Cohomology of groups and splendid equivalences of derived categories

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To Professor Klaus W. Roggenkamp on the occasion of his sixtieth birthday

Abstract

In an earlier paper we studied the impact of equivalences between derived categories of group rings on their cohomology rings. Especially the group of auto-equivalences $TrPic(RG)$ of the derived category of a group ring RG as introduced by Raphaël Rouquier and the author defines an action on the cohomology ring of this group. We study this action with respect to the restriction map, transfer, conjugation and the local structure of the group G.

Introduction

Let R be a complete discrete valuation domain or a field. The cohomology ring of a finite group G with values in the trivial module R has a very rich structure. One of these structures is the structure of a Mackey functor (see e.g. [14, (53.3)] or [1]) with structure mappings being the transfer, the restriction and the conjugation. Let $TrPic(B_0(RG))$ be the group of isomorphism classes of auto-equivalences of standard type of the derived category of the principal block of RG as defined and studied by Raphaël Rouquier and the author in [13]. Let $HD_R(G)$ be the subgroup of those auto-equivalences which fix the trivial module up to isomorphism. In [16] an action of $HD_R(G)$ on $H^*(G,R)$ is defined so that the cohomology ring is an R $(HD_R(G))$-module. We study in the present paper the connection between the functor 'cohomology of a finite group G' and, for fixed G, the action of $HD_R(G)$.

A major problem is that derived equivalences are not well suited for restriction to subrings or quotient rings. But, if one restricts to splendid equivalences introduced by J. Rickard [9], then, at least for centralizers $C_G(Q)$ of p-subgroups Q and in case R is a field, restriction maps may be defined using the Brauer construction [3].

In the present paper we define the respective structures and show compatibility of the Mackey functor structure with respect to local subgroups in a certain sense and under some additional hypothesis. We show that this action of the group of splendid auto-equivalences may be interpreted as action of the sheaf $HSplen_k(-)$ of splendid auto-equivalences fixing the trivial module on the Mackey functor $H^*(-,\mathbb{F}_p)$, "mod p group cohomology".

It should be noted that it is in general not possible to modify an element in $TrPic(RG)$ in the natural way so that the modified element fixes the trivial module. An example is given in section 1.

The paper is organized as follows. In Section 1 the most basic definitions concerning derived equivalences are given. Section 2 defines the subgroup of splendid auto-equivalences and in Section 3 we prove the compatibility of the Mackey functor structure with the structure as modules for the group of splendid auto-equivalences of the derived category. Finally Section 4 states the main theorem. The notations and definitions used in the paper follow the conventions of [7].

1 Recall some facts

1.1 Equivalences and auto-equivalences of derived categories

In [13] Raphaël Rouquier and the author defined and studied the group of auto-equivalences of a derived category.

Recall that, following Bernhard Keller [6], a *two-sided tilting complex* X for two R-algebras Λ and Γ so that Λ is flat as an R-module is an element X in $D^b(\Lambda\otimes_R\Gamma^{op})$ so that $X\otimes^{\mathbb{L}}_{\Gamma}-$ is an equivalence of categories.

We denote by $[X]$ the isomorphism class of a complex X in the derived category.

In case one of Λ or Γ is a symmetric R-algebra, then by [15] the other is as well. So, if Λ is a classical symmetric order over a Dedekind domain, then Γ is a classical symmetric order as well. In this case the inverse to $X\otimes^{\mathbb{L}}_{\Gamma}-$ is $Hom_R(X,R)\otimes^{\mathbb{L}}_{\Gamma}-$. If R is a field, analogous statements hold as was already observed by Rickard.

Definition 1.1 [13] Suppose Λ is an R-projective R-algebra. Then,

$$TrPic_R(RG):=\{\text{isomorphism classes } [X]\mid X\in D^b(\Lambda\otimes_R\Lambda^{op}) \text{ is a two-sided tilting complex}\}$$

is a group under $-\otimes^{\mathbb{L}}_{\Lambda}-$.

Let G be a group and R a commutative ring. In [16] it is shown that the group

$$HD_R(G):=Stab_{TrPic_R(RG)}(R):=\{[X]\in TrPic(RG)\mid X\otimes^{\mathbb{L}}_{RG}R\simeq R\}$$

acts on the cohomology ring $H^*(G,R)$. This action is functorial in R.

1.2 Definition of splendid equivalences

We recall Rickard's definition of a splendid equivalence [9]. Let p be a rational prime and let R be either a field of characteristic p or a complete discrete valuation domain of characteristic 0 with field of fractions of characteristic p. Let G and H be finite groups with a common Sylow p subgroup P. Let $B_0(RG)$ be the principal block of the group ring RG and $B_0(RH)$ likewise. A *p-permutation module* is a direct factor of a permutation module. Let ΔP be the diagonal embedding of P in $G\times H$.

A bounded complex X of finitely generated $B_0(RG)\otimes_R B_0(RH)^{op}$-modules is said to be a *splendid tilting complex* if

- the complex $Hom_{B_0(G)}(X,X)$ is isomorphic to $B_0(H)$ in the category $K^b(B_0(H))$ and $Hom_{B_0(H)}(X,X)$ is isomorphic to $B_0(G)$ in the category $K^b(B_0(G))$
- each homogeneous component of X is projective as $B_0(RG)$ as well as $B_0(RH)$-module
- all homogeneous components are relatively ΔP-projective p-permutation modules.

