Group Automorphism Action on Group Cohomology

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Abstract

An automorphism α of a group G is called almost inner if $\alpha(g)$ is conjugate to g in G for any $g \in G$. Obviously, there exist p-groups G and non inner automorphisms of G inducing the identity on the mod p cohomology ring $H^*(G, \mathbb{F}_p)$. We pose the question if there are p-groups G and almost inner automorphisms which are not inner but which induce the identity on $H^*(G, \mathbb{F}_p)$. In the present paper we treat some examples to give evidence that there are no such groups G.

Introduction

Let G be a finite group and R be a commutative ring with trivial G-action. Then, any automorphism α of G induces by functoriality an automorphism of the cohomology ring $H^*(G, R)$. Jackowski and Marciniak proved [8] that in case α induces an inner automorphism of the group ring RG, then α^* is the identity. Furthermore, they posed the following question ([8, Question 4.10] and [7]) for a p-group G: is any automorphism α of G necessarily inner whenever α induces the identity on $H^*(G, \mathbb{Z})$? Note that for non p-groups a generalization of this question has a negative solution by an example of Hertweck [4]. Indeed, recently, in [4] Hertweck constructed a non p-group G and a non inner automorphism α of G which becomes inner in $\mathbb{Z}G$.

What might be the above Jackowski's question for mod p cohomology? The Quillen stratification of the cohomology variety implies that an automorphism α of the p-group G which is the identity on the mod p cohomology ring fixes each conjugacy class of maximal elementary abelian subgroups. Obviously, there exist p-groups G and non inner automorphisms inducing the identity on $H^*(G, \mathbb{F}_p)$. So, for mod p cohomology, it seems natural to *restrict to almost inner group automorphisms* α of a p-group G, that is, $\alpha(g)$ is conjugate to g for any $g \in G$. As mentioned by Jackowski and Marciniak [8] the group $C_8 \rtimes \mathrm{Aut}(C_8) = C_8 \rtimes (C_2 \times C_2)$ has an almost inner automorphism α that is not inner; however α^* is not trivial. Here we denote by C_n the cyclic group of order n.

Let A be an abelian p-group acting on the cyclic p-group C_{p^n}. In this paper we prove that any almost inner automorphism of $C_{p^n} \rtimes A$ which induces the identity on $H^*(C_{p^n} \rtimes A, \mathbb{F}_p)$ is inner. Actually we show the following result.

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Theorem 1 *Let G be a semidirect product of a cyclic p-group by an abelian p-group. Let α be an almost inner but not inner automorphism of G. Then, α induces a non trivial automorphism of $H^3(G, \mathbb{F}_p)$.*

In the final remark we mention that this result remains valid for the group $G = C_8 \rtimes Q_8$ (and the field $\mathbb{F}_2$). Here Q_8 denotes the quaternion group of order 8. Note that this group has been used in [9] to construct a group H with a non inner automorphism but which induces an inner automorphism on the group ring SH over some semilocal ring of algebraic integers (thus a semilocal version of Hertweck's example).

The paper is organized as follows. In section 1 and 2 we introduce the notation and recall some known facts. In section 3 we restrict our group theoretical setup, and in section 4 we do the cohomology constructions.

1 Notations

Let G be a finite group and let α be an automorphism of G. For any $\mathbb{Z}G$-module M we denote by ${}^{\alpha}M$ the $\mathbb{Z}G$-module with the same additive structure as M but $g \in G$ acts on ${}^{\alpha}M$ as $\alpha(g)$ acts on M. Let R be a commutative ring and view R as an RG-module via the trivial action. Then, α acts on the cohomology ring $H^*(G, R)$ via the following construction (see e.g. [2, page 80]). Take a projective resolution P of $\mathbb{Z}$ as $\mathbb{Z}G$-module. Then, P is a chain complex with homology $\mathbb{Z}$ in degree 0 and exact in all the other degrees. The complex ${}^{\alpha}P$ has homology ${}^{\alpha}\mathbb{Z} \simeq \mathbb{Z}$ in degree 0. Hence, there is a chain map

$$\tau : P \longrightarrow {}^{\alpha}P$$

with

$$\tau = \bigoplus_{n \in \mathbb{N}} \tau_n,$$

where

$$\tau_n : P_n \longrightarrow {}^{\alpha}P_n$$

is the mapping between the homogenous components. Any two such mappings τ differ by a homotopy. Applying $Hom_{\mathbb{Z}G}(-, R)$ to P and to ${}^{\alpha}P$ the mapping

$$\tau : P \longrightarrow {}^{\alpha}P$$

induces a cochain map

$$\tau^* := Hom(\tau, R) : Hom_{\mathbb{Z}G}({}^{\alpha}P, R) \longrightarrow Hom_{\mathbb{Z}G}(P, R)$$

Moreover, the identity map gives an identification

$$Hom_{\mathbb{Z}G}({}^{\alpha}P, R) \simeq Hom_{\mathbb{Z}G}(P, {}^{\alpha^{-1}}R) \simeq Hom_{\mathbb{Z}G}(P, R)$$

Since τ^* is a cochain map, it induces a mapping on the cohomology of the complexes:

$$\tau^* : H^*(Hom_{\mathbb{Z}G}({}^{\alpha}P, R)) \longrightarrow H^*(Hom_{\mathbb{Z}G}(P, R))$$

The map

$$\tau^n : H^n(Hom_{\mathbb{Z}G}({}^{\alpha}P, R)) \simeq H^n(G, R) \longrightarrow H^n(G, R) \simeq H^n(Hom_{\mathbb{Z}G}(P, R))$$

is the mapping induced by α on the cohomology.

