Two-sided tilting complexes for Gorenstein orders

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Abstract

Work of J. Rickard proves that the derived module categories of two rings A and B are equivalent as triangulated categories if and only if there is a particular object T, a so-called tilting complex, in the derived category of A such that B is the endomorphism ring of T. The functor inducing the equivalence however is not explicit by the knowledge of T. Suppose the derived categories of A and B are equivalent. If A and B are R-algebras and projective of finite type over the commutative ring R, then Rickard proves the existence of a so-called two-sided tilting complex X, which is an object in the derived category of bimodules. The left derived tensor product by X is then an equivalence between the derived categories of A and B. There is no general explicit construction known to derive X from the knowledge of T. In an earlier paper S. König and the author gave for a class of algebras a tilting complex T by a general procedure with prescribed endomorphism ring. Under some mild additional hypothesis we construct in the present paper an explicit two-sided tilting complex whose restriction to one side is any given one-sided tilting complex of the type described in the above cited paper. This provides two-sided tilting complexes for various cases of derived equivalences, making the functor inducing this equivalence explicit. In particular the perfect isometry induced by such a derived equivalences is determined.

Introduction

A well known process due to Grothendieck and Verdier [16] associates to any abelian category, e.g. category of modules over a ring A, a bounded derived category which is a so-called triangulated category. In case of the abelian category to be the category of modules over a ring A, the bounded derived category is denoted by $D^b(A)$. For more details of this procedure we refer to [16] or [8]. The question of when two rings A and B induce equivalent bounded derived module categories $D^b(A)$ and $D^b(B)$ as triangulated categories is treated by Rickard in [10]. There he proves that $D^b(A)$ is equivalent to $D^b(B)$ as triangulated categories if and only if there is a particular object in $D^b(A)$, a so-called *tilting complex* T, such that

$$End_{D^b(A)}(T) \simeq B \ .$$

This description is completely general, but the functor between $D^b(A)$ and $D^b(B)$ inducing the equivalence, called *derived equivalence*, remains nevertheless not at all easy to handle and is up to some extent even not constructable just from the knowledge of T. Rickard proves in [11] for algebras A and B over a commutative ring R which are flat as R-modules the existence of a complex X in $D^b(A \otimes_R B^{op})$ such that

$$X \otimes^{\mathbb{L}}_B - : D^b(B) \longrightarrow D^b(A)$$

is an equivalence of triangulated categories. The complex X is called *two-sided tilting complex.* One obtains the tilting complex T as image of the rank one free B-module:

$$T \simeq X \otimes^{\mathbb{L}}_B B \ .$$

B. Keller gives a significantly simpler construction of X in terms of differential graded algebras and without the hypothesis that B is flat over R in [6, 8]. To distinguish between tilting complexes and two-sided tilting complexes we call T a *one-sided tilting complex* or just tilting complex. By this description, the equivalence between $D^b(A)$ and $D^b(B)$ is as explicit as the knowledge X. However, a module theoretic construction of X out of a knowledge of T in general is not known so far, even though Keller constructs X as a certain projective resolution of T as differential graded module over a certain differential graded algebra [8].

In [7] a one-sided tilting complex T is constructed for a Gorenstein order Λ over a complete discrete valuation domain R with residue field of characteristic p such that the endomorphism ring T is a pullback of the endomorphism rings of the homology of the complex T over the endomorphism ring of the homology in the stable category. Recall that an R-algebra Λ is called an *R-order* if Λ is finitely generated projective as R-module such that $K \otimes_R \Lambda$ is semisimple for the field of fractions K of R. An order Λ is *Gorenstein* if $Ext^1_\Lambda(L, \Lambda) = 0$ for any Λ-module L which is finitely generated projective as R-module. Gorenstein orders are well suited for the above problem. In fact, a lot of effort is undertaken in recent years to construct derived equivalences between blocks of group rings RG and RH for two finite groups G and H. Recall that the principal block of a group ring RG is the indecomposable ring direct factor of RG which acts on the trivial G-module. Broué conjectured [1], see also [8], that if the two groups G and H have isomorphic abelian Sylow p-subgroups and if moreover the normalizers of a Sylow p-subgroup of G and H are isomorphic, then the principal blocks of G and of H should have equivalent derived categories. Broué explains many for a long time conjectured and many known but not sufficiently explained phenomenons out of this derived equivalence. Now, principal blocks of group rings of finite groups are Gorenstein orders, they are even symmetric. Most of the complexes used so far in proving Broué's conjecture satisfy the hypotheses of [7].

What we do in the present paper is to give an *explicit two-sided tilting complex* X such that the image of the rank one free $End_{D^b(\Lambda)}(T)$-module is the tilting complex T discussed in [7]. To do this we have to make an additional hypotheses which is not very restrictive if one is interested in Broué's conjecture.

Twosided tilting complexes do not only provide the explicit equivalence between the derived categories but may be used also for other purposes. In fact, finding a one-sided tilting complex T of the type discussed in [7] together with verification of some hypothesis implies by our paper the explicit knowledge of a two-sided tilting complex X restriction of which to the left is isomorphic to T. The work of Marcus [9] reduces Broué's conjecture to its validity between the principal blocks of finite simple groups with abelian Sylow p-subgroups and the principal block of the normalizer of a Sylow p-subgroup together with a technical condition to be verified on the two-sided tilting complex providing this derived equivalence. The two-sided tilting complex has to be known very explicitly for to be tested if it verifies these technical condition. Hence, our construction gives a tool for verifying these.

A perfect isometry [1] between two finite groups G and H is a special kind of isometry of the character ring of the characters belonging to the principal block of G and the character ring of the characters belonging to the principal block of H. As a further application we give explicitly the perfect isometry induced by the two-sided tilting complex X. Moreover, we are able to control up to a certain extent all perfect isometries who come from derived equivalences $F : D^b(\Gamma) \longrightarrow D^b(\Lambda)$ such that $F(\Gamma) \simeq T$ for T as in [7].

Besides the interest coming from Broué's conjecture we feel that the construction itself deserves attention. The fact that two rings are derived equivalent gives many ring theoretic information, and controlling the equivalence explicitly should give even more.

At the very end we give examples where the theorem applies. Amongst them are algebras of semidihedral type in the sense of Erdmann [2].

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1 Recalling the one-sided situation

Our conventions composing mappings $\alpha : A \longrightarrow B$ and $\beta : B \longrightarrow C$ are as follows. We write $\beta \circ \alpha$ when we look at images of particular elements of A. We write $\alpha\beta$ when we write mappings on the right, what we usually do when we discuss commutative diagrams.

Throughout, we fix a complete discrete valuation domain R with field of fractions K and a Gorenstein order Λ. We recall that an order Λ is an R-algebra which is finitely generated projective as R-module and $K \otimes_R \Lambda$ is a semisimple K-algebra. A Λ-lattice is a Λ-module which is finitely generated projective as R-module. An R-order Λ is called a Gorenstein R-order if $Ext^1_\Lambda(L, \Lambda) = 0$ for any Λ-lattice L. A morphism $U \longrightarrow V$ between two lattices U and V is called pure if its cokernel is a lattice. A sublattice U of a lattice V is a pure sublattice if the embedding is a pure homomorphism.

When we regard complexes we mean chain complexes T with differential $d_i : T_i \longrightarrow T_{i-1}$ with $d_i \circ d_{i+1} = 0$ for all $i \in \mathbb{Z}$ and we regard homology $H_*(T)$ unless otherwise stated. For further conventions and definitions concerning derived categories we refer to Verdier [16] and to [8].

In [7] we constructed a one-sided tilting complex T with endomorphism ring Γ as follows. Take a Λ-lattice L with projective cover

$$Q \stackrel{\lambda}{\longrightarrow} L$$

and a projective cover

$$P \stackrel{\pi}{\longrightarrow} \Omega L$$

of $\Omega L := ker\ \lambda$ with $C := ker\ \pi$. Denote by φ the resulting homomorphism $P \longrightarrow Q$. We hence get a four term exact sequence

$$0 \longrightarrow C \stackrel{\iota}{\longrightarrow} P \stackrel{\varphi}{\longrightarrow} Q \stackrel{\lambda}{\longrightarrow} L \longrightarrow 0.$$

We choose a finitely generated projective Λ-module $\tilde{P}$ such that $Q \oplus P \oplus \tilde{P}$ is a progenerator for Λ.

Theorem 1 [7, 8] *Let R, Λ, P, Q and $\tilde{P}$ be as above. Suppose that $Hom_\Lambda(P \oplus \tilde{P}, L) = 0$. Then, the complex T*

$$\ldots \longrightarrow 0 \longrightarrow P \oplus P \oplus \tilde{P} \stackrel{(\varphi,0,0)}{\longrightarrow} Q \longrightarrow 0 \longrightarrow \ldots$$

with homology concentrated in degrees 0 and 1 is a tilting complex. Denoting by $\overline{End}_\Lambda(L)$ the quotient of the ring of Λ-linear endomorphisms of L modulo those endomorphisms factoring over a projective module, then $End_{D^b(\Lambda)}(T) \simeq \Gamma$ occurs in the following pullback diagram.

$$\begin{array}{ccc} \Gamma & \longrightarrow & End_\Lambda(H_0(T)) \\ \downarrow & & \downarrow \\ End_\Lambda(H_1(T)) & \longrightarrow & \overline{End}_\Lambda(L) \end{array}$$

2 Exploring Γ

We fix L, Q, P, $\tilde{P}$, φ, π and ι as in Theorem 1.