It is a consequence of Rickard's theory of equivalences between derived categories [8] that

$$X\otimes_{B_0(RG)}-:D^b(B_0(RH))\longrightarrow D^b(B_0(RH))$$

is an equivalence of triangulated categories, and hence X is a two-sided tilting complex.

1.3 The Brauer construction

Let R be a discrete valuation ring with residue field $k := R/rad\ R$ of characteristic p. Let Q be a p-subgroup of G. Then, (see [9]) the map

$$\begin{array}{rcl} -(Q) : kG - mod & \longrightarrow & k\ N_G(Q) - mod \\ M & \longrightarrow & M^Q/(\sum_{R<Q} Tr_R^Q(M^R)) \end{array}$$

extends to a functor. For a permutation module $M = k\Omega$ one has $M(Q) \simeq k(\Omega^Q)$. Moreover

$$B_0(kG)(Q) \simeq B_0(kC_G(Q)).$$

We have the following

Lemma 1.2 (Rickard [9]) $(-^G)(Q)$ *is isomorphic to* $(-(\Delta(Q))^{C_G(Q)}$ *as functors from the category of relatively* $\Delta(Q)$*-projective* p*-permutation* $k(G \times Q)$ *modules to the category of* k*-vector spaces.*

2 Splendid auto-equivalences

Suppose that R is a complete discrete valuation ring of characteristic 0 and residue field of characteristic p or a field of characteristic p. Let $B_0(RG)$ be the principal block of RG.

We denote by (X) the homotopy equivalence class of a complex X.

Definition 2.1

$$SplenPic_R(G) := \{(X) \mid X \text{ is splendid and } [X] \in TrPic_R(B_0(RG))\}$$

Then, Rickard proves

Theorem 1 (Rickard [9]) *For any* $(X) \in SplenPic_k(B_0(kG))$*, applying the Brauer functor one has* $(X(\Delta Q)) \in SplenPic_k(B_0(C_G(Q)))$*.*

Lemma 2.2 $SplenPic_R(G)$ *is a group which maps to* $TrPic_R(B_0(RG))$*.*

Proof. If X is splendid, then $Hom_R(X, R)$ is splendid as well. In fact, $Hom_R(X, R)$ is $B_0(RG)$-projective on either side as the homogeneous components of this complex are the R-duals of the homogeneous components of X. These are projective as the group ring is a symmetric algebra.

Moreover,

$$\begin{array}{rcl} Hom_{B_0(RG)}(Hom_R(X,R), Hom_R(X,R)) & \simeq & Hom_R(Hom_R(X,R) \otimes_{RG} X, R) \\ & \simeq & Hom_R(B_0(RG), R) \\ & \simeq & B_0(RG) \end{array}$$

is a chain of isomorphisms in the *homotopy* category of $B_0(RG)$ bimodules. Similarly,

$$\begin{array}{rcl} Hom_{B_0(RG)^{op}}(Hom_R(X,R), Hom_R(X,R)) & \simeq & Hom_R(X \otimes_{RG} Hom_R(X,R), R) \\ & \simeq & Hom_R(B_0(RG), R) \\ & \simeq & B_0(RG) \end{array}$$

Finally, all homogeneous terms of $Hom_R(X,R)$ are $\Delta(P)$-projective permutation modules, as they are just the duals of the terms of X.

We shall have to prove that if X and Y are splendid, also $Z := X \otimes_{B_0(RG)} Y$ is splendid. In fact, it is clear that Z is a complex of left and right projective modules since these are direct sums of tensor products of left and right projective modules. Moreover,

$$\begin{array}{rcl} Hom_R(Z,R) \otimes_{B_0(RG)} Z & \simeq & Hom_R(Y,R) \otimes_{B_0(RG)} Hom_R(X,R) \otimes_{B_0(RG)} X \otimes_{B_0(RG)} Y \\ & \simeq & Hom_R(Y,R) \otimes_{B_0(RG)} B_0(RG) \otimes_{B_0(RG)} Y \\ & \simeq & B_0(RG) \end{array}$$

and likewise $Z \otimes_{B_0(RG)} Hom_R(Z,R) \simeq B_0(RG)$ in the homotopy category of $B_0(RG)$-bimodules.

Since the homogeneous terms of Z are direct sums of tensor products of $\Delta(P)$-projective p-permutation modules, they are $\Delta(P)$-projective p-permutation modules as well.

This concludes the proof. ■

Definition 2.3 We define

$$HSplen_R(G) := Stab_{SplenPic_R(G)}(R) = \{(X) \in SplenPic_R(G) \mid X \otimes_{RG} R \simeq R\}$$

The group $HSplen_k(G)$ contains those outer automorphisms of G which is the identity on a fixed Sylow subgroup. Denote by $Inn(RG)$ the group of inner automorphisms of RG, denote by $Aut^P(G)$ the group of automorphisms of G whose restriction to P is the identity on P and denote by $Inn(G)$ the group of inner automorphisms of G.

Lemma 2.4 $Out_R^P(G) := Aut^P(G)/(Inn(RG) \cap Aut^P(G)) \leq HSplen_R(G)$.

Proof. The observation that an automorphism of G is trivial in the Picard group if and only if it is inner in RG rather than in G is a known fact. What we have to show is therefore the following: Let α be an automorphism of G fixing a Sylow p subgroup P and let $[\alpha]$ be its image in $Pic_R(G)$. Then, $[\alpha] \in HSplen_R(G)$.