Observe that the map τ is defined as soon as ${}^{\alpha}R \simeq R$. This in turn is true not only for trivial G-modules R.

We discuss the possible form of τ_n.

Lemma 1 *Let P be a free RG-module of rank n and let α be an automorphism of RG. Then, choosing a basis of P, any isomorphism ${}^{\alpha}P \xrightarrow{\varphi} P$ can be decomposed as $\varphi(x) = \alpha^{-1}(x)M$ for a $n \times n$-matrix M with coefficients in RG.*

Proof. Choose a basis for P. We then obtain an RG-isomorphism $\alpha_0 : P \longrightarrow {}^{\alpha}P$ by applying α to each of the components (with respect to this basis). The automorphism $\alpha_0 \circ \varphi$ is determined by a $n \times n$-matrix N with coefficients in RG; that is, $\alpha_0(\varphi(x)) = xN$. So the result follows. ■

2 Constructing projective resolutions

For the reader's convenience, we recall a method due to C. T. C. Wall [10] for constructing projective resolutions for group extensions.

Let

$$1 \longrightarrow N \longrightarrow G \longrightarrow H \longrightarrow 1$$

be an exact sequence of finite groups and let B be a free resolution of the $\mathbb{Z}N$-module $\mathbb{Z}$. Let C be a free resolution for the $\mathbb{Z}H$-module $\mathbb{Z}$. Then, $\mathbb{Z}G \otimes_{\mathbb{Z}N} B$ is a free resolution for the $\mathbb{Z}G$-module

$$\mathbb{Z}G \otimes_{\mathbb{Z}N} \mathbb{Z} \simeq \mathbb{Z}H.$$

Take a free resolution $C = (C_n, d_1)$ of $\mathbb{Z}H$. Then, each of the homogeneous components C_n of C is a free $\mathbb{Z}H$-module and can be replaced by the above argument by a direct sum of the projective resolution $\mathbb{Z}G \otimes_{\mathbb{Z}N} B$. The corresponding identification map is the augmentation and will be denoted by ϵ as well. Call the $k-$th homogeneous component of the free resolution of the $n-$th homogeneous component of C the $\mathbb{Z}G$-module $A_{n,k}$. Let d_0 denote the differential of each of the free resolutions of C_s. Since B is exact in non zero degrees, there exist $\mathbb{Z}G$ linear maps

$$d_1 = (d_1)_{n,m} : A_{n,m} \to A_{n-1,m}$$

so that $d_1 d_0 + d_0 d_1 = 0$. In other words, there is a chain map

$$d_1 : (\mathbb{Z}G \otimes_{\mathbb{Z}N} B)^{\dim_{\mathbb{Z}H} C_s} \longrightarrow (\mathbb{Z}G \otimes_{\mathbb{Z}N} B)^{\dim_{\mathbb{Z}H} C_{s-1}}$$

which lifts the differential d_1 of C; but one takes alternate signs for differentials resolving C_s and C_{s+1}. Of course, d_1 is not necessarily a differential again. Nevertheless, we have the following statement.

Proposition 2 *(C. T. C. Wall [10]) There exists $\mathbb{Z}G$-linear endomorphisms d_r of bidegree $(-r, r-1)$, that is, there are $\mathbb{Z}G$-linear maps $d_r : A_{n,m} \to A_{n+r-1,m-r}$ ($r \geq 1$, $m \geq r$), such that*

$$d_1 \epsilon = \epsilon d_1 \quad \textit{and} \quad \sum_{i=0}^{k} d_i d_{k-i} = 0$$

for each k. Moreover, $(\bigoplus_{k=1}^{n} A_{k,n-k}, \sum_{k=0}^{n} d_k)$ is a free resolution of the $\mathbb{Z}G$-module $\mathbb{Z}$. Furthermore, any set of d_r satisfying the above properties gives rise to a free resolution of the $\mathbb{Z}G$-module $\mathbb{Z}$.

Remark 3 The proof of proposition 2 is done inductively. That is, if one has found d_k which satisfies $\sum_{i=0}^{k} d_i d_{k-i} = 0$, then there are d_n with $d_0, d_1, \ldots, d_k$ already determined.

3 Almost inner automorphisms and metabelian groups

We shall discuss p-groups G which have a cyclic normal subgroup $N = C_{p^n}$ and an abelian complement Ab. That is, we get a split extension of finite groups

$$1 \longrightarrow N \longrightarrow G \longrightarrow Ab \longrightarrow 1 \ .$$

As is well known (cf e.g. [6, Satz I.13.9]), the group of automorphisms of the cyclic p-group C_{p^n} has a Sylow p-subgroup which is cyclic in case p is odd and is isomorphic to $C_2 \times C_{2^{n-2}}$ in case $p = 2$ (and $n \geq 2$).