Lemma 1 *If $Hom_\Lambda(P \oplus \tilde{P}, L) = 0$, then there is an idempotent $e^2 = e$ in the centre $Z(K \otimes_R \Lambda)$ of $K \otimes_R \Lambda$ such that $L \simeq e \cdot Q$ and $e \cdot (P \oplus \tilde{P}) = 0$.*

Proof. Set $A := K \otimes_R \Lambda$. Since $Hom_\Lambda(P \oplus \tilde{P}, L) = 0$, for any primitive central idempotent f in A with $f \cdot (P \oplus \tilde{P}) \neq 0$ one gets $f \cdot L = 0$. In fact, take any $\alpha \in Hom_A(K \otimes_R (P \oplus \tilde{P}), K \otimes_R L)$. Then $\alpha(P \oplus \tilde{P})$ is a Λ-lattice in $K \otimes_R L$, and hence there is a non zero element r in R, such that

$$r \cdot \alpha(P \oplus \tilde{P}) \subseteq L.$$

Then,

$$r \cdot \alpha \in Hom_\Lambda(P \oplus \tilde{P}, L) = 0$$

and therefore $r\alpha = 0$, which yields $\alpha = 0$.

Let

$$E := \{\epsilon \in A \mid \epsilon \text{ is a central primitive idempotent in } A \text{ with } \epsilon \cdot L \neq 0\}$$

Set $e := \sum_{\epsilon \in E} \epsilon$. Then, by the above, $e \cdot (P \oplus \tilde{P}) = 0$. We look at the short exact sequence

$$0 \longrightarrow \Omega L \longrightarrow Q \longrightarrow L \longrightarrow 0$$

and we apply $e\Lambda \otimes_\Lambda -$ to it. We get the exact sequence

$$\ldots \longrightarrow e\Omega L \longrightarrow eQ \longrightarrow eL \longrightarrow 0$$

where $e\Omega L = 0$ by the above and $eL = L$ by the definition of E. This concludes the proof of the lemma. ■

Lemma 2 *L is free of rank 1 as $End_\Lambda(L)$-module.*

Proof.
The following sequence is exact:

$$0 \longrightarrow \Omega L \longrightarrow Q \longrightarrow L \longrightarrow 0$$

We apply $Hom_\Lambda(-,L)$ to it and see

$$L \simeq Hom_\Lambda(\Lambda, L) \simeq \begin{pmatrix} Hom_\Lambda(Q,L) \\ Hom_\Lambda(P,L) \\ Hom_\Lambda(\tilde{P},L) \end{pmatrix} = \begin{pmatrix} Hom_\Lambda(Q,L) \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} Hom_\Lambda(L,L) \\ 0 \\ 0 \end{pmatrix} \simeq End_\Lambda(L)$$

as $End_\Lambda(L)$-module. ■

Assume that $Hom_\Lambda(C, \Omega L) = 0$.

Since from the very beginning we assumed that $Hom_\Lambda(P \oplus \tilde{P}, L) = 0$, Lemma 1 provides us with a central idempotent e in $A = K \otimes_R \Lambda$ with $e \cdot Q = L$ and $e \cdot (P \oplus \tilde{P}) = 0$. Moreover, if we define $e_{\Omega L}$ the sum of all central primitive idempotents e_i of A with $e_i \cdot \Omega L \neq 0$, and define

$$e_C := 1 - e - e_{\Omega L}.$$

The above hypotheses ensure that

$$e \cdot e_{\Omega L} = e \cdot e_C = e_{\Omega L} \cdot e_C = 0\ .$$

We define $\tilde{P}(c)$ by the following exact sequence.

$$0 \longrightarrow \tilde{P}(c) \longrightarrow \tilde{P} \longrightarrow e_{\Omega L}\tilde{P} \longrightarrow 0$$

Analogously to Lemma 1 we have

Lemma 3 *Assume that $Hom_\Lambda(C, \Omega L) = 0$. Then, we have natural isomorphisms*

- $e_{\Omega L} \cdot P \simeq \Omega L$
- $End_\Lambda(C) = Hom_\Lambda(C, P)$
- $Hom_\Lambda(P, Q) \simeq Hom_\Lambda(\Omega L, \Omega L)$
- $Hom_\Lambda(\tilde{P}, Q) \simeq Hom_\Lambda(e_{\Omega L}\tilde{P}, \Omega L)$
- $Hom_\Lambda(C, \tilde{P}(c)) \simeq Hom_\Lambda(C, \tilde{P})$

Proof.

$$0 \longrightarrow C \longrightarrow P \longrightarrow \Omega L \longrightarrow 0$$

is exact. Multiplying this short exact sequence by $e_{\Omega L}$, i.e. applying $e_{\Omega L}\Lambda \otimes_\Lambda -$, proves the first statement.

Applying $Hom_\Lambda(C, -)$ to

$$0 \longrightarrow C \longrightarrow P \longrightarrow \Omega L \longrightarrow 0$$

yields the second isomorphism.

Applying $Hom_\Lambda(P, -)$ to

$$0 \longrightarrow \Omega L \longrightarrow Q \longrightarrow L \longrightarrow 0$$

yields $Hom_\Lambda(P, Q) \simeq Hom_\Lambda(P, \Omega L)$. Applying $Hom_\Lambda(-, \Omega L)$ to

$$0 \longrightarrow C \longrightarrow P \longrightarrow \Omega L \longrightarrow 0$$

proves the third statement.

Applying $Hom_\Lambda(-, \Omega L)$ to

$$0 \longrightarrow \tilde{P}(c) \longrightarrow \tilde{P} \longrightarrow e_{\Omega L}\tilde{P} \longrightarrow 0$$

proves $Hom_\Lambda(e_{\Omega L}\tilde{P},\Omega L)\simeq Hom_\Lambda(\tilde{P},\Omega L)$. Applying $Hom_\Lambda(\tilde{P},-)$ to

$$0\longrightarrow \Omega L\longrightarrow Q\longrightarrow L\longrightarrow 0$$

gives the fourth statement.

Applying $Hom_\Lambda(C,-)$ to

$$0\longrightarrow \tilde{P}(c)\longrightarrow \tilde{P}\longrightarrow e_{\Omega L}\tilde{P}\longrightarrow 0$$

proves the fifth statement.

■

3 Inverting T

We recall from [8, Lemma 5.2.5].

Lemma 4 *Let S be a ring and let M and X be S-modules. Assume that X is a direct summand of M. Then, the functor*

$$Hom_S(M,-): S-Mod\longrightarrow End_S(M)-Mod$$

induces for any S-module Y an isomorphism

$$Hom_S(X,Y)\longrightarrow Hom_{End_S(M)}(Hom_S(M,X),Hom_S(M,Y))$$

Likewise, the functor

$$Hom_S(-,M): S-Mod\longrightarrow End_S(M)-Mod$$

induces for any S-module Y an isomorphism

$$Hom_S(Y,X)\longrightarrow Hom_{End_S(M)}(Hom_S(X,M),Hom_S(Y,M))$$

Proof. The first part is proven in [8, Lemma 5.2.5].

We give a different proof due to B. Keller for the reader's convenience.

The first statement follows from the counit of the following pair of adjunctions. The functors

$$Hom_\Lambda(M,-):\ \Lambda-mod\longrightarrow End_\Lambda(M)-mod$$

and

$$M\otimes_{End_\Lambda(M)}-:\ End_\Lambda(M)-mod\longrightarrow \Lambda-mod$$

form an adjoint pair. The counit of this adjunction

$$\eta: M\otimes_{End_\Lambda(M)} Hom_\Lambda(M,-)\longrightarrow Id$$

has the property that its evaluation η_M on M is an isomorphism:

$$M\otimes_{End_\Lambda(M)} Hom_\Lambda(M,M)\simeq M$$

Since the two functors $Hom_\Lambda(M,-)$ and $M\otimes_{End_\Lambda(M)}-$ are additive, the same is true for any direct summand X. Hence,

$$\begin{array}{rcl} Hom_{End_S(M)}(Hom_S(M,X),Hom_S(M,Y)) & \simeq & Hom_\Lambda(M\otimes_{End_\Lambda(M)} Hom_\Lambda(M,X),Y)\\ & \simeq & Hom_\Lambda(X,Y)\end{array}$$

For the second isomorphism we proceed as follows. Clearly, the functor $Hom_S(-,M)$ induces a mapping

$$\begin{array}{rcl}\Theta: Hom_S(Y,X) & \longrightarrow & Hom_{End_S(M)}(Hom_S(X,M),Hom_S(Y,M))\\ \alpha & \longrightarrow & (\chi\longrightarrow \alpha\chi)\end{array}$$

Take $\pi \in Hom_\Lambda(M,X)$ the natural projection and denote by $\iota : X \longrightarrow M$ the natural embedding. Define

$$\begin{array}{rcl}\Psi : Hom_{End_S(M)}(Hom_S(X,M), Hom_S(Y,M)) & \longrightarrow & Hom_S(Y,X) \\ \rho & \longrightarrow & \rho(\iota)\pi\end{array}$$

Now, for any $\alpha \in Hom_S(Y,X)$,

$$\begin{array}{rcl}\Psi(\Theta(\alpha)) & = & \Psi(\chi \longrightarrow \alpha\chi) \\ & = & \alpha\iota\pi \\ & = & \alpha\end{array}$$

Moreover,

$$\begin{array}{rcl}\Theta\Psi(\rho)) & = & \Theta(\rho(\iota)\pi) \\ & = & (\chi \longrightarrow \rho(\iota)\pi\chi) \\ & = & (\chi \longrightarrow \rho(\iota\pi\chi)) \\ & = & (\chi \longrightarrow \rho(\chi)) \\ & = & \rho\end{array}$$

This proves the lemma completely. ∎

Take a Gorenstein order Λ and a Λ-module L as in Theorem 1. Then, choose a projective module $\tilde{P}$ to form a tilting complex T as in Theorem 1.