It is clear that ${}_{\alpha}RG_1$ is projective from either side.

Let P be a Sylow p-subgroup of G. Since $\alpha|_P = id_P$, the bimodule ${}_{\alpha}RG_1$ is ΔP-projective. Hence, we proved the statement.

■

Remark 2.5 In general $Inn(RG) \cap Aut(G) \neq Inn(G)$. See [11] for a finite group G where this strict inequality holds for any semi-local coefficient domain R. Martin Hertweck recently gave an example where one can remove the condition of R being semi-local [5]. For cohomology of finite groups one may always assume that R is local though.

2.1 The trivial module may not be preserved

Let A be an algebra over a field k.

Lemma 2.6 *Let Y and Z be right bounded complexes of A-modules. Then,*

$$Hom_k(Y,k) \otimes_k Z \simeq RHom_k(Y,Z)$$

as $A \otimes_k A^{op}$-modules.

Proof. Let M and N be A-modules. Then,

$$\begin{array}{rcl} Hom_k(M,k)\otimes_k N & \stackrel{\sim}{\longrightarrow} & Hom_k(M,N) \\ \phi\otimes_k n & \mapsto & m\mapsto\phi(m)n \end{array}$$

is an isomorphism of $A\otimes_k A^{op}$-modules.

Moreover, the degree (i,j) homogeneous component of the bicomplex $Hom_k(Y,Z)$ is $Hom_k(Y_{-i},Z_j)$ while the degree i homogeneous component of $Hom_k(Y,k)$ is $Hom_k(Y_{-i},k)$. Therefore, $Hom_k(Y_{-i},Z_j)$ has degree (i,j) in the bicomplex $Hom_k(Y,k)\otimes_k Z$. It is moreover clear that the bi-differentials of the bicomplexes $Hom_k(Y,Z)$ and of $Hom_k(Y,k)\otimes_k Z$ correspond under the above isomorphism at the level of modules. Therefore, using again the above isomorphisms on the level of modules, the total complexes of $Hom_k(Y,Z)$ and of $Hom_k(Y,k)\otimes_k Z$ are isomorphic. This completes the proof. ∎

Example: Denoting by $\mathcal{T}(G)$ the épaisse sub-categorie of the stable module categorie which contains the trivial module, Carlson and Rouquier prove [4] that any functor $F:\mathcal{T}(G)\longrightarrow\mathcal{T}(H)$ of triangulated categories has the property that the functor $\tilde{F}:=(Hom_k(F(k),k)\otimes_k -)\circ F$ sends the trivial module to the trivial module (see also Roggenkamp [10]).

We shall show with an example that the analogous statement is not true for splendid equivalences and by consequence for derived equivalences neither.

Let $k=\mathbb{F}_3$ be the field with three elements and let $G=H=\mathfrak{S}_3$ be the symmetric group on three letters.

Then there are two indecomposable projective kG-modules P_+ and P_- up to isomorphism and the complex X_+ defined by

$$\ldots\longrightarrow 0\longrightarrow P_+\otimes_k Hom_k(P_+,k)\longrightarrow kG\longrightarrow 0\longrightarrow\ldots$$

with homology concentrated in degree 0 and 1 is a splendid tilting complex with isomorphism class in $SplenPic_k(G)$. If P_+ denotes the projective cover of the trivial module, then

$$X_+\otimes_{kG} k\simeq(\ldots\longrightarrow 0\longrightarrow P_+\otimes_k k\longrightarrow k\longrightarrow 0\longrightarrow\ldots)\simeq\Omega^1(k)[1]$$

where Ω is the syzygy operator. This module is two-dimensional and represents the non trivial element in $Ext^1(k_-,k)$ where k_- is the sign representation.

Denoting the k-dual of an object Y by $Y^{\check{}}$, we shall prove

$$(X\otimes_B k)^{\check{}}\simeq k^{\check{}}\otimes_B X^{\check{}}.$$

Hence, by adjointness of tensor products and covariant homomorphism functor

$$\begin{array}{rcl} Hom_k(X\otimes_B k,k) & \simeq & Hom_B(X,Hom_k(k,k)) \\ & \simeq & Hom_B(X,k^{\check{}}) \end{array}$$

Since $Hom_B(X,-)$ is an inverse to the functor $-\otimes_A X$, as well as $-\otimes_B Hom_k(X,k)$, we get that

$$Hom_B(X,k^{\check{}})\simeq k^{\check{}}\otimes_B Hom_k(X,k)\ .$$

Hence,

$$\begin{array}{rcl} (X\otimes_{kG} k)^{\check{}}\otimes_k(X\otimes_{kG} k) & \simeq & End_k(X\otimes_{kG} k) \\ & \simeq & End_k(\Omega^1(k)) \end{array}$$

and the latter is a four-dimensional, non trivial module.

3 Group cohomology as module and as Mackey functor

As is well known, $H^*(G,k)$ together with restriction, transfer and conjugation is a Mackey functor. We shall discuss the compatibility of this structure with respect to the action of splendid auto-equivalences of the derived category.

For the convenience of the reader we shall first recall the definition of a Mackey functor and define what we understand by an action of a group sheaf on a Mackey functor.

3.1 Action on Mackey functors

Let G be a group and let R be a commutative ring. We recall the definition of a Mackey functor (see e.g. [1, 14]). A *Mackey functor* is a pair of functors (M^*, M_*), the first contravariant and the second covariant, from the category of G-sets to the category of R-modules so that M_* and M^* are identical on objects subject to the following conditions.

