A counterexample for the Isomorphism-problem of polycyclic groups

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1 Introduction

This note is very closely related to to the paper [4] in this issue.

It is a long outstanding problem in representation theory of groups, to find non–isomorphic groups, which have isomorphic group rings. For finite groups some progress in the positive direction was achieved in [2] and [3].

Here we shall give an example of two non–isomorphic polycyclic groups G_1 and G_2, such that the integral group rings $\mathbb{Z}G_1$ and $\mathbb{Z}G_2$ are Morita equivalent. This implies that RG_1 and RG_2 are isomorphic for a suitabley choosen [1] ring of algebraic integers R in a global algebraic number field, such that

$$RG_1 \simeq RG_2\,.$$

2 The example

In our paper [4] we have constructed a finite group G and an automorphism α of G, which is not inner, but which lies in the kernel of the natural homomorphism

$$Aut(G) \to Out(\mathbb{Z}_\pi G)\,;$$

here $\mathbb{Z}_\pi$ is the semilocalization of $\mathbb{Z}$ at a finite set of rational primes, which contains all the prime divisors of $|G|$. This group G and the automorphism α are the crucial ingredients for our counterexample.

M. Mazur (cf. [1]) brought the following construction of a certain type of semidirect products to our interest.

Let H be a group and let β be an automorphism of H. Then, we define the group $H \rtimes_\beta C_\infty$ as follows.

As a set $H \rtimes_\beta C_\infty$ is the product $H \times C_\infty$, where C_∞ is the infinite cyclic group. The multiplication in it is defined for $g, g' \in G$, $z, z' \in C_\infty$ as

$$(g, z) \cdot (g', z') = (g \cdot \beta^z(g'), z \cdot z')\,.$$

As a word of explanation for the symbol β^z, we identify C_∞ with $\mathbb{Z}$, and under this identification the above expression makes sense.

It was noted by Mazur [1], that $H_\beta := H \rtimes_\beta C_\infty \simeq H_0 := H \times C_\infty$ for a finite group H if and only if β is inner. For the convenience of the reader we include the argument in the direction we need:

[1] R has to be big enough to kill a certain ideal class.

Assume that $H_0 \simeq H_\beta$. The group C_∞ in $H \times C_\infty$ is identified with $\mathbb{Z}$. The element $(1,1)$ in H_0 is then mapped by the above isomorphism to an element (h_0, x_0) in the centre of H_β. For any $(g,y) \in H_\beta$ we have

(1) $\quad (g \cdot \beta^y(h_0), x_0 + y) = (g,y) \cdot (h_0, x_0) = (h_0, x_0) \cdot (g,y) = (h_0 \cdot \beta^{x_0}(g), x_0 + y).$

Choosing $g = 1$, we see that β centralizes h_0. Now, $(1,y)$ is mapped onto $(h^y, y \cdot x_0)$ and since H is the maximal finite normal subgroup in H_0 and H_β, $H \leq H_0$ is mapped onto $H \leq H_\beta$. Fixing the isomorphism above, an automorphism α of H is induced. Since an element $(h,z) = (h,0) \cdot (1,z) \in H_0$ is mapped onto $(\alpha(h), 0) \cdot (h_0^z, z x_0)$, the element $(1,1)$ can be in the image only if $x_0 \in \{1,-1\}$. Equation 1 tells that β is conjugation by h_0 or by h_0^{-1}.

On the other hand, Mazur [1] noted, that for β which becomes inner in RH for an intergral domain R of characteristic zero the group rings RH_0 and RH_β are isomorphic; in fact, the following map induces an isomorphism:
Let $\beta(g) = u \cdot g \cdot u^{-1}$ with a fixed unit $u \in U(RH)$. Then [2], $(h,z) \longrightarrow (h,z) \cdot u^{-z}$ induces an isomorphism

$$RH_0 \xrightarrow{\varphi(R)} RH_\alpha\,,$$

as is easily seen. Putting this together we obtain

Theorem 1 *Let R be an integral domain of characteristic zero, such that α is inner in RG, then for the group G and the automorphism α from [4] we have:*

1. *G_0 is not ismorphic to G_α.*
2. *$RG_0 \simeq RG_\alpha$.*

Remark 1 One might be tempted to ask, why this construction does not work for finite groups; i. e. why can't we repalce C_∞ by C_n, a cyclic group of order n. An analysis of the above arguments shows, that then u [3] must have order dividing n. On the othe hand, one knows, that for finite groups F_1 and F_2, an isomorphism $\mathbb{Z}F_1 \simeq \mathbb{Z}F_2$ implies that F_1 is a direct product of two groups if and only if F_2 is a direct product of two groups of the same order.
A consequence of these considerations is that u cannot have finite order.

We point out that most likely, the automorphism α is also inner in $\mathbb{Z}G$, and not only in the semilocalization; however, we did not elaborate on this.

Corollary 1 *The group rings $\mathbb{Z}G_0$ and $\mathbb{Z}G_\alpha$ are Morita equivalent.*

Proof: We consider the bimodule $M := {}_\alpha\mathbb{Z}\mathbb{G}_1$, which is surely a progenerator for $\mathbb{Z}G$, note that it need not be free, but it just may be projectve. Now the induced module $M{\uparrow}^{H_0}$ is a bimodule for $\mathbb{Z}G_\alpha, \mathbb{Z}G_0$, which induces a Morita equivalence. q.e.d.

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[2] Note that we have identified C_∞ with $\mathbb{Z}$.
[3] Recall the α is conjugation by u.

References

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