If p is odd,

$$G \simeq (C_{p^n} \rtimes C_{p^k}) \times \widetilde{Ab}$$

for some k and $\widetilde{Ab}$ an abelian group. Now, if α is an almost inner automorphism of G, then any element $x \in G$ is conjugate to its image $\alpha(x)$. So, since $\widetilde{Ab}$ is abelian, it follows that α acts as the identity on $\widetilde{Ab}$ and α induces an automorphism on $(C_{p^n} \rtimes C_{p^k})$.

Lemma 4 *Let G be a split metacyclic group. Then, any almost inner automorphism of G is inner.*

Proof. Let α be an almost inner automorphism on $G = C \rtimes A$, with C and A cyclic groups. Suppose $Ab = \langle a \rangle$. As C is normal in G, α induces an automorphism on C. Since C is cyclic, we may use an inner automorphism of G to modify α in such a way that α is the identity on C. As G/C is abelian, we get $\alpha(a) = ac = xax^{-1}$ for some $x \in G$ and $c \in C$. Again because Ab is abelian, we may also assume that $x \in C$. It follows that α is conjugation by x. ■

Because of Lemma 4 and the preceding discussion we get at once the following application.

Corollary 5 *If p is odd and if G is a split extension of a cyclic p-group and an abelian p-group, then any almost inner automorphism of G is inner.*

Let now $p = 2$ ($n \geq 2$) and thus $\mathrm{Aut}(N) = C_2 \times C_{2^{n-2}}$. Let α be an almost inner automorphism of G. If the image of $Ab \to \mathrm{Aut}(N)$ is cyclic, then we again may apply Lemma 4.

Lemma 6 *Let $G = C_{2^n} \rtimes Ab$ for an abelian group Ab and let α be an almost inner automorphism of G. If the induced homomorphism $Ab \longrightarrow Aut(C_{2^n})$ has a cyclic image, then any almost inner automorphism of G is inner.*

Proof. In this case,

$$G \simeq (C_{2^n} \rtimes C_{2^k}) \times \widetilde{Ab}$$

for a certain k and an abelian group $\widetilde{Ab}$. Therefore, any almost inner automorphism α induces an almost inner automorphism on each of the factors. Since $\widetilde{Ab}$ is abelian and since $C_{2^n} \rtimes C_{2^k}$ is split metacyclic, the induced automorphisms on the factors are inner. Thus α is inner. ∎

We now shall discuss the remaining case where in fact almost inner automorphisms can occur. So $p = 2$, $G = C_{2^n} \rtimes Ab$ for an abelian group Ab, and the induced mapping φ of Ab to the automorphism group $\mathrm{Aut}(C_{2^n}) = C_{2^{n-2}} \rtimes C_2$ does not have a cyclic image. Thus $\varphi(Ab) = C_{2^m} \times C_2$ with $m \leq n-2$. Here the factor group C_2 is generated by the automorphism that maps a to a^{-1}, where a is the generator of C_{2^n}. It follows that $Ab = \langle b \rangle \times \langle c \rangle \times \widetilde{Ab}$, so that $bab^{-1} = a^{-1}$ and c induces the automorphism of order 2^m on C_{2^n}. Any almost inner automorphism α of G induces the identity on $\widetilde{Ab}$ and an almost inner automorphism on $C_{2^n} \rtimes (\langle b \rangle \times \langle c \rangle)$. So we shall now discuss the action of α on the latter group.

Lemma 7 *Let $G = C_{2^n} \rtimes (C_{2^k} \times C_{2^l})$. Suppose $C_{2^n} = \langle a \rangle$, $C_{2^k} = \langle b \rangle$, $C_{2^l} = \langle c \rangle$ and $bab^{-1} = a^{-1}$. Then G has an almost inner automorphism α_0 so that $\alpha_0(c) = a^{2^{n-1}}c$ and α_0 acts trivially on $\langle a, b \rangle$. Furthermore, α_0 is not inner and for any almost inner automorphism α of G there exists an inner automorphism γ so that $\alpha\gamma$ either equals α_0 or the identity.*

Proof. Let α be an almost inner automorphism of G. Because of Lemma 4, we may modify α by an inner automorphism so that

$$\alpha(a) = a \text{ and } \alpha(b) = b.$$

Hence, α induces an almost inner automorphism on the abelian group $G/\langle a, b \rangle$

$$\alpha(c) = a^s \cdot c$$

for some s. As $bc = cb$, we get $b \cdot a^s \cdot c = \alpha(b)\alpha(c) = \alpha(c) \cdot \alpha(b) = a^s \cdot c \cdot b$. Therefore, $a^{-s} = a^s$ which implies $s \in \{0, 2^{n-1}\}$.