(M1) Let X and Y be G-sets and i_X resp. i_Y be the injections of X or Y in $X\coprod Y$. Then they are supposed to satisfy $M^*(i_X)\oplus M^*(i_Y)$ and $M_*(i_X)\oplus M_*(i_Y)$ are mutually inverse isomorphisms between $M_*(X\coprod Y)$ and $M_*(X)\coprod M_*(Y)$.

(M2) Moreover, they satisfy the following condition on pullbacks: If

$$\begin{array}{ccc} T & \xrightarrow{\gamma} & Y \\ \delta\downarrow & & \downarrow\alpha \\ Z & \xrightarrow{\beta} & X \end{array}$$

is a pullback of G-sets, then $M^*(\beta)M_*(\alpha)=M_*(\delta)M^*(\gamma)$.

A *morphism of Mackey functors* $\theta:(M^*,M_*)\longrightarrow(N^*,N_*)$ is a natural transformation $\theta:M_*\longrightarrow N_*$ and $\theta:M^*\longrightarrow N^*$ (the same for M^* and for M_*).

A group sheaf on a partially ordered set $\mathfrak{C}$ of subgroups of G is a contravariant functor $\mathcal{G}:\mathfrak{C}\longrightarrow\mathfrak{Group}$ where $\mathfrak{Group}$ is the category of groups with group homomorphisms.

Definition 3.1 An operation of a group sheaf on a Mackey functor M is an operation of $\mathcal{G}(S)$ on $M(S)$ for any $S\in\mathfrak{C}$ by morphisms of Mackey functors.

3.2 Elementary properties

Lemma 3.2 *We have a homomorphism of groups*

$$\begin{array}{rcl} -(\Delta Q): SplenPic_k(G) & \longrightarrow & SplenPic_k(C_G(Q)) \\ (X) & \longrightarrow & (X(\Delta Q)) \end{array}$$

Proof.
Let $(X),(Y)\in HSplen_k(G)$. By Theorem 1 one has $(X(\Delta Q))\in SplenPic_k(B_0(C_G(Q)))$. To simplify notation denote for a moment $H:=C_G(Q)$.

One has the following isomorphism of functors.

$$Hom_{B_0(kG)}(X,-)\simeq X^{-1}\otimes_{B_0(kG)}-$$

Moreover, using Lemma 1.2,

$$\begin{array}{rcl} X^{-1}\otimes_{B_0(kG)}- & \simeq & (Hom_{B_0(kG)}(X,-))(\Delta Q) \\ & \simeq & Hom_{B_0(kH)}(X(\Delta Q),-(\Delta Q)) \end{array}$$

which may be applied to kG since kG is ΔQ-projective. Then, since group rings are symmetric and therefore the k-dual is isomorphic to the $B_0(kH)$-dual on the principal block,

$$X^{-1}(\Delta Q)\simeq X(\Delta Q)^{-1}\ .$$

Moreover,

$$(Hom_{B_0(kG)}(X,Y))(\Delta Q) \simeq Hom_{B_0(kH)}(X(\Delta Q), Y(\Delta Q))$$

as $kH \otimes_k kH^{op}$-bimodules again by Lemma 1.2. But this may be interpreted as

$$\begin{aligned} ((X^{-1} \otimes_{B_0(kG)} Y) \otimes_{B_0(kG)} kG)(\Delta Q) &\simeq ((X^{-1}(\Delta Q) \otimes_{B_0(kH)} Y(\Delta Q)) \otimes_{B_0(kH)} kH) \\ &\simeq ((X(\Delta Q)^{-1} \otimes_{B_0(kH)} Y(\Delta Q)) \otimes_{B_0(kH)} kH) \end{aligned}$$

and therefore

$$(X^{-1} \otimes_{B_0(kG)} Y)(\Delta Q) \simeq (X(\Delta Q)^{-1} \otimes_{B_0(kH)} Y(\Delta Q))$$

This concludes the proof. ■

Lemma 3.3 *If X is a splendid tilting complex, then*

$$X(\Delta Q) \otimes_{kC(Q)} k \simeq X^G(Q)$$

Proof. Let $X \in SplenPic_k(G)$. An inverse to the auto-equivalence $X \otimes_{kG} -$ of $D^b(B_0(kG))$ is $Hom_k(X,k) \otimes_{kG} -$ which is isomorphic to $Hom_{kG}(X,-)$. It is clear that if X is splendid, also $Y := Hom_k(X,k)$ is splendid. So, Lemma 1.2 implies that

$$(Y(\Delta Q))^{C_G(Q)} \simeq Y^G(Q)$$

But, Raphaël Rouquier proves in [12, Lemma 2.2] that for p-permutation modules taking k-duals commutes up to functorial isomorphism with the Brauer functor. Hence,

$$Hom_{kC_G(Q)}(X(\Delta Q), k) \simeq (Hom_{kG}(X,k))(Q)$$

Since $Y(\Delta Q) \otimes_{kC_G(Q)} -$ is an inverse to $Hom_{kC_G(Q)}(Y(\Delta Q), -)$ which in turn is an inverse to $Hom_{kC_G(Q)}(X(\Delta Q), -)$, we get

$$Y^G(Q) \simeq Y(\Delta Q) \otimes_{kC_G(Q)} k$$

Replacing X by Y taking double duals, the lemma is proved. ■

Remark 3.4 Let $\alpha \in Aut^P(G)$. Then,

$$(\,_\alpha kG_1)(\Delta Q) \otimes_{C_G(Q)} k \simeq (\,_\alpha kG_1)^G(Q) = k(Q) = k$$

3.3 The transfer with respect to local subgroups

We will prove in this section that the transfer in group cohomology is compatible with the action of $HSplen_k(G)$. For a subgroup H of a group G denote by tr_H^G the transfer map.