We are now able to prove that under certain circumstances there is an 'inverse' to T which again is of the form described in Theorem 1.

A first step to this direction is in fact the observation that L has a natural structure of an $End_\Lambda(L)$-module. Moreover, $End_\Lambda(L)$ is an epimorphic image of Γ as rings, as follows by the description of Γ as pullback and by the observation that $H_0(T) \simeq L$.

The projective cover of $End_\Lambda(L)$ as Γ-module is computed as follows. Set $C := ker\phi$. Then,

$$End_\Lambda(H_1(T)) \simeq \begin{pmatrix} End_\Lambda(C) & Hom_\Lambda(C,P) & Hom_\Lambda(C,\tilde{P}) \\ Hom_\Lambda(P,C) & End_\Lambda(P) & Hom_\Lambda(P,\tilde{P}) \\ Hom_\Lambda(\tilde{P},C) & Hom_\Lambda(\tilde{P},P) & End_\Lambda(\tilde{P}) \end{pmatrix}$$

The pullback diagram

$$\begin{array}{ccc}\Gamma & \longrightarrow & End_\Lambda(H_0(T)) \\ \downarrow & & \downarrow \\ End_\Lambda(H_1(T)) & \longrightarrow & \overline{End}_\Lambda(L)\end{array}$$

gives us a morphism of the rank one free Γ-right module to $End_\Lambda(L)$. We can even determine the projective cover of $End_\Lambda(L)$ as Γ-module. Observe that

$$\begin{pmatrix} Hom_\Lambda(P,C) & End_\Lambda(P) & Hom_\Lambda(P,\tilde{P}) \\ Hom_\Lambda(\tilde{P},C) & Hom_\Lambda(\tilde{P},P) & End_\Lambda(\tilde{P}) \end{pmatrix}$$

maps as 0 to $\overline{End}_\Lambda(L)$. In fact, only the component $End_\Lambda(C)$ gives a contribution to the homomorphism, the mapping to $\overline{End}_\Lambda(L)$ being induced by the exact sequence

$$0 \longrightarrow C \longrightarrow P \longrightarrow Q \longrightarrow L \longrightarrow 0.$$

Hence, since the above short exact sequence is the projective cover sequence for L as Λ-module, the projective cover of $\overline{End}_\Lambda(L)$ as Γ-module is Q^* defined as pullback as follows:

$$\begin{array}{ccc}Q^* & \longrightarrow & End_\Lambda(L) \\ \downarrow & & \downarrow \\ Hom_\Lambda(C, C \oplus P \oplus \tilde{P}) & \longrightarrow & \overline{End}_\Lambda(L)\end{array}$$

The kernel of $Q^* \longrightarrow End_\Lambda(L)$ is isomorphic to the kernel of $Hom_\Lambda(C, C \oplus P \oplus \tilde{P}) \longrightarrow \overline{End}_\Lambda(L)$ using that Q^* is defined as a pullback. Since again the latter is defined by pulling mappings along the exact sequence

$$0 \longrightarrow C \longrightarrow P \longrightarrow Q \longrightarrow L \longrightarrow 0,$$

already $Hom_\Lambda(C, 0 \oplus P \oplus \tilde{P})$ is in the kernel of the mapping in question. Moreover, using the fact that Λ is a Gorenstein order, we see that

$$\overline{End}_\Lambda(L) \simeq \overline{End}_\Lambda(C).$$

We shall need also for later use the following.

Lemma 5 *Let Λ be a Gorenstein order, let U be a pure sublattice of a projective module V and denote by α the embedding. Then, for any lattice W the set of Λ-homomorphisms from U to W factoring through any projective module is $\alpha \cdot Hom_\Lambda(V, W)$.*

Proof. Let $U \longrightarrow W$ be a homomorphism which factors through a projective module X. We form the pushout diagram as below:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U & \longrightarrow & V & \longrightarrow & V/U & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & X & \longrightarrow & PO & \longrightarrow & V/U & \longrightarrow & 0 \end{array}$$

Since U is pure in V, also X is pure in PO. Since X is projective, the Gorenstein property gives us that the lower short exact sequence splits. Hence there is a mapping $PO \longrightarrow X$ such that $X \longrightarrow PO \longrightarrow X$ is the identity on X. Hence,

$$\begin{array}{rcl} (U \longrightarrow W) & = & (U \longrightarrow X \longrightarrow W) \\ & = & (U \longrightarrow X \longrightarrow PO \longrightarrow X \longrightarrow W) \\ & = & (U \longrightarrow V \longrightarrow PO \longrightarrow X \longrightarrow W) \\ & = & (U \longrightarrow V) \longrightarrow (PO \longrightarrow X \longrightarrow W) \end{array}$$

and $U \longrightarrow W$ already factors through the embedding of U into V. ■

We apply Lemma 5 to compute the kernel of the projective cover mapping. The endomorphisms of C factoring over any projective module are those factoring over the embedding $C \stackrel{\iota}{\longrightarrow} P$. Hence,

$$0 \longrightarrow (\ \iota \cdot Hom_\Lambda(P, C), Hom_\Lambda(C, P), Hom_\Lambda(C, \tilde{P})\ ) \longrightarrow Q^* \longrightarrow End_\Lambda(L) \longrightarrow 0$$

is an exact sequence of Γ-modules.

We look for a projective Γ-module mapping onto this kernel. We have the short exact sequence

$$0 \longrightarrow C \longrightarrow P \longrightarrow \Omega L \longrightarrow 0$$

of Λ-modules. We apply $Hom_\Lambda(-, C \oplus P \oplus \tilde{P})$ to it and get the exact sequence

$$0 \longrightarrow Hom_\Lambda(\Omega L, C \oplus P \oplus \tilde{P}) \longrightarrow Hom_\Lambda(P, C \oplus P \oplus \tilde{P}) \longrightarrow Hom_\Lambda(C, C \oplus P \oplus \tilde{P}) \longrightarrow Ext^1_\Lambda(\Omega L, C)$$

of Γ-modules where

$$im(\ Hom_\Lambda(P, C \oplus P \oplus \tilde{P}) \longrightarrow Hom_\Lambda(C, C \oplus P \oplus \tilde{P})\ ) = \iota \cdot Hom_\Lambda(P, C) \oplus Hom_\Lambda(C, P \oplus \tilde{P}).$$

Hence, the projective $End_\Lambda(C \oplus P \oplus \tilde{P})$-module $Hom_\Lambda(P, C \oplus P \oplus \tilde{P})$ maps onto our kernel. In fact it is not only a projective $End_\Lambda(C \oplus P \oplus \tilde{P})$-module but also a projective Γ-module: In fact, the pullback construction does only affect the $End_\Lambda(C)$ entry in $End_\Lambda(C \oplus P \oplus \tilde{P})$.

We hence get a projective resolution of $End_\Lambda(L)$ as Γ-module out of the following diagram.

$$\begin{array}{ccccc} 0 & & & & \\ \downarrow & & & & \\ Hom_\Lambda(\Omega L, C \oplus P \oplus \tilde{P}) & & & & \\ \downarrow & & & & \\ Hom_\Lambda(P, C \oplus P \oplus \tilde{P}) & \rightarrow & Q^* & \rightarrow & End_\Lambda(L) \rightarrow 0 \\ \downarrow & & \downarrow & \text{p.b.} & \downarrow \\ 0 \rightarrow \iota \cdot Hom_\Lambda(P, C) \oplus Hom_\Lambda(C, P \oplus \tilde{P}) & \rightarrow & Hom_\Lambda(C, C \oplus P \oplus \tilde{P}) & \rightarrow & \overline{End}_\Lambda(L) \rightarrow 0 \\ \downarrow & & & & \\ 0 & & & & \end{array}$$

We regard the complex T^*

$$\begin{array}{ccccccc} 0 & \longrightarrow & Hom_\Lambda(P, C\oplus P\oplus \tilde{P}) & \longrightarrow & Q^* & \longrightarrow & 0 \\ & & \oplus & & \oplus & & \\ 0 & \longrightarrow & Hom_\Lambda(P, C\oplus P\oplus \tilde{P}) & \longrightarrow & 0 & & \\ & & \oplus & & \oplus & & \\ 0 & \longrightarrow & Hom_\Lambda(\tilde{P}, C\oplus P\oplus \tilde{P}) & \longrightarrow & 0 & & \end{array}$$

of Γ-right modules with homology concentrated in degree 0 and 1.