Clearly α_0 is an automorphism. We now prove that it is almost inner. Write $cac^{-1} = a^v$ and $v = 2^r v' + 1$ with $0 \leq r < 2^n$ and $(v', 2) = 1$. Let $q = 2^{n-1-r}$. Then $q(v-1) = 2^{n-1}v'$. So it follows that

$$a^q c a^{-q} c^{-1} = a^q a^{-vq} = a^{q(1-v)} = a^{2^{n-1}}$$

and thus $\alpha_0 = a^{2^{n-1}}c = a^q c a^{-q}$. Hence, for any integers x, y,

$$\alpha_0(a^x b^{2y} c) = a^x b^{2y} a^{2^{n-1}} c = a^x b^{2y} a^q c a^{-q} = a^q (a^x b^{2y} c) a^{-q}.$$

Now to deal with odd exponents of b, write $v + 1 = 2^{r'} v''$ with $1 \leq r' \leq n$ and $(v'', 2) = 1$. It follows that

$$\begin{aligned}
\alpha_0(a^x b^{2y+1} c) &= a^x b^{2y+1} a^{2^{n-1}} c \\
&= a^x b^{2y+1} (a^{2^{n-1}})^{v''} c \\
&= a^x b^{2y+1} a^{-2^{n-1-r'} 2^{r'} v''} c \\
&= a^x b^{2y+1} a^{-2^{n-1-r'}(1+v)} c \\
&= a^{2^{n-1-r'}} a^x b^{2y+1} c (c^{-1} a^{-v 2^{n-1-r'}} c) \\
&= a^{2^{n-1-r'}} (a^x b^{2y+1} c) a^{-2^{n-1-r'}}
\end{aligned}$$

Hence this proves that α_0 is indeed almost inner.

Finally, suppose α_0 is inner. Since $\langle b, c\rangle$ is abelian, there exists $x \in \mathbf{N}$ so that

$$a^x c a^{-x} = a^{2^{n-1}} c$$

and

$$a^x b c a^{-x} = a^{2^{n-1}} bc.$$

Therefore $a^x c a^{-x} = b^{-1} a^x b c a^{-x}$ and thus $a^x c a^{-x} = a^{-x} c a^{-x}$. So $a^{2x} = 1$ and thus $2^{n-1} | x$. But then a^x is central and this yields a contradiction. ∎

4 Computing $H^*(G, \mathbb{F}_p)$

In this section we will prove Theorem 1. So we consider a p-group G which is a semidirect product of a cyclic p-group by an abelian p-group. In order that G has an almost inner automorphism α that is not inner, the results in Section 3 yield that $p = 2$ and

$$G \simeq (C_{2^n} \rtimes (C_{2^k} \times C_{2^l})) \times \widetilde{Ab},$$

for some abelian group $\widetilde{Ab}$. Moreover, up to inner automorphisms α is unique.

4.1 $H^*(G, \mathbb{F}_p)$ as vector space

We are only interested in cohomology with values in $\mathbb{F}_p$. By the Künneth formula,

$$H^n(G, \mathbb{F}_p) = \bigoplus_{r=0}^{n} H^r(\widetilde{Ab}, \mathbb{F}_p) \otimes_{\mathbb{Z}} H^{n-r}(C_{p^n} \rtimes (C_{p^k} \times C_{p^l}), \mathbb{F}_p)).$$

Note that for integral cohomology the situation would be more difficult. Since our automorphism is the identity on $\widetilde{Ab}$, we may trace the action of α by the knowledge of the action of α on $H^*(C_{p^n} \rtimes (C_{p^k} \times C_{p^l}), \mathbb{F}_p)$.

So, we may assume

$$G = C_{2^n} \rtimes (C_{2^k} \times C_{2^l})$$

and we use the same notation as in the statement of Lemma 7. Let m be a positive integer so that $c^{-1}ac = a^m$. For an element $g \in G$ of order n we denote $\Delta_g := \sum_{i=0}^{n-1} g^i$. Put, $A := a - 1$, $B := b - 1$ and $C := c - 1$.

We shall have to give an explicit projective resolution of $\mathbb{Z}G$-modules of the trivial module $\mathbb{Z}$. For this we shall use Proposition 2 several times. We start with the standard projective resolution of the trivial $\mathbb{Z}C_{2^n}$-module $\mathbb{Z}$:

$$\mathbb{Z} \xleftarrow{\epsilon} \mathbb{Z}C_{2^n} \xleftarrow{A} \mathbb{Z}C_{2^n} \xleftarrow{\Delta_a} \mathbb{Z}C_{2^n} \xleftarrow{A} \mathbb{Z}C_{2^n} \xleftarrow{\Delta_a} \mathbb{Z}C_{2^n} \xleftarrow{A} \mathbb{Z}C_{2^n} \xleftarrow{\Delta_a} \dots$$

A complex giving a free resolution of the trivial $\mathbb{Z}G/C_{2^n}$-module $\mathbb{Z}$ is obtained from the total complex of the tensor product of the standard projective resolutions, as $\mathbb{Z}C_{2^k}$ and $\mathbb{Z}C_{2^l}$ respectively, of the trivial module $\mathbb{Z}$.