Proposition 3.5 *Let G be a finite group and let Q be a p-subgroup. Let k be a field of characteristic p. For any X with $[X] \in HSplen_k(G)$ we denote by F_X the action of $[X]$ on $H^*(G,k)$. Suppose that $X(\Delta Q) \in HSplen_k(C_G(Q))$. Then the diagram*

$$\begin{array}{ccc} H^*(G,k) & \xrightarrow{F_X} & H^*(G,k) \\ tr^G_{C_G(Q)} \uparrow & & \uparrow tr^G_{C_G(Q)} \\ H^*(C_G(Q),k) & \xrightarrow{F_{X(\Delta Q)}} & H^*(C_G(Q),k) \end{array}$$

is commutative.

Proof. The transfer map $tr^G_{C_G(Q)}$ can be defined as composite

$$Hom_{kC_G(Q)}(k,k[n]) \simeq Hom_{kG}(k,kG\otimes_{kC_G(Q)}k[n]) \to Hom_{kG}(k,kG\otimes_{kG}k[n]) \simeq Hom_{kG}(k,k[n])$$

and so we have to show that two diagrams are commutative. The first one is

$$\begin{array}{ccc} Hom_{kG}(k,kG\otimes_{kC}k[n]) & \longrightarrow & Hom_{kG}(k,kG\otimes_{kG}k[n]) \\ F_X\downarrow & & \downarrow F_X \\ Hom_{kG}(X^G,X\otimes_{kC}k[n]) & \longrightarrow & Hom_{kG}(X^G,X\otimes_{kG}k[n]) \end{array}$$

where the commutativity is clear. The second diagram is

$$\begin{array}{ccc} Hom_{kC_G(Q)}(k,k[n]) & \stackrel{\sim}{\longrightarrow} & Hom_{kG}(k,kG\otimes_{kC}k[n]) \\ F_{X(\Delta Q)}\downarrow & & \downarrow F_X \\ Hom_{kC_G(Q)}(X^G(Q),X^G(Q)[n]) & \stackrel{\alpha}{\longrightarrow} & Hom_{kG}(X^G,X\otimes_{kC_G(Q)}k[n]) \end{array}$$

where commutativity is less clear. But, all mappings beside α are defined as isomorphisms, so there is only one α possible which makes the diagram commutative. We have to show that this α is the mapping coming from the natural isomorphism from Frobenius reciprocity.

Since $X^G \simeq k$ in the derived category of G-modules, the complex X^G is exact in any degree besides in degree 0, where its homology is the trivial module. Hence, the restriction of X^G to $C_G(Q)$ is exact as well, besides in degree 0, where its homology is the trivial module k. Since $X^G(Q) \simeq k$ in the derived category of $C_G(Q)$-modules, the complex $X^G(Q)$ is exact in any degree besides in degree 0, where its homology is the trivial module. So, in the derived category of complexes of $C_G(Q)$-modules, we get the following isomorphism:

$$res^G_{C_G(Q)}(X^G) \stackrel{\varphi}{\simeq} X^G(Q) \simeq k$$

Now, α decomposes as follows:

$$\begin{array}{cclr} Hom_{kC_G(Q)}(k,k[n]) & \stackrel{F_{X(\Delta Q)}}{\longrightarrow} & Hom_{kC_G(Q)}(X^G(Q),X^G(Q)[n]) & \\ & \stackrel{\varphi^*(\varphi_*)^{-1}}{\longleftarrow} & Hom_{kC_G(Q)}(X^G,X^G[n]) & \\ & \simeq & Hom_{kG}(X^G,X^G[n]\uparrow^G_{C_G(Q)}) & \text{by Frobenius reciprocity} \end{array}$$

Now, we see that

$$\begin{array}{rcl} X\otimes_{kG} Hom_{kC_G(Q)}(kG,k) & \simeq & Hom_{kC_G(Q)}(kG,X\otimes_{kG}k) \\ x\otimes\phi & \mapsto & (m\mapsto x\otimes\phi(m)) \end{array}$$

are naturally isomorphic. Using again Lemma 3.3 we get that $F_X^{-1}\circ\alpha\circ F_{X(\Delta Q)}$ equals Frobenius reciprocity $Hom_{kC_G(Q)}(k,k[n]) \simeq Hom_{kG}(k,k\uparrow^G_{C_G(Q)})$.

This proves Proposition 3.5. ∎

3.4 The restriction with respect to local subgroups

We shall prove sensibly the same thing as for transfer now for restriction.

Proposition 3.6 *Let G be a finite group and let Q be a p-subgroup. Let k be a field of characteristic p. For any X with $[X] \in HSplen_k(G)$ we denote by F_X the action of $[X]$ on $H^*(G,k)$. Suppose that $X(\Delta Q) \in HSplen_k(C_G(Q))$. Then the diagram*

$$\begin{array}{ccc} H^*(G,k) & \stackrel{F_X}{\longrightarrow} & H^*(G,k) \\ res^G_{C_G(Q)}\downarrow & & \downarrow res^G_{C_G(Q)} \\ H^*(C_G(Q),k) & \stackrel{F_{X(\Delta Q)}}{\longrightarrow} & H^*(C_G(Q),k) \end{array}$$

is commutative.