We are ready to prove that T^* verifies the hypotheses of Theorem 1.

Lemma 6 *1.* $H_0(T^*) \simeq End_\Lambda(L)$ *and* $H_1(T^*) \simeq Hom_\Lambda(\Omega L\oplus P\oplus\tilde{P}, C\oplus P\oplus\tilde{P})$

2. $Q^*\oplus Hom_\Lambda(P\oplus\tilde{P}, C\oplus P\oplus\tilde{P})$ *is a progenerator of* Γ.

3. $Hom_\Gamma(\ Hom_\Lambda(P\oplus\tilde{P}, C\oplus P\oplus\tilde{P}), End_\Lambda(L)\ ) = 0.$

Proof. Part 1 of the proposition is clear from the construction of T^*.
Since the module in question is in fact free of rank 1, the second part follows also.
It is clear that Γ again is an order. We apply $K\otimes_R -$ to $Hom_\Lambda(P\oplus\tilde{P}, C\oplus P\oplus\tilde{P})$ and to $End_\Lambda(L)$. Now, as a general fact, $\overline{End}_\Lambda(L)$ is an R-torsion module and hence $K\otimes_R \overline{End}_\Lambda(L) = 0$. The idempotent e reappears in Γ as identity endomorphism on L. Since $K\otimes_R \overline{End}_\Lambda(L) = 0$,

$$K\otimes_R \Gamma \simeq (\ K\otimes_R End_\Lambda(L)\ )\oplus(\ K\otimes_R Hom_\Lambda(P\oplus\tilde{P}, C\oplus P\oplus\tilde{P})\ )$$

is a decomposition into two-sided ideals. Hence, also 3. follows. ∎

Corollary 1 *If* Γ *is again a Gorenstein order, then* $\Lambda^{op}\simeq End_{D^b(\Gamma)}(T^*)$.

Remark If Λ is symmetric, then also Γ is symmetric and symmetric orders are Gorenstein.

For the proof we use Lemma 4.

$$Hom_\Gamma(\ Hom_\Lambda(P\oplus\tilde{P}, C\oplus P\oplus\tilde{P}), Hom_\Lambda(P\oplus\tilde{P}, C\oplus P\oplus\tilde{P})\ ) \simeq Hom_\Lambda(P\oplus\tilde{P}, P\oplus\tilde{P})^{op}$$

and

$$Hom_\Gamma(\ Hom_\Lambda(P\oplus\tilde{P}, C\oplus P\oplus\tilde{P}), Hom_\Lambda(\Omega L, C\oplus P\oplus\tilde{P})\ ) \simeq Hom_\Lambda(\Omega L, P\oplus\tilde{P}) \simeq Hom_\Lambda(Q, P\oplus\tilde{P})$$

Those Γ-linear mappings from $Hom_\Lambda(P, C\oplus P\oplus\tilde{P})$ to $Hom_\Lambda(P\oplus\tilde{P}, C\oplus P\oplus\tilde{P})$ which factor through Q^* are precisely those which factor through $Hom_\Lambda(C, C\oplus P\oplus\tilde{P})$ since $End_\Lambda(L)$ lies in another component. Lemma 4 then gives us that

$$Hom_\Gamma(Hom_\Lambda(P, C\oplus P\oplus\tilde{P}), Hom_\Lambda(P\oplus\tilde{P}, C\oplus P\oplus\tilde{P}) \simeq Hom_\Lambda(P\oplus\tilde{P}, P)$$

and that

$$Hom_\Gamma(Hom_\Lambda(C, C\oplus P\oplus\tilde{P}), Hom_\Lambda(P\oplus\tilde{P}, C\oplus P\oplus\tilde{P}) \simeq Hom_\Lambda(P\oplus\tilde{P}, C).$$

Applying $Hom_\Lambda(P\oplus\tilde{P}, -)$ to the short exact sequence

$$0\longrightarrow C\longrightarrow P\longrightarrow \Omega L\longrightarrow 0$$

gives us that

$$0\longrightarrow Hom_\Lambda(P\oplus\tilde{P}, C)\longrightarrow Hom_\Lambda(P\oplus\tilde{P}, P)\longrightarrow Hom_\Lambda(P\oplus\tilde{P}, \Omega L)\longrightarrow 0$$

is exact. But,

$$Hom_\Lambda(P\oplus\tilde{P}, \Omega L)\simeq Hom_\Lambda(P\oplus\tilde{P}, Q).$$

We have to compute the endomorphism ring over Γ of $Hom_\Lambda(\Omega L, C \oplus P \oplus \tilde{P})$. But, by Lemma 3, we get $e_{\Omega L} P \simeq \Omega L$. Hence,

$$\begin{array}{rcl} End_\Gamma(Hom_\Lambda(\Omega L, C \oplus P \oplus \tilde{P})) & \simeq & End_\Gamma(Hom_\Lambda(e_{\Omega L} P, C \oplus P \oplus \tilde{P})) \\ & \simeq & End_\Gamma(e_{\Omega L} Hom_\Lambda(P, C \oplus P \oplus \tilde{P})) \\ & \simeq & End_\Lambda(e_{\Omega L} P)^{op} \\ & \simeq & End_\Lambda(\Omega L)^{op} \end{array}$$

applying Lemma 4. Even more,

$$End_\Gamma(End_\Lambda(L)) \simeq End_{End_\Lambda(L)}(End_\Lambda(L)) \simeq End_\Lambda(L)^{op} \ .$$

Now, Γ is Gorenstein and hence the first three parts of Proposition 6 together justify that we may apply Theorem 1. Lemma 5 gives us that those Γ-linear endomorphisms of $Hom_\Lambda(\Omega L, C \oplus P \oplus \tilde{P})$ factoring over a projective module are those factoring over $Hom_\Lambda(P, C \oplus P \oplus \tilde{P})$. These correspond then to those endomorphisms of ΩL factoring over a projective module and hence, the endomorphism ring of

$$T_r : \ 0 \longrightarrow Hom_\Gamma(P, C \oplus P \oplus \tilde{P}) \longrightarrow Q^* \longrightarrow 0.$$

is isomorphic to the opposite of the endomorphism ring of Q.

Hence,

$$End_{D^b(\Gamma)}(T^*) \simeq \left(\begin{array}{cc} End_\Lambda(Q)^{op} & Hom_\Lambda(P \oplus \tilde{P}, Q) \\ Hom_\Lambda(Q, P \oplus \tilde{P}) & End_\Lambda(P \oplus \tilde{P})^{op} \end{array} \right) \simeq \Lambda^{op}$$

■

We recall a lemma of Rickard from [8, 18].

Lemma 7 (J. Rickard) *Let R be a complete discrete valuation ring and let Λ and Γ be R-orders. Assume that we have a complex X in $D^b(\Lambda \otimes_R \Gamma^{op})$ such that $X \simeq T$ in $D^b(\Lambda)$ and $X \simeq T^* \in D^b(\Gamma^{op})$ where T is a tilting complex with endomorphism ring isomorphic to Γ and T^* is a tilting complex with endomorphism ring isomorphic to Λ^{op}. Then, X is a two-sided tilting complex.*

By Lemma 7 we are done if we have constructed a complex X in $D^b(\Lambda \otimes_R \Gamma^{op})$ which is isomorphic to T in $D^b(\Lambda)$ and to T^* in $D^b(\Gamma^{op})$.

4 How to construct the two-sided tilting complex

We shall construct a complex X in $D^b(\Lambda \otimes_R \Gamma^{op})$ such that $X \simeq T$ in $D^b(\Lambda)$ and $X \simeq T^*$ in $D^b(\Gamma^{op})$ for our complexes T and T^* from section 3. Lemma 7 then tells us that X is indeed a two-sided tilting complex.

The complex will be constructed by the following commutative diagram.

$$\begin{array}{ccccccccc} & & & & \widehat{\Omega L} & \hookrightarrow & X_0 & \twoheadrightarrow & End_\Lambda(L) \\ & & & & \| & & \downarrow & P.B. & \downarrow \\ H_1(X) & \hookrightarrow & X_1 & \twoheadrightarrow & \widehat{\Omega L} & \hookrightarrow & \overline{X_0} & \twoheadrightarrow & \overline{End_\Lambda}(L) \\ \| & & \downarrow & P.B. & \downarrow \alpha & & & & \\ H_1(X) & \hookrightarrow & \overline{X_1} & \twoheadrightarrow & \overline{\Omega} & & & & \end{array}$$

Here all sequences $\bullet \hookrightarrow \bullet \twoheadrightarrow \bullet$ are assumed to be exact. $\widehat{\Omega L}$ denotes the kernel of this map $X_0 \longrightarrow L$ as $\Lambda \otimes_R \Gamma^{op}$-module.