Set $\Lambda := \mathbb{Z}C_{2^k} \times \mathbb{Z}C_{2^l}$. Using Proposition 2 we get a projective resolution

$$0 \longleftarrow \mathbb{Z} \longleftarrow \Lambda \longleftarrow \Lambda^2 \longleftarrow \Lambda^3 \longleftarrow \Lambda^4 \longleftarrow \Lambda^5 \longleftarrow$$

The differentials are

$$d^1 = \begin{pmatrix} C \\ B \end{pmatrix}; d^2 := \begin{pmatrix} \Delta_c & 0 \\ -B & C \\ 0 & \Delta_b \end{pmatrix}; d^3 := \begin{pmatrix} C & 0 & 0 \\ B & \Delta_c & 0 \\ 0 & -\Delta_b & C \\ 0 & 0 & B \end{pmatrix}; d^4 := \begin{pmatrix} \Delta_c & 0 & 0 & 0 \\ -B & C & 0 & 0 \\ 0 & \Delta_b & \Delta_c & 0 \\ 0 & 0 & -B & C \\ 0 & 0 & 0 & \Delta_b \end{pmatrix}; \ldots$$

Let $\Gamma := \mathbb{Z}G$. To compute a complex giving a free resolution of the trivial $\mathbb{Z}G$-module $\mathbb{Z}$ we consider the total complex of the tensor product of the standard resolution of $\mathbb{Z}C_{2^n}$-modules and the resoution obtained for Λ-modules. So we obtain the diagram:

$$\begin{array}{ccccccccccc}
\Gamma_{0,0} & \leftarrow & \Gamma^2_{1,0} & \leftarrow & \Gamma^3_{2,0} & \leftarrow & \Gamma^4_{3,0} & \leftarrow & \Gamma^5_{4,0} & \leftarrow & \Gamma^6_{5,0} \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
\Gamma_{0,1} & \leftarrow & \Gamma^2_{1,1} & \leftarrow & \Gamma^3_{2,1} & \leftarrow & \Gamma^4_{3,1} & \leftarrow & \Gamma^5_{4,1} & \leftarrow & \Gamma^6_{5,1} \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
\Gamma_{0,2} & \leftarrow & \Gamma^2_{1,2} & \leftarrow & \Gamma^3_{2,2} & \leftarrow & \Gamma^4_{3,2} & \leftarrow & \Gamma^5_{4,2} & \leftarrow & \Gamma^6_{5,2} \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
\Gamma_{0,3} & \leftarrow & \Gamma^2_{1,3} & \leftarrow & \Gamma^3_{2,3} & \leftarrow & \Gamma^4_{3,3} & \leftarrow & \Gamma^5_{4,3} & \leftarrow & \Gamma^6_{5,3} \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
\Gamma_{0,4} & \leftarrow & \Gamma^2_{1,4} & \leftarrow & \Gamma^3_{2,4} & \leftarrow & \Gamma^4_{3,4} & \leftarrow & \Gamma^5_{4,4} & \leftarrow & \Gamma^6_{5,4} \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
\Gamma_{0,5} & \leftarrow & \Gamma^2_{1,5} & \leftarrow & \Gamma^3_{2,5} & \leftarrow & \Gamma^4_{3,5} & \leftarrow & \Gamma^5_{4,5} & \leftarrow & \Gamma^6_{5,5} \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
\Gamma_{0,6} & \leftarrow & \Gamma^2_{1,6} & \leftarrow & \Gamma^3_{2,6} & \leftarrow & \Gamma^4_{3,6} & \leftarrow & \Gamma^5_{4,6} & \leftarrow & \Gamma^6_{5,6}
\end{array}$$

The vertical differentials

$$d_0 : \Gamma^{i+1}_{i,j} \longrightarrow \Gamma^{i+1}_{i,j-1}$$

are multiplication by A if j is odd and multiplication by Δ_a if j is even.

To define d_1 we proceed in several steps. The morphisms

$$d_1 : \Gamma^{i+1}_{i,0} \longrightarrow \Gamma^{i}_{i-1,0}$$

are determined by the same matrix as the morphisms

$$d_1 : \Lambda^{i+1}_{i,0} \longrightarrow \Lambda^{i}_{i-1,0}\ .$$

The morphisms

$$d_1 : \Gamma^{i+1}_{i,1} \longrightarrow \Gamma^{i}_{i-1,1}$$

and

$$d_1 : \Gamma^{i+1}_{i,2} \longrightarrow \Gamma^{i}_{i-1,2}$$

are defined by matrices obtained from the matrices in degree $(*, 0)$ by replacing c by $c\nabla^m_a$ and b by $b\nabla^{2^n-1}_a$. Continuing this way, the morphisms

$$d_1 : \Gamma^{i+1}_{i,2f-1} \longrightarrow \Gamma^{i}_{i-1,2f-1}$$

and

$$d_1 : \Gamma^{i+1}_{i,2f} \longrightarrow \Gamma^{i}_{i-1,2f}$$

are defined by matrices obtained from the matrices in degree $(*,0)$ by replacing c by $c(\nabla_a^m)^f$ and b by $b(\nabla_a^{2^n-1})^f$. Now note that

$$Ab = b(a-1)\nabla_a^{2^n-1} \text{ and } Ac = c(a-1)\nabla_a^m$$

and thus

$$A\left(c(\nabla_a^m)^{f-1}-1\right) = \left(c(\nabla_a^m)^f-1\right)A \text{ and } A\left(b(\nabla_a^{2^n-1})^{f-1}-1\right) = \left(b(\nabla_a^{2^n-1})^f-1\right)A.$$