Proof. Again using Lemma 3.3 and the hypothesis, $res^G_{C_G(Q)}(X^G) \simeq k \simeq X^G(Q)$ in the derived category of bounded complexes of $res^G_{C_G(Q)}$-modules. Let $\varphi : X^G \simeq X^G(Q)$ be this isomorphism. Let $\beta : X^G \simeq k$ and $\alpha : X^G(Q) \longrightarrow k$ so that $\varphi = \alpha \circ \beta^{-1}$. Then F_X acts on $\xi \in H^*(G,k)$ as $(\beta^*)^{-1}\beta_*(X \otimes \chi)$ and $F_{X(\Delta Q)}$ acts on $\eta \in H^*(C_G(Q),k)$ as $(\alpha^*)^{-1}\alpha_*(X(\Delta Q) \otimes \eta)$. Therefore, the following diagram is commutative:

$$\begin{array}{ccc}
Hom_{D^b(kG)}(k,k[n]) & \longrightarrow & Hom_{D^b(kG)}(X^G,X^G[n]) \\
res^G_{C_G(Q)} \downarrow & & \downarrow res^G_{C_G(Q)} \\
Hom_{D^b(kC_G(Q))}(k,k[n]) & & Hom_{D^b(kC_G(Q))}(res^G_{C_G(Q)}X^G, res^G_{C_G(Q)}X^G[n]) \\
\| & & \downarrow (\varphi^*)(\varphi_*)^{-1} \\
Hom_{D^b(kC_G(Q))}(k,k[n]) & \longrightarrow & Hom_{D^b(kG)}(X^G(Q),X^G(Q)[n])
\end{array}$$

Now, since $\varphi = \alpha \circ \beta^{-1}$,

$$\begin{array}{ccc}
Hom_{D^b(kG)}(X^G,X^G[n]) & \stackrel{(\beta^*)^{-1}\beta_*}{\longrightarrow} & Hom_{D^b(kG)}(k,k[n]) \\
\downarrow res^G_{C_G(Q)} & & \downarrow res^G_{C_G(Q)} \\
Hom_{D^b(kC_G(Q))}(X^G,X^G[n]) & \stackrel{(\beta^*)^{-1}\beta_*}{\longrightarrow} & Hom_{D^b(kC_G(Q))}(k,k[n]) \\
\varphi^*(\varphi_*)^{-1} \downarrow & & \| \\
Hom_{D^b(kG)}(X^G(Q),X^G(Q)[n]) & \stackrel{(\alpha^*)^{-1}\alpha_*}{\longrightarrow} & Hom_{D^b(kC_G(Q))}(k,k[n])
\end{array}$$

is commutative. Combining these two commutative diagrams we get a commutative diagram

$$\begin{array}{ccc}
Hom_{D^b(kG)}(k,k[n]) & \stackrel{F_X}{\longrightarrow} & Hom_{D^b(kG)}(k,k[n]) \\
\downarrow res^G_{C_G(Q)} & & \downarrow res^G_{C_G(Q)} \\
Hom_{D^b(kC_G(Q))}(k,k[n]) & \stackrel{F_{X(\Delta Q)}}{\longrightarrow} & Hom_{D^b(kC_G(Q))}(k,k[n])
\end{array}$$

This concludes the proof. ■

3.5 An example and an application

Remark 3.7 Proposition 3.5 implies that for $R = \mathbb{F}_p$ and $G = \mathfrak{S}_p$ the symmetric group on p letters, any element in $HSplen_R(G)$ acts trivially on $H^*(G,R)$. This fact was proven in [16] by explicit calculation.

In fact,

$$tr^{\mathfrak{S}_p}_{C_p} \circ res^{\mathfrak{S}_p}_{C_p} : H^*(\mathfrak{S}_p,\mathbb{F}_p) \longrightarrow H^*(\mathfrak{S}_p,\mathbb{F}_p)$$

is just multiplication by $[\mathfrak{S}_p : C_p] = (p-1)!$ and is hence invertible since we are working in characteristic p. So, $res^{\mathfrak{S}_p}_{C_p}$ is linear with respect to the action of splendid auto-equivalences $[X] \in HSplen_k(\mathfrak{S}_p)$ with $(X(\Delta C_p)) \in HSplen_k(C_p)$ if and only if $tr^{\mathfrak{S}_p}_{C_p}$ is.

Denoting $C_p :=< (1,2,3,\ldots,p) >$ a p-Sylow subgroup of $\mathfrak{S}_p$, it is well known that

$$res^G_{C_p} : H^*(\mathfrak{S}_p,R) \longrightarrow H^*(C_p,R)$$

is injective since the p-Sylow subgroup of $\mathfrak{S}_p$ is abelian. Hence, the fact that the restriction to the centralizer of C_p is compatible with the action of $HSplen_k(\mathfrak{S}_p)$ proves that the action of $HSplen_R(G)$ on $H^*(\mathfrak{S}_p,R)$ is trivial if the action of $HSplen_R(C_p)$ on $H^*(C_p,R)$ is trivial and one may prove that the trivial module is fixed also when one takes the Brauer functor of the auto-equivalence in question. This can be checked individually for the auto-equivalences considered in [16]. Moreover,