Remark 1 *In general X_0 is not projective neither as Λ nor as Γ^{op}-module.*

What we have to do, is to define α, the $\Lambda \otimes_R \Gamma^{op}$-module $\overline{\Omega}$, the module $\overline{X_1}$ and its projection onto $\overline{\Omega}$.

Our complex X will then be

$$\ldots \longrightarrow 0 \longrightarrow X_1 \longrightarrow X_0 \longrightarrow 0 \longrightarrow \ldots$$

where the non zero differential is induced by the mapping

$$\begin{array}{ccc} & \widehat{\Omega L} & \hookrightarrow & X_0 \\ & \| & & \\ X_1 & \longrightarrow\!\!\!\!\rightarrow & \widehat{\Omega L} & \end{array}$$

We begin with the parts which are easy to describe.

$$\overline{X_0} = \begin{pmatrix} End_\Lambda(Q) \otimes_R End_\Lambda(C) & End_\Lambda(Q) \otimes_R Hom_\Lambda(C,P) & End_\Lambda(Q) \otimes_R Hom_\Lambda(C,\tilde{P}) \\ Hom_\Lambda(P,Q) \otimes_R End_\Lambda(C) & Hom_\Lambda(P,Q) \otimes_R Hom_\Lambda(C,P) & Hom_\Lambda(P,Q) \otimes_R Hom_\Lambda(C,\tilde{P}) \\ Hom_\Lambda(\tilde{P},Q) \otimes_R End_\Lambda(C) & Hom_\Lambda(\tilde{P},Q) \otimes_R Hom_\Lambda(C,P) & Hom_\Lambda(\tilde{P},Q) \otimes_R Hom_\Lambda(C,\tilde{P}) \end{pmatrix}$$

The mapping to $\overline{End}_\Lambda(L)$ is defined as follows. Since there is a central idempotent e in $K \otimes_R \Lambda$ with $e \cdot Q = L$, each endomorphism ϕ of Q induces an endomorphism $e\phi$ of L and hence the mapping $\overline{X_0} \longrightarrow \overline{End}_\Lambda(L)$ is defined as

$$\begin{pmatrix} \phi \otimes \psi & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{pmatrix} \longrightarrow \overline{\phi \cdot \psi}$$

denoting by $\overline{}$ the residue class in $\overline{End}_\Lambda(L)$.

We see immediately that $\widehat{\Omega L}$ is

$$\widehat{\Omega L} = \begin{pmatrix} \tilde{Y} & End_\Lambda(Q) \otimes_R Hom_\Lambda(C,P) & End_\Lambda(Q) \otimes_R Hom_\Lambda(C,\tilde{P}) \\ Hom_\Lambda(P,Q) \otimes_R End_\Lambda(C) & Hom_\Lambda(P,Q) \otimes_R Hom_\Lambda(C,P) & Hom_\Lambda(P,Q) \otimes_R Hom_\Lambda(C,\tilde{P}) \\ Hom_\Lambda(\tilde{P},Q) \otimes_R End_\Lambda(C) & Hom_\Lambda(\tilde{P},Q) \otimes_R Hom_\Lambda(C,P) & Hom_\Lambda(\tilde{P},Q) \otimes_R Hom_\Lambda(C,\tilde{P}) \end{pmatrix}$$

for $\tilde{Y}$ is defined via the exact sequence

$$0 \longrightarrow \tilde{Y} \longrightarrow End_\Lambda(Q) \otimes_R End_\Lambda(C) \longrightarrow \overline{End}_\Lambda(L) \longrightarrow 0$$

We define

$$H_1(X) := \begin{pmatrix} Hom_\Lambda(\Omega L, C) & Hom_\Lambda(\Omega L, P) & Hom_\Lambda(\Omega L, \tilde{P}) \\ Hom_\Lambda(P,C) & End_\Lambda(P) & Hom_\Lambda(P,\tilde{P}) \\ Hom_\Lambda(\tilde{P},C) & Hom_\Lambda(\tilde{P},P) & End_\Lambda(\tilde{P}) \end{pmatrix}$$

Lemma 8 *$H_1(X) \simeq H_1(T)$ in $\Lambda - mod$ and $H_1(X) \simeq H_1(T^*)$ in $mod - \Gamma$*

Proof. Using that $Hom_\Lambda(\Omega L, C \oplus P \oplus \tilde{P}) \simeq Hom_\Lambda(Q, C \oplus P \oplus \tilde{P})$ by Lemma 1, we recognize readily the direct summands. ■

5 Defining the remaining components

We shall assume from now on throughout that $Hom_\Lambda(C, \Omega L) = 0$

Let $\tilde{P}(c) := e_C \tilde{P} \cap \tilde{P}$.

5.1 Defining $\overline{X}_1$ and $\overline{\Omega}$ as R-modules

We define

$$\overline{X_1} := \overline{X_1}^{\Omega L} \oplus \overline{X_1}^C$$

with

$$\overline{X_1}^{\Omega L} := \begin{pmatrix} 0 & Hom_\Lambda(P, \Omega L) & Hom_\Lambda(P, e_{\Omega L}\tilde{P}) \\ 0 & Hom_\Lambda(P, \Omega L) & Hom_\Lambda(P, e_{\Omega L}\tilde{P}) \\ 0 & Hom_\Lambda(\tilde{P}, \Omega L) & Hom_\Lambda((\tilde{P}, e_{\Omega L}\tilde{P}) \end{pmatrix}$$

and

$$\overline{X_1}^C := \begin{pmatrix} 0 & 0 & 0 \\ Hom_\Lambda(C,P) & Hom_\Lambda(C,P) & Hom_\Lambda(C,\tilde{P}) \\ Hom_\Lambda(\tilde{P}(c),P) & Hom_\Lambda(\tilde{P}(c),P) & Hom_\Lambda(\tilde{P}(c),\tilde{P}) \end{pmatrix}$$

We shall discover a $\Lambda \otimes_R \Gamma^{op}$-module structure on $\overline{X_1}$ in the sequel. However, want first define $\overline{\Omega}$.

To be able to define $\overline{\Omega}$ we discuss various pushout diagrams.

First, we state a surely well known lemma.

Lemma 9 *Let $X \xrightarrow{\chi} Z \xleftarrow{\eta} Y$ be the pushout of $X \xleftarrow{\alpha} W \xrightarrow{\beta} Y$. Assume that $ker\alpha \cap ker\beta = 0$. Then, α induces an isomorphism $ker\chi \simeq ker\beta$.*

Proof. $Z \simeq (X \oplus Y)/\{(\alpha(w), -\beta(w))|w \in W\}$. And

$$ker\chi = \{x \in X|\chi(x) \in \{(\alpha(w), -\beta(w))|w \in W\}\} = \{x \in X|x = \alpha(w) \text{ and } w \in ker\beta\} = \alpha(ker\beta)$$

Hence, α is surjective as mapping $ker\beta \longrightarrow ker\chi$. Since $ker\alpha \cap ker\beta = 0$, we get the statement. ■

We apply $Hom_\Lambda(P, -)$ and $Hom_\Lambda(-, P)$ to

$$0 \longrightarrow C \longrightarrow P \longrightarrow \Omega L \longrightarrow 0.$$

Since $Hom_\Lambda(\Omega L, C) = 0$, the hypothesis of Lemma 9 are fulfilled and we obtain a commutative diagram, which is completed to a pushout diagram in form of Ω, as follows.

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & Hom_\Lambda(P, C) & = & Hom_\Lambda(P, C) & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Hom_\Lambda(\Omega L, P) & \longrightarrow & Hom_\Lambda(P, P) & \longrightarrow & Hom_\Lambda(C, P) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow \sigma & & \\
0 & \longrightarrow & Hom_\Lambda(\Omega L, P) & \longrightarrow & Hom_\Lambda(P, \Omega L) & \xrightarrow{\tau} & \Omega & \longrightarrow 0 & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Observe that by Lemma 9 this diagram is at once a pullback and a pushout diagram!