Since also Δ_a is central in $\mathbb{Z}G$ and one then easily verifies that the diagram is commutative, i.e., $d_0d_1 = d_1d_0$. The diagram does not have exact lines, however. Nevertheless, since the extension for G splits, in degree $(*,0)$ the sequence is exact. Next, define

$$0 = d_2 : \Gamma_{i,2f}^{i+1} \longrightarrow \Gamma_{i-2,2f+1}^{i-1}$$

and

$$d_2 : \Gamma_{i,2f-1}^{i+1} \longrightarrow \Gamma_{i-2,2f}^{i-1}$$

is determined by the $(i+1)\times(i-1)$-matrix

$$d_2 := -\begin{pmatrix} \lambda_f & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & \lambda_f & 0 & & & & \vdots \\ \kappa_f & 0 & \lambda_f & 0 & & & \vdots \\ 0 & \kappa_f & 0 & \lambda_f & 0 & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & & 0 & \kappa_f & 0 & \lambda_f \\ \vdots & & & & 0 & \kappa_f & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & \kappa_f \end{pmatrix}$$

with

$$\kappa_f := \frac{(2^n-1)^{f\cdot 2^k}-1}{2^n} \text{ and } \lambda_f := \frac{m^{f\cdot 2^l}-1}{2^n},$$

That the latter numbers are integers follows from the following calculations.

Since $c\nabla_a^m A = Ac$ we get

$$\left((c\nabla_a^m)^f\right)^{2^l}\cdot A \quad = \quad Ac^{2^l} = A$$

and hence $\left((c\nabla_a^m)^f\right)^{2^l} - 1 = y\Delta_a$ for some $y \in \mathbb{Z} < b, c >$. Putting $a = 1$ one obtains

$$(cm)^{f2^l} - 1 = m^{f2^l} - 1 = y2^n.$$

This gives

$$\left((c\nabla_a^m)^f\right)2^l - 1 = \frac{m^{f2^l}-1}{2^n}\Delta_a \; .$$

Similarly, as $\nabla_a^{2^n-1} = \Delta_a - a^{-1}$,

$$\begin{aligned}
(b\nabla_a^{2^n-1})^{2^k} - 1 &= b^{2^k}\ \left((\Delta_a - a)(\Delta_a - a^{-1})\right)^{2^{k-1}} - 1 \\
&= \left((2^n-2)\Delta_a + 1\right)^{2^{k-1}} - 1 \\
&= \left((\Delta_a - 1)(\Delta_a - 1)\right)^{2^{k-1}} - 1 \\
&= (\Delta_a - 1)^{2^k} - 1 \\
&= \sum_{i=1}^{2^k} (-1)^{2^k-i}\binom{2^k}{i}\Delta_a^i \\
&= \sum_{i=1}^{2^k} (-1)^{2^k-i}\binom{2^k}{i} 2^{n(i-1)}\Delta_a \\
&= 2^{-n}\left(\sum_{i=1}^{2^k} (-1)^{2^k-i}\binom{2^k}{i}(2^n)^i\right)\Delta_a \\
&= \left(\frac{(2^n-1)^{2^k} - 1}{2^n}\right)\Delta_a
\end{aligned}$$

Thus $\left(b\Delta_a^{2^n-1}\right)^{2^k} A = A$. It follows that $\left(b\Delta_a^{2^n-1}\right)^{2^k} - 1 \in \mathbb{Z}\langle b, c\rangle\Delta_a$. So we obtain for any f,

$$\left(b(\nabla_a^{2^n-1})^f\right)^{2^k} - 1 = \frac{(2^n-1)^{f\cdot 2^k} - 1}{2^n}\Delta_a$$

Further one easily verifies that

$$\begin{aligned}
c\,(\nabla_a^m)^f\, b\ \left(\nabla_a^{2^n-1}\right)^f &= cb(m\Delta_a - a^{-1}\nabla_{a^{-1}}^m) \\
&= cb(m\Delta_a - (a^{-1} + a^{-2} + \cdots + a^{-m}) \\
&= b\left(\nabla_a^{2^n-1}\right)^f c\,(\nabla_a^m)^f
\end{aligned}$$

Making use of all this information one now verifies that

$$d_2d_0 + d_1d_1 + d_0d_2 = d_1d_1 + d_0d_2 = 0$$

in odd rows and

$$d_2d_0 + d_1d_1 + d_0d_2 = d_1d_1 + d_2d_0 = 0$$

in even rows. So we found the first two terms of the spectral sequence leading to Wall's criterion (Proposition 2) for a projective resolution.

Since the matrix for the differentials d^k in $\mathbb{Z} < b, c >$ have nonzero entries only in the diagonal and the lower diagonal, and because the matrix defining d_2 never has two consecutive non zero entries, one observes by elementary computation that one can pose $d_3 = 0$.