$$HSplen_R(C_p) \subseteq \{\phi \in Aut_R(RC_p) |\ {}^{\phi}R \simeq R\}$$

as $C_p =< c >$ is a p-group. Any such automorphism α_x is fixed by $\alpha_x(c) = 1 + (c-1)x$ for an $x \in FC_p$. Since $H^*(C_p, R)$ is generated by an element ζ in degree 2, and an element χ in degree 1 we have to evaluate the action in degree 1 and 2 only. Ones sees readily that $\alpha_x(\zeta) = x \cdot \zeta$ and $\alpha_x(\chi) = x \cdot \chi$. Now, the image of $res^{\mathfrak{S}_p}_{C_p}$ in $H^*(C_p, R)$ is the fix point ring $H^*(C_p, R)^{C_{p-1}}$ under the normalizer action. This action is multiplication by a primitive $p-1$-th root of unity in R/pR and $H^*(C_p, R)^{C_{p-1}}$ is generated in degree $2(p-1)$ and $2p-3$. The operation of α_x is by multiplication by x^{p-1}, but the unit group of R is cyclic of order $p-1$.

One would like to get the same result for R being the p-adic integers. But, using the Brauer construction one is forced to work over a field as ring of coefficients (but see [17], where the lifting question to a complete discrete valuation domain is studied and where parts of the present results are announced). Moreover, what we proved is valid only for splendid auto-equivalences. We plan in the near future to give a handy criterion for deciding whether an equivalence is splendid.

3.6 Conjugation

Proposition 3.8 *Let $x \in G$ and Q be a p-subgroup of G. Then, denoting by c_x the conjugation morphism, for any X with $(X) \in HSplen_R(G)$ the following diagram is commutative:*

$$\begin{array}{ccc} H^*(C(Q), R) & \stackrel{c_x}{\longrightarrow} & H^*(C(Q)^x, R) \\ X(\Delta Q) \otimes_{RG} - \downarrow & & \downarrow X(\Delta Q^x) \otimes_{RG} - \\ H^*(C(Q), R) & \stackrel{c_x}{\longrightarrow} & H^*(C(Q)^x, R) \end{array}$$

Proof. Lemma 3.3 implies that we have to prove that the diagram

$$\begin{array}{ccc} Hom_{RC_G(Q)}(R, R[n]) & \stackrel{c_x}{\longrightarrow} & Hom_{RC_G(Q)^x}(xR, xR[n]) \\ X(\Delta Q) \otimes_{RG} - \downarrow & & \downarrow X(\Delta Q^x) \otimes_{RG} - \\ Hom_{RC_G(Q)}(X^G(Q), X^G(Q)[n]) & \stackrel{c_x}{\longrightarrow} & Hom_{RC_G(Q)}(X^G(Q^x), X^G(Q^x)[n]) \end{array}$$

is commutative. But this is clear. ■

4 Group sheaf action on the Mackey functor $H^*(-, k)$

We are ready to state and to prove our main theorem.

Denote by $\mathfrak{G}roup$ the category of groups with morphisms being group homomorphisms and denote by $k - \mathfrak{A}lg$ the category of k-algebras with morphisms being algebra homomorphisms. Let k be a field of characteristic p and let G be a finite group. Denote for any p-subgroup Q of G by $HSplen_k(C_G(Q))$ the group of splendid auto-equivalences of the bounded derived category $D^b(kC_G(Q))$ which fix the trivial module.

Fix a set $\mathfrak{P}_{p,G}$ of p-subgroups of G which may contain all p-subgroups of G or not. Suppose that for any $Q_1, Q_2 \in \mathfrak{P}_{p,G}$ we have $Q_1 < Q_2 \Rightarrow Q_2 \leq C_G(Q_1)$. This is the case for example if all elements of $\mathfrak{P}_{p,G}$ are abelian. In fact this is also necessary for all but the maximal elements of $\mathfrak{P}_{p,G}$. This gives a category whose objects is $\mathfrak{P}_{p,G}$ and whose morphism sets are inclusion (that is $Hom_{\mathfrak{P}_{p,G}}(Q_1, Q_2)$ contains one element, the inclusion, if $Q_1 \leq Q_2$ and $Hom_{\mathfrak{P}_{p,G}}(Q_1, Q_2) = \emptyset$ if $Q_1 \not\subseteq Q_2$). Define

$$\mathfrak{C}_{p,G} := \{C_G(Q) |\; Q \in \mathfrak{P}_{p,G}\} \cup \{\{1\}\}$$

with morphisms being again induced by inclusion. Taking centralizers $C_G(-) : \mathfrak{P}_{p,G} \longrightarrow \mathfrak{C}_{p,G}$ is a contravariant functor: $Q_1 \subseteq Q_2 \Rightarrow C_G(Q_1) \supseteq C_G(Q_2)$. Define a contravariant functor by

$$\begin{array}{ccc} \mathfrak{C}_{p,G} & \longrightarrow & \mathfrak{G}roup \\ C_G(Q) & \mapsto & HSplen_k(C_G(Q)) \end{array}$$

and which maps an inclusion $C_G(Q_1) < C_G(Q_2)$ to the Brauer construction:

$$HSplen_k(C_G(Q_2))) \ni (X) \mapsto (X(\Delta Q_1)) \in HSplen_k(C_G(Q_1)))\ .$$

Then, the functor $\mathfrak{C}_{p,G} \longrightarrow \mathfrak{Group}$ which is $HSplen_k(-)$ on objects and $HSplen_k(C_G(Q_1) \leq C_G(Q_2))(X) := (X(\Delta Q_1))$ on morphisms is a sheaf of groups.