In the same way we introduce a commutative diagram associated to $Hom_\Lambda(\tilde{P}, \tilde{P})$ with respect to e_C and $e_{\Omega L}$ and complete it to a pushout diagram.

$$\begin{array}{ccccc}
Hom_\Lambda(\tilde{P}, \tilde{P}) & \longrightarrow & Hom_\Lambda(\tilde{P}(c), \tilde{P}) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \\
Hom_\Lambda(\tilde{P}, e_{\Omega L}\tilde{P}) & \longrightarrow & \tilde{\Omega} & \longrightarrow 0 & \\
\downarrow & & \downarrow & & \\
0 & & 0 & &
\end{array}$$

We shall do the same with the commutative diagram associated to $Hom_\Lambda(P, \tilde{P})$ with respect to e_C and $e_{\Omega L}$ as well as with $Hom_\Lambda(\tilde{P}, P)$ with respect to the same central idempotents e_C and $e_{\Omega L}$ of A. Observe that e acts as 0 on each of these homomorphism sets.

$$\begin{array}{ccccc}
Hom_\Lambda(P, \tilde{P}) & \longrightarrow & Hom_\Lambda(C, \tilde{P}) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \\
Hom_\Lambda(P, e_{\Omega L}\tilde{P}) & \longrightarrow & \tilde{\Omega}_P & \longrightarrow 0 & \\
\downarrow & & \downarrow & & \\
0 & & 0 & &
\end{array}$$

and

$$\begin{array}{ccccc}
Hom_\Lambda(\tilde{P}, P) & \longrightarrow & Hom_\Lambda(\tilde{P}(c), P) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \\
Hom_\Lambda(\tilde{P}, e_{\Omega L}P) & \longrightarrow & \tilde{\Omega}^P & \longrightarrow 0 & \\
\downarrow & & \downarrow & & \\
0 & & 0 & &
\end{array}$$

are the corresponding pullback/pushout diagrams.

The same procedure applied to $Hom_\Lambda(\tilde{P}, C)$ yields the following. The sequence

$$0 \longrightarrow \tilde{P}(c) \longrightarrow \tilde{P} \longrightarrow e_{\Omega L}\tilde{P} \longrightarrow 0$$

is exact. We apply $Hom_\Lambda(-, C)$ to it and, using that $Hom_\Lambda(e_{\Omega L}\tilde{P}, C) = 0$, we obtain

$$0 \longrightarrow Hom_\Lambda(\tilde{P}, C) \longrightarrow Hom_\Lambda(\tilde{P}(c), C) \longrightarrow Ext^1_\Lambda(e_{\Omega L}\tilde{P}, C) \longrightarrow 0$$

is exact. Since $Hom_\Lambda(\tilde{P}(c), \Omega L) = 0$, we have

$$Hom_\Lambda(\tilde{P}(c), C) \simeq Hom_\Lambda(\tilde{P}(c), P)\ .$$

Lemma 10 *Let A be a ring and let*

$$\mathbb{M} \ : \ 0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$$

as well as

$$\mathbb{N} \ : \ 0 \longrightarrow N_1 \longrightarrow N_2 \longrightarrow N_3 \longrightarrow 0$$

be short exact sequences of A-modules. Assume that

$$Ext^1_A(N_2, M_1) = Ext^1_A(N_3, M_2) = 0 \text{ and that } Hom_A(N_3, M_1) = Hom_A(N_1, M_3) = 0.$$

Then, there is a short exact sequence

$$0 \longrightarrow Hom_A(N_2, M_2) \longrightarrow Hom_A(N_1, M_2) \oplus Hom_A(N_2, M_3) \longrightarrow Ext^1_A(N_3, M_1) \longrightarrow 0$$

induced by the natural maps.

Proof. We apply $Hom_A(-, M_1)$ to $\mathbb{N}$ and obtain, using that $Hom_A(N_3, M_1) = 0$ and that $Ext^1(N_2, M_1) = 0$, a short exact sequence

$$0 \longrightarrow Hom_A(N_2, M_1) \longrightarrow Hom_A(N_1, M_1) \longrightarrow Ext^1_A(N_3, M_1) \longrightarrow 0$$

whence applying $Hom(N_2, -)$ to $\mathbb{M}$ we get, using that $Ext^1_A(N_2, M_1) = 0$ a short exact sequence

$$0 \longrightarrow Hom_A(N_2, M_1) \longrightarrow Hom_A(N_2, M_2) \longrightarrow Hom_A(N_2, M_3) \longrightarrow 0$$

Since $Hom_A(N_1, M_3) = 0$, we get $Hom_A(N_1, M_1) \simeq Hom_A(N_1, M_2)$. We hence get a commutative diagrams with exact rows and columns.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & Hom_A(N_2, M_1) & \longrightarrow & Hom_A(N_1, M_1) & \longrightarrow & Ext^1_A(N_3, M_1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & Hom_A(N_2, M_2) & \longrightarrow & Hom_A(N_1, M_2) & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & Hom_A(N_2, M_3) & & 0 & & & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}$$

The sequence

$$0 \longrightarrow Hom_A(N_3, M_2) \cap Hom_A(N_2, M_1) \longrightarrow Hom_A(N_2, M_2) \longrightarrow Hom_A(N_1, M_2) \oplus Hom_A(N_2, M_3)$$

is exact; where the left term is identified with its image in $Hom_A(N_2, M_2)$. An element in $\chi \in Hom_A(N_3, M_2) \cap Hom_A(N_2, M_1)$ is a mapping $N_2 \longrightarrow M_2$ which factorizes via M_1 and via N_3. The factorizing property via M_1 implies that $im\ \chi \subseteq M_1$. Since by the surjectivity of $N_2 \longrightarrow N_3$ the mapping $\chi : N_2 \longrightarrow N_3 \longrightarrow M_2$ hence is in fact a mapping $N_2 \longrightarrow N_3 \longrightarrow M_1$. The last part is zero since $Hom_A(N_3, M_1) = 0$. We get that $Hom_A(N_2, M_2) \longrightarrow Hom_A(N_1, M_2) \oplus Hom_A(N_2, M_3)$ is a monomorphism.

This induces a commutative diagram

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & 0 & & 0 & \longrightarrow & ker & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Hom_A(N_2, M_1) & \longrightarrow & Hom_A(N_1, M_1) & \longrightarrow & Ext^1_A(N_3, M_1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Hom_A(N_2, M_2) & \longrightarrow & Hom_A(N_1, M_2) \oplus Hom_A(N_2, M_3) & \longrightarrow & coker & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & Hom_A(N_2, M_3) & \stackrel{\varphi}{\longrightarrow} & Hom_A(N_2, M_3) & \longrightarrow & cocoker & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

with exact rows and columns. Since $\varphi = id$ makes the diagram commutative, the snake lemma implies that $cocoker = ker = 0$ and that $coker \simeq Ext^1_A(N_3, M_1)$. This proves the lemma. ■

We apply lemma 10 to determine $\widetilde{\Omega}_P, \widetilde{\Omega}^P$ and $\widetilde{\Omega}$.

We shall use the sequences

$$\tilde{\mathbb{P}} \ : \ 0 \longrightarrow \tilde{P}(c) \longrightarrow \tilde{P} \longrightarrow e_{\Omega L}\tilde{P} \longrightarrow 0$$

and

$$\mathbb{P} \ : \ 0 \longrightarrow C \longrightarrow P \longrightarrow \Omega L \longrightarrow 0$$

The following identifications give the results below.

$\mathbb{M} = \mathbb{P}$	$\mathbb{N} = \widetilde{\mathbb{P}}$	$\tilde{\Omega}_P = Ext^1_\Lambda(e_{\Omega_L}\tilde{P}, C)$
$\mathbb{M} = \widetilde{\mathbb{P}}$	$\mathbb{N} = \mathbb{P}$	$\tilde{\Omega}^P = Ext^1_\Lambda(\Omega_L, \tilde{P}(c))$
$\mathbb{M} = \widetilde{\mathbb{P}}$	$\mathbb{N} = \widetilde{\mathbb{P}}$	$\tilde{\Omega} = Ext^1_\Lambda(e_{\Omega_L}\tilde{P}, \tilde{P}(c))$
$\mathbb{M} = \mathbb{P}$	$\mathbb{N} = \mathbb{P}$	$\Omega = Ext^1_\Lambda(\Omega_L, C)$

Summarizing, the sequences

$$0 \longrightarrow Hom_\Lambda(\tilde{P}, \tilde{P}) \longrightarrow Hom_\Lambda(\tilde{P}(c), \tilde{P}) \oplus Hom_\Lambda(\tilde{P}, e_{\Omega L}\tilde{P}) \longrightarrow Ext^1_\Lambda(e_{\Omega L}\tilde{P}, \tilde{P}(c)) \longrightarrow 0$$

$$0 \longrightarrow Hom_\Lambda(\tilde{P}, P) \longrightarrow Hom_\Lambda(\tilde{P}(c), P) \oplus Hom_\Lambda(\tilde{P}, \Omega L) \longrightarrow Ext^1_\Lambda(e_{\Omega L}\tilde{P}, C) \longrightarrow 0$$

$$0 \longrightarrow Hom_\Lambda(P, \tilde{P}) \longrightarrow Hom_\Lambda(C, \tilde{P}) \oplus Hom_\Lambda(P, e_{\Omega L}\tilde{P}) \longrightarrow Ext^1_\Lambda(\Omega L, \tilde{P}(c)) \longrightarrow 0$$

$$0 \longrightarrow Hom_\Lambda(P, P) \longrightarrow Hom_\Lambda(C, P) \oplus Hom_\Lambda(P, \Omega L) \longrightarrow Ext^1_\Lambda(\Omega L, C) \longrightarrow 0$$

are exact. These give the entries $(3,3)$, $(2,3)$, $(3,2)$ and $(2,2)$ of $\overline{\Omega}$. Moreover, by the pullback property of the pushout diagrams we get that

$$0 \longrightarrow Hom_\Lambda(\tilde{P}, C) \longrightarrow Hom_\Lambda(\tilde{P}(c), P) \longrightarrow Ext^1_\Lambda(e_{\Omega L}\tilde{P}, C) \longrightarrow 0$$

$$0 \longrightarrow Hom_\Lambda(\Omega L, \tilde{P}) \longrightarrow Hom_\Lambda(P, e_{\Omega L}\tilde{P}) \longrightarrow Ext^1(\Omega L, \tilde{P}(c)) \longrightarrow 0$$

$$0 \longrightarrow Hom_\Lambda(\Omega L, P) \longrightarrow Hom_\Lambda(P, \Omega L) \longrightarrow Ext^1_\Lambda(\Omega L, C) \longrightarrow 0$$

$$0 \longrightarrow Hom_\Lambda(P, C) \longrightarrow Hom_\Lambda(C, P) \longrightarrow Ext^1_\Lambda(\Omega L, C) \longrightarrow 0$$

are exact. These sequences give the $(1,3)$, $(3,1)$, $(2,1)$ and $(1,2)$ entry of $\overline{\Omega}$.