Since $d_2d_2 = 0$ the defining equation for d_4 degenerates to $d_0d_4 + d_4d_0 = 0$ and here $d_4 = 0$ is possible. Consequently, we can take $d_r = 0$ for any $r \geq 3$. Hence,

$$d := d_0 + (-1)^i d_1 + d_2$$

already is a differential (here the $(-1)^i$ means that one applies d_1 on even columns and one applies $-d_1$ on odd columns).

Lemma 8 *The differential d as defined above yields a projective resolution of the trivial $\mathbb{Z}G$-module $\mathbb{Z}$.*

Proof. This is an immediate consequence of the above discussion and the theorem of C. T. C. Wall. ∎

Corollary 9 *The spectral sequence provided by the bigraded module used in C.T.C. Wall's construction for this group degenerates in degree 3.* ∎

We write down the first terms of this resolution explicitly.

$$\Gamma \leftarrow \Gamma^2 \oplus \Gamma \leftarrow \Gamma^3 \oplus \Gamma^2 \oplus \Gamma \leftarrow \Gamma^4 \oplus \Gamma^3 \oplus \Gamma^2 \oplus \Gamma \leftarrow \Gamma^5 \oplus \Gamma^4 \oplus \Gamma^3 \oplus \Gamma^2 \oplus \Gamma \leftarrow \dots$$

the differentials are

$$d_1 := \begin{pmatrix} C \\ B \\ A \end{pmatrix}; d_2 := \begin{pmatrix} \Delta_c & 0 & 0 \\ -B & C & 0 \\ 0 & \Delta_b & 0 \\ A & 0 & 1-c\nabla_a^m \\ 0 & A & 1-b\nabla_a^{2^n-1} \\ 0 & 0 & \Delta_a \end{pmatrix};$$

$$d_3 := \begin{pmatrix} C & 0 & 0 & 0 & 0 & 0 \\ B & \Delta_c & 0 & 0 & 0 & 0 \\ 0 & -\Delta_b & C & 0 & 0 & 0 \\ 0 & 0 & B & 0 & 0 & 0 \\ A & 0 & 0 & -\Delta_{c\nabla_a^m} & 0 & -\lambda_1 \\ 0 & A & 0 & b\nabla_a^{2^n-1}-1 & 1-c\nabla_a^m & 0 \\ 0 & 0 & A & 0 & -\Delta_{b\nabla_a^{2^n-1}} & -\kappa_1 \\ 0 & 0 & 0 & \Delta_a & 0 & c\nabla_a^m-1 \\ 0 & 0 & 0 & 0 & \Delta_a & b\nabla_a^{2^n-1}-1 \\ 0 & 0 & 0 & 0 & 0 & A \end{pmatrix};$$

$d_4 :=$

$$\begin{pmatrix} \Delta_c & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -B & C & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \Delta_b & \Delta_c & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -B & C & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Delta_b & 0 & 0 & 0 & 0 & 0 & 0 \\ A & 0 & 0 & 0 & 1-c\nabla_a^m & 0 & 0 & -\lambda_1 & 0 & 0 \\ 0 & A & 0 & 0 & 1-b\nabla_a^{2^n-1} & -\Delta_{c\nabla_a^m} & 0 & 0 & -\lambda_1 & 0 \\ 0 & 0 & A & 0 & 0 & \Delta_{b\nabla_a^{2^n-1}} & 1-c\nabla_a^m & -\kappa_1 & 0 & 0 \\ 0 & 0 & 0 & A & 0 & 0 & 1-b\nabla_a^{2^n-1} & 0 & -\kappa_1 & 0 \\ 0 & 0 & 0 & 0 & \Delta_a & 0 & 0 & \Delta_{c\nabla_a^m} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \Delta_a & 0 & 1-b\nabla_a^{2^n-1} & c\nabla_a^m-1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \Delta_a & 0 & \Delta_{b\nabla_a^{2^n-1}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & A & 0 & 1-c(\nabla_a^m)^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A & 1-b(\nabla_a^{2^n-1})^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Delta_a \end{pmatrix}$$

4.2 The action of the automorphism

We shall have to define homomorphisms

$$\tau_k : \oplus_{j=1}^{k+1}\Gamma^j \longrightarrow {}^{\alpha}(\oplus_{j=1}^{k-1}\Gamma^j)$$

and then τ_k defines the action of α_0 on the cohomology in degree k. Because of Lemma 1 we shall define matrices M_k such that $\tau_k(x) = \alpha(x) \cdot M_k$.

For the moment we put $\nabla := \nabla_a^{2^{n-1}}$ and $\nabla^x := \nabla_a^x$. It is clear that we may put $M_0 = 1$. Now, for $k \geq 1$ the matrix k has to be such that

$$\alpha_0(d_k) = M_k d_k.$$

It is easy to see that

$$M_1 = \begin{pmatrix} a^{2^{n-1}} & 0 & \nabla \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$M_2 := \begin{pmatrix} 1 & 0 & 0 & \Delta_{c^2}\nabla & 0 & \frac{1+m}{2}c\Delta_{c^2} \\ 0 & a^{2^{n-1}} & 0 & 0 & \nabla & (2^{n-1}-1)b \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & a^{2^{n-1}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

satisfy the desired equalities.