Furthermore,

$$H^*(-,k) : \mathfrak{C}_{p,G} \longrightarrow k - \mathfrak{Alg}$$

is a contravariant functor of k-algebras, a ringed space. Moreover, $(H^*(-,k), res, Tr, c)$ is a Mackey functor on the set of all subgroups of G. The Mackey functor structure may be restricted to $\mathfrak{C}_{p,G}$ if this set is stable under intersection. It is clear that $\mathfrak{C}_{p,G}$ is stable under conjugation. So, the condition to be stable under intersection implies that the Mackey formula makes sense for functors on $\mathfrak{C}_{p,G}$.

Define the sub-functor $HSplen_{k,\mathfrak{P}_{p,G}}(-)$ of $HSplen_k(-)$ by

$$\begin{aligned} &HSplen_{k,\mathfrak{P}_{p,G}}(C_G(Q)) \\ &\quad := \ \{(X) \in HSplen_k(C_G(Q))|\ (X(\Delta\hat{Q})) \in HSplen_k(C_G(\hat{Q})) \text{ for all } \hat{Q} \in \mathfrak{P}_{p,G} \text{ and } \hat{Q} \geq Q\} \end{aligned}$$

Remark 4.1 Let P be a p-Sylow subgroup of G. By Remark 3.4 one has

$$\{[\phi] \in Out_R(G)|\ \phi|_P = id_P\} \leq HSplen_{k,\mathfrak{P}_{p,G}}(C_G(Q)) \text{ for all } Q \in \mathfrak{P}_{p,G}.$$

We are ready to state our main results.

Theorem 2 *Let $\mathfrak{P}_{p,G}$ be a partially ordered set of p-subgroups of the finite group G. Suppose that if $Q_1 < Q_2$ and $Q_1, Q_2 \in \mathfrak{P}_{p,G}$, then Q_2 centralizes Q_1.*
Let Res: $\mathfrak{C}_{p,G} \longrightarrow k - Mod$ *and* Trans: $\mathfrak{C}_{p,G} \longrightarrow (k - Mod)^{op}$ *be two functors which are identical on objects:* $H^*(-,k) : \mathfrak{C}_{p,G} \longrightarrow k - \mathfrak{Alg}$ *and which are on morphisms*

$$\mathsf{Res}(C_G(Q_1) \leq C_G(Q_2)) := res^{C_G(Q_2)}_{C_G(Q_1)}$$

while

$$\mathsf{Trans}(C_G(Q_1) \leq C_G(Q_2)) := tr^{C_G(Q_2)}_{C_G(Q_1)}$$

Then,

$$HSplen_{k,\mathfrak{P}_{p,G}}(-) : \mathfrak{C}_{p,G} \longrightarrow \mathfrak{Group}$$

acts by natural transformations on Res *and on* Trans*. That means that the functor giving the group operation structure on each subgroup*

$$HSplen_{k,\mathfrak{P}_{p,G}}(-) \times H^*(-,k) \longrightarrow H^*(-,k)$$

is functorial in the sense that for any $Q_1 < Q_2$ one has

$$\begin{array}{ccccc} HSplen_{k,\mathfrak{P}_{p,G}}(C_G(Q_1))) & \times & H^*(C_G(Q_1)),k) & \longrightarrow & H^*(C_G(Q_1)),k) \\ Br^{C_G(Q_1)}_{C_G(Q_2)} \downarrow & & res^{C_G(Q_1)}_{C_G(Q_2)} \downarrow & & res^{C_G(Q_1)}_{C_G(Q_2)} \downarrow \\ HSplen_{k,\mathfrak{P}_{p,G}}(C_G(Q_2))) & \times & H^*(C_G(Q_2)),k) & \longrightarrow & H^*(C_G(Q_2)),k) \end{array}$$

is commutative and similarly for Trans*.*

Proof. This follows from Proposition 3.5 and Proposition 3.6. ∎

Theorem 3 *Suppose $\mathfrak{C}_{p,G}$ is stable under intersection. Then, the constant sheaf*

$$HSplen_{k,\mathfrak{P}_{p,G}}(G) : \mathfrak{C}_{p,G} \longrightarrow \mathfrak{G}roup$$

acts by morphisms of Mackey functors on

$$H^*(-,k) : \mathfrak{C}_{p,G} \longrightarrow k - \mathfrak{A}lg$$

Proof. This is exactly the statement of Proposition 3.5, Proposition 3.6 and Proposition 3.8. ∎

Remark 4.2

1. It is interesting to note that Serge Bouc used in [2] functors M which satisfy only two third of the Mackey functor axioms to construct resolutions of Mackey functors. More precisely, he used functors F which satisfy all the Mackey functor axioms which may be formulated by use of conjugation and restriction only. Dually one could use transfer instead of restriction. In our situation in Theorem 2 this is exactly the type of functors we consider.
2. It would be useful to know when

   $$(X) \in HSplen_k(G) \text{ implies } (X(\Delta Q)) \in HSplen_k(C_G(Q))].$$

   This would imply considerable simplifications in the formulation of Theorem 2 and Theorem 3.
3. In case $\mathfrak{C}_{p,G}$ has only one non-trivial element, Theorem 2 is exactly Proposition 3.6 and Proposition 3.5.

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