We see that

$$\overline{\Omega} = \begin{pmatrix} 0 & Ext^1_\Lambda(\Omega L, C) & Ext^1_\Lambda(\Omega L, \tilde{P}(c)) \\ Ext^1_\Lambda(\Omega L, C) & Ext^1_\Lambda(\Omega L, C) & Ext^1_\Lambda(\Omega L, \tilde{P}(c)) \\ Ext^1_\Lambda(e_{\Omega L}\tilde{P}, C) & Ext^1_\Lambda(e_{\Omega L}\tilde{P}, C) & Ext^1_\Lambda(e_{\Omega L}\tilde{P}, \tilde{P}(c)) \end{pmatrix}$$

What we have to define is the module structure of $\overline{X_1}$ and to verify that the mapping $H_1(X) \longrightarrow \overline{X_1}$ and the mapping $\widehat{\Omega L} \longrightarrow \overline{\Omega}$ are $\Lambda \otimes_R \Gamma^{op}$-module homomorphisms.

5.2 Defining the $\Lambda \otimes_R \Gamma^{op}$-module structure on $\overline{X}_1$ and $\overline{\Omega}$

For defining a left Λ-module structure on $\overline{X_1}$, we define the structure on $\overline{X_1}^{\Omega L}$ and on $\overline{X_1}^{C}$ separately.

Observe that

$$\Lambda = End_\Lambda(\Lambda) = \begin{pmatrix} Hom_\Lambda(Q, Q) & Hom_\Lambda(Q, P) & Hom_\Lambda(Q, \tilde{P}) \\ Hom_\Lambda(P, Q) & Hom_\Lambda(P, P) & Hom_\Lambda(P, \tilde{P}) \\ Hom_\Lambda(\tilde{P}, Q) & Hom_\Lambda(\tilde{P}, P) & Hom_\Lambda(\tilde{P}, \tilde{P}) \end{pmatrix}$$

We begin with $\overline{X_1}^{\Omega L}$.

Since

$$Hom_\Lambda(C, \Omega L) = Hom_\Lambda(\Omega L, L) = Hom_\Lambda(C, L) = 0,$$

any endomorphism of Q induces an endomorphism of ΩL which in turn induces an endomorphism of P. Two endomorphisms of P induced this way differ by an element in $\iota \cdot Hom_\Lambda(P, C)$.

Therefore, the second column of $\overline{X_1}^{\Omega L}$ is isomorphic to

$$\begin{pmatrix} End_\Lambda(\Omega L) \\ Hom_\Lambda(P, \Omega L) \\ Hom_\Lambda(\tilde{P}, \Omega L) \end{pmatrix}$$

which is certainly a Λ-module. It is an extension of ΩL and $Ext^1(\Omega L \oplus P \oplus \tilde{P}, \Omega L)$.

The third column is as Λ left-module isomorphic to

$$\begin{pmatrix} Hom_\Lambda(Q, e_{\Omega L}\tilde{P}) \\ Hom_\Lambda(P, e_{\Omega L}\tilde{P}) \\ Hom_\Lambda(\tilde{P}, e_{\Omega L}\tilde{P}) \end{pmatrix} \simeq e_{\Omega L}\tilde{P}$$

which as well is a Λ-left module.

The right module structure is seen as follows. We use again Lemma 3. Then,

$$\begin{pmatrix} 0 & Hom_\Lambda(P, \Omega L) & Hom_\Lambda(P, e_{\Omega L}\tilde{P}) \end{pmatrix} \simeq Hom_\Lambda(P, C \oplus P \oplus \tilde{P}) \cdot e_{\Omega L}$$

which is clearly a Γ-right module. Moreover,

$$\begin{pmatrix} 0 & Hom_\Lambda(\tilde{P}, e_{\Omega L}P) & Hom_\Lambda(\tilde{P}, e_{\Omega L}\tilde{P}) \end{pmatrix} \simeq Hom_\Lambda(\tilde{P}, C \oplus P \oplus \tilde{P}) \cdot e_{\Omega L}$$

which also is a Γ-right module.

We postpone for the moment the question if these left- and right module structure fit together to a bimodule structure.

We shall define the module structure on $\overline{X_1}^C$. The left Λ-structure on $\overline{X_1}^C$ is defined by the following.

Lemma 3 gives us that $Hom_\Lambda(C, P)$ is an $End_\Lambda(P)$-left module in a natural way. Also,

$$Hom_\Lambda(P, \tilde{P}) \cdot Hom_\Lambda(\tilde{P}(c), P) \subseteq Hom_\Lambda(C, P).$$

Moreover, since $C = e_C P \cap P$ and $\tilde{P}(c) = e_C \tilde{P} \cap \tilde{P}$,

$$Hom_\Lambda(\tilde{P}, P) Hom_\Lambda(C, P) + End_\Lambda(\tilde{P}) Hom_\Lambda(\tilde{P}(c), P) \subseteq Hom_\Lambda(\tilde{P}(c), P)$$

This makes the matrix multiplication of

$$\Lambda = End_\Lambda(\Lambda) = \begin{pmatrix} Hom_\Lambda(Q, Q) & Hom_\Lambda(Q, P) & Hom_\Lambda(Q, \tilde{P}) \\ Hom_\Lambda(P, Q) & Hom_\Lambda(P, P) & Hom_\Lambda(P, \tilde{P}) \\ Hom_\Lambda(\tilde{P}, Q) & Hom_\Lambda(\tilde{P}, P) & Hom_\Lambda(\tilde{P}, \tilde{P}) \end{pmatrix}$$

from the left on $\overline{X_1}^C$ well defined.

The Γ right module structure on $\overline{X_1}^C$ is defined as follows. By Lemma 3 one has $Hom_\Lambda(C, P) \simeq Hom_\Lambda(C, C)$. Hence, for the second line in $\overline{X_1}^C$ we get therefore

$$\begin{pmatrix} Hom_\Lambda(C, P) & Hom_\Lambda(C, P) & Hom_\Lambda(C, \tilde{P}) \end{pmatrix} \simeq Hom_\Lambda(C, C \oplus P \oplus \tilde{P}).$$

The third line in the matrix representation of $\overline{X_1}^C$ is a Γ-right module in the following way. A Λ-linear mapping from $\tilde{P}(c) \longrightarrow P$ has image in P and in $e_C P$ as well. Therefore, it has image in C. this proves that

$$Hom_\Lambda(\tilde{P}(c), P) = Hom_\Lambda(\tilde{P}(c), C).$$

The third line of $\overline{X_1}^C$ is therefore isomorphic to

$$Hom_\Lambda(\tilde{P}(c), C \oplus P \oplus \tilde{P}).$$

We shall now discuss why these left and right action are compatible. For proving this we form the module $K\otimes_R\overline{X_1}$ and prove that this is now by extension of scalars a $K\otimes_R\Lambda\otimes\Gamma^{op}$ module. When we have proved this, again restriction to $\Lambda\otimes\Gamma^{op}$ of $K\otimes_R\overline{X_1}$ gives us that the submodule $\overline{X_1}$ is already a $\Lambda\otimes\Gamma^{op}$ module.

But,

$$K\otimes_R\overline{X_1}^{\Omega L}\simeq K\otimes_R Hom_\Lambda(e_{\Omega L}\Lambda, e_{\Omega L}C\oplus P\oplus\tilde{P})$$

and the operation of $\Lambda\otimes_R\Gamma^{op}$ is the regular one. Analogously,

$$K\otimes_R\overline{X_1}^{C}\simeq K\otimes_R Hom_\Lambda(e_C\Lambda, e_C C\oplus P\oplus\tilde{P})$$

with regular operation of $\Lambda\otimes_R\Gamma^{op}$.

This observation proves at the same time that the embedding

$$H_1(X)\hookrightarrow\overline{X_1}$$

is a $\Lambda\otimes_R\Gamma^{op}$-module homomorphism.