The matrix M_3 can be taken as the following.

$$\begin{pmatrix} a^{2^{n-1}} & 0 & 0 & 0 & \nabla & 0 & 0 & \frac{1}{2}(c-1)\Delta_{c^2} + \Delta_{cm} & 0 & \sigma \\ 0 & 1 & 0 & 0 & 0 & \Delta_{c^2}\nabla & 0 & -\Delta_{c^2}b(2^{n-1}-1) & \frac{1+m}{2}c\Delta_{c^2} & \rho \\ 0 & 0 & a^{2^{n-1}} & 0 & 0 & 0 & \nabla & 0 & (2^{n-1}-1)b & (2^{n-1}-1)(\nabla)^2 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \frac{1+m}{2}c\nabla^m\Delta_{(c\nabla^m)^2} \\ 0 & 0 & 0 & 0 & 0 & a^{2^{n-1}} & 0 & a^{2^{n-1}} & 0 & (2^{n-1}-1)b\nabla^{2^n-1} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a^{2^{n-1}} & 0 & \nabla \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Here, ρ is a solution of the following equation (this obtained from computing the (2,6)-position of the respective matrices)

$$\begin{aligned} -\Delta_{c^2}b(2^{n-1})(c\nabla^m - 1) + \frac{1+m}{2}c\Delta_{c^2}(b\nabla^{2^n-1} - 1) + \rho(a-1) \\ = (b-1)\frac{1+m}{2}c\Delta_{c^2} + (1 + a^{2^{n-1}}c)\Delta_{c^2}(2^{n-1}-1)b \end{aligned}$$

This is equivalent to

$$bc\Delta_{c^2}((2^{n-1}-1)(\nabla^m + a^{2^{n-1}} + \frac{1+m}{2}(-\nabla^{2^n-1}+1))) = \rho(1-a)$$

Since the equation on the left hand side has augmentation 0 with respect to a, the equation is solvable in the augmentation ideal with respect to a, hence the existence of ρ is proved.

The corresponding equation for σ is (this is obtained from computing the (1,6)-position):

$$\sigma(a-1) + \frac{1}{2}((c-1)\Delta_{c^2} + \Delta_{cm})(c\nabla^m - 1) - \lambda\nabla^{2^{n-1}} \quad = \quad (ca^{2^{n-1}} - 1)\frac{1+m}{2}c\Delta_{c^2}$$

Since $-\frac{1}{2}((c-1)\Delta_{c^2} + \Delta_{cm})(c\nabla^m - 1) + \lambda\nabla^{2^{n-1}} + (ca^{2^{n-1}} - 1)\frac{1+m}{2}c\Delta_{c^2}$ has augmentation 0 with respect to a, this equation is solvable for σ.

In d_3 only λ might be of odd augmentation. The rest of the coefficients of $Hom_{\mathbb{Z}G}(d_3, \mathbb{F}_2)$ are 0 regarded as mapping between two copies of $Hom_{\mathbb{Z}G}(\Gamma, \mathbb{F}_2)$. Thus, if we write the elements of $Hom_{\mathbb{Z}G}(\Gamma^n, \mathbb{F}_2) \cong (Hom_{\mathbb{Z}G}(\Gamma, \mathbb{F}_2))^n$ as column vectors, then the image of $Hom_{\mathbb{Z}G}(d_3, \mathbb{F}_2)$ is contained in the 5-th row. The kernel of d_4 has, for the same reason, a complement that is contained in the direct sum of the 8-th and the 9-th row. In any case, the $10-th$ row (as well as the 6-th row) is fully contained in $H^3(G, \mathbb{F}_2)$. Because $(2^{n-1}-1)b\nabla_a^{2^n-1}$ has odd augmentation, we get that M_3 does not act trivially on $H^3(G, \mathbb{F}_2)$.

This finishes the proof of Theorem 1. ■

Remark 10 We remark that with the very same methods one can see that with $G = C_8 \rtimes Q_8$ and Q_8 being the quaternion group of order 8, and this acting on C_8 via the natural epimorphism

$$Q_8 \twoheadrightarrow C_2 \times C_2 \simeq Aut(C_8)$$

any non inner but almost inner automorphism of G induces a non trivial automorphism of $H^3(G, \mathbb{F}_2)$. This is interesting since in [9] one constructs a group

$$H = (C_p^2 \times C_q^2) \rtimes (G \rtimes C_2)$$

and a non inner automorphism α of H which becomes inner in SH for either a suitably big ring of algebraic integers S or for any semilocalization S of $\mathbb{Z}$. Of course, it follows that α is almost inner and thus α induces the identity on $H^*(H, \mathbb{Z})$.

We remark that D. Benson computes in [1] the first terms of a projective resolution of the trivial module $\mathbb{Z}$ for the group $(C_p \times C_q) \rtimes Q_8$ in view of a possible counterexample to the Poincaré conjecture.

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