and L. L. Scott, *Isomorphisms of $\wp$-adic group rings,* Ann. of Math. **126** (1987), 593–647.

[34] R. Rouquier, *From stable equivalences to Rickard equivalences for blocks with cyclic defect*, pp.512–523 in Campbell, Hurley, Robertson, Tobin and Ward (eds.) 'Groups '93 Galway/ St. Andrews London Math. Soc. Lecture Notes **212**, Cambridge (UK) 1995.

[35] R. Rouquier and A. Zimmermann, *Picard groups of derived categories,* manuscript 1993.

[36] G. Thompson, *On the conjugacy of group bases*, Trans. Amer. Math. Soc. **341** (1) (1994), 423–447.

[37] J.L. Verdier, Catégories dérivées, état 0, in SGA $4\frac{1}{2}$, Springer Lecture Notes in Math. **569**, Berlin 1977.

Mathematisches Institut B, Universität Stuttgart, 70550 Stuttgart, Germany

Current address: UFR de Mathématiques, Université de Paris 7 Denis Diderot, 2 place Jussieu, F–75251 Paris Cedex 05, France

E-mail address: alex@machnix.mathematik.uni-stuttgart.de; alex@mathp7.jussieu.fr