The $\Lambda\otimes_R\Gamma^{op}$-module structure is defined via the mapping $\overline{X}_1\longrightarrow\!\!\!\!\!\rightarrow\overline{\Omega}_1$.

5.3 Defining the mapping $\widehat{\Omega L}\longrightarrow\overline{\Omega}$

We shall have to define a homomorphism $\widehat{\Omega L}\longrightarrow\overline{\Omega}$.

Lemma 3 gives canonical isomorphisms

$$\begin{array}{rcl} End_\Lambda(C) & \simeq & Hom_\Lambda(C,P)\\ Hom_\Lambda(P,Q) & \simeq & End_\Lambda(\Omega L)\\ Hom_\Lambda(\tilde{P},Q) & \simeq & Hom_\Lambda(e_{\Omega L}\tilde{P},\Omega L)\\ Hom_\Lambda(C,\tilde{P}(c)) & \simeq & Hom_\Lambda(C,\tilde{P}) \end{array}$$

Moreover, any endomorphism of Q induces a unique endomorphism of ΩL.

The following is the well known Baer construction.

Lemma 11 *Let G and D be two Λ-modules. Then, there is a mapping*

$$Hom_\Lambda(D,\Omega L)\otimes_R Ext^1_\Lambda(\Omega L,C)\otimes_R Hom_\Lambda(C,G)\longrightarrow Ext^1_\Lambda(D,G)$$

Proof. Let $0\longrightarrow C\longrightarrow X\longrightarrow\Omega L\longrightarrow 0$ be an element in $Ext^1_\Lambda(\Omega L,C)$. Then, forming a pushout via an element $\gamma\in Hom_\Lambda(C,G)$ and a pullback via $\delta\in Hom_\Lambda(D,\Omega L)$ we get a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C & \longrightarrow & X & \longrightarrow & \Omega L & \longrightarrow & 0\\ & & \downarrow\gamma & & \downarrow & & \| & & \\ 0 & \longrightarrow & G & \longrightarrow & X^\gamma & \longrightarrow & \Omega L & \longrightarrow & 0\\ & & \| & & \uparrow & & \uparrow\delta & & \\ 0 & \longrightarrow & G & \longrightarrow & X^\gamma_\delta & \longrightarrow & D & \longrightarrow & 0 \end{array}$$

The lower sequence is the image in $Ext^1_\Lambda(D,G)$. The diagram defines the mapping as stated and has the appropriate properties. ∎

To define the mapping $\widehat{\Omega L}\longrightarrow\overline{\Omega}$ we just apply lemma 11 and lemma 3. The image of the short exact sequence

$$0\longrightarrow C\longrightarrow P\longrightarrow\Omega L\longrightarrow 0$$

under the homomorphism set actions gives the desired mapping. The fact that the so defined mapping is $\Lambda\otimes_R\Gamma^{op}$-linear is then clear.

6 Looking at X one-sided

Recall the construction of X. The complex X is defined to be the composite of the mapping

$$X_1 \longrightarrow\!\!\!\!\rightarrow \widehat{\Omega L} \hookrightarrow X_0$$

regarded as complex. Moreover, the second and the third line in the matrix

$$\widehat{\Omega L} = \begin{pmatrix} \tilde{Y} & End_\Lambda(Q) \otimes_R Hom_\Lambda(C,P) & End_\Lambda(Q) \otimes_R Hom_\Lambda(C,\tilde{P}) \\ Hom_\Lambda(P,Q) \otimes_R End_\Lambda(C) & Hom_\Lambda(P,Q) \otimes_R Hom_\Lambda(C,P) & Hom_\Lambda(P,Q) \otimes_R Hom_\Lambda(C,\tilde{P}) \\ Hom_\Lambda(\tilde{P},Q) \otimes_R End_\Lambda(C) & Hom_\Lambda(\tilde{P},Q) \otimes_R Hom_\Lambda(C,P) & Hom_\Lambda(\tilde{P},Q) \otimes_R Hom_\Lambda(C,\tilde{P}) \end{pmatrix}$$

are mapped isomorphically to X_0. In fact, this was our observation that the pullback construction yielding $\widehat{\Omega L}$ as kernel, does only affect the position $(1,1)$ in the corresponding matrix rings. Since X_1 maps surjectively to $\widehat{\Omega L}$, the differential restricted to the second and third line of the matrix X_1 are mapped surjectively to the lower lines of X_0, the two lower matrix lines of which are projective right Γ-modules. Hence, regarded as complex with two non zero entries, the complex restricted to the two lower lines is quasi-isomorphic to its homology. This is isomorphic to

$$Hom_\Lambda(P \oplus \tilde{P}, C \oplus P \oplus \tilde{P}).$$

Completely analogously to the above, X restricted to the right has as direct summands $P \oplus \tilde{P}$ as left Λ-modules.

Lemma 12 • *$Hom_\Lambda(P \oplus \tilde{P}, C \oplus P \oplus \tilde{P})[1]$ is a direct summand of X regarded as complex of right Γ-modules.*

- *$(P \oplus \tilde{P})[1]$ is a direct summand of X regarded as complex of left Λ-modules.* ∎

We shall prove the following lemma.

Lemma 13 • *$X/(P \oplus \tilde{P})[1]$ is isomorphic to $0 \longrightarrow P \longrightarrow Q \longrightarrow 0$ as complex of left Λ-modules.*

- *$X/Hom_\Lambda(P \oplus \tilde{P}, C \oplus P \oplus \tilde{P})[1]$ is isomorphic to $T_r :\ 0 \longrightarrow Hom_\Lambda(P, C \oplus P \oplus \tilde{P}) \longrightarrow Q^* \longrightarrow 0$ as complex of right Γ-modules.*

Proof. We shall give a morphism of complexes of Γ-right modules

$$T_r \longrightarrow X/Hom_\Lambda(P \oplus \tilde{P}, C \oplus P \oplus \tilde{P})[1]$$

which we shall prove to be a quasi-isomorphism. Set

$$X_r := X/Hom_\Lambda(P \oplus \tilde{P}, C \oplus P \oplus \tilde{P})[1].$$

Set

$$T_l := T/(P \oplus \tilde{P})[1]$$

and

$$X_l := X/(P \oplus \tilde{P})[1]$$

as complexes of left Λ-modules. We will, analogously to the situation for Γ-right modules, give a quasi-isomorphism of complexes of left Λ-modules

$$T_l \longrightarrow X_l$$

Let us prove first the statement for the left Λ-structure.

$$P = \begin{pmatrix} Hom_\Lambda(Q,P) \\ Hom_\Lambda(P,P) \\ Hom_\Lambda(\tilde{P},P) \end{pmatrix} \longrightarrow \begin{pmatrix} 0 \\ Hom_\Lambda(C,P) \\ Hom_\Lambda(\tilde{P}(c),P) \end{pmatrix}$$

in the obvious way. Moreover,

$$\begin{pmatrix} Hom_\Lambda(Q,\varphi) \\ Hom_\Lambda(P,\varphi) \\ Hom_\Lambda(\tilde{P},\varphi) \end{pmatrix} \otimes_R id_C : \ P = \begin{pmatrix} Hom_\Lambda(Q,P) \\ Hom_\Lambda(P,P) \\ Hom_\Lambda(\tilde{P},P) \end{pmatrix} \longrightarrow \begin{pmatrix} Hom_\Lambda(Q,Q) \otimes_R Hom_\Lambda(C,C) \\ Hom_\Lambda(P,Q) \otimes_R Hom_\Lambda(C,C) \\ Hom_\Lambda(\tilde{P},Q) \otimes_R Hom_\Lambda(C,C) \end{pmatrix}$$

is Λ-linear from the left. Since

$$im(\begin{pmatrix} Hom_\Lambda(Q,\varphi) \\ Hom_\Lambda(P,\varphi) \\ Hom_\Lambda(\tilde{P},\varphi) \end{pmatrix} \otimes_R id_C \ ) \subseteq ker(X_0 \longrightarrow L),$$

we get a mapping

$$P \longrightarrow \widehat{\Omega L}\ .$$

Moreover, the two mappings $P \longrightarrow \widehat{\Omega L}$. and $P \longrightarrow \overline{X_1}$ coincide in $\overline{\Omega}$, and hence the pullback property ensures a unique morphism $P \longrightarrow X_1$ which makes the corresponding diagrams

$$\begin{array}{ccc} P & \longrightarrow & X_1 \\ \| & & \downarrow \\ P & \longrightarrow & \overline{X_1} \end{array} \quad \text{and} \quad \begin{array}{ccc} P & \longrightarrow & X_1 \\ \| & & \downarrow \\ P & \longrightarrow & \widehat{\Omega L} \end{array}$$

commutative. Define a homomorphism $Q \longrightarrow X_0$ by means of

$$1_Q \otimes_R id_C : \begin{pmatrix} Hom_\Lambda(Q,Q) \\ Hom_\Lambda(P,Q) \\ Hom_\Lambda(\tilde{P},Q) \end{pmatrix} \longrightarrow \begin{pmatrix} Hom_\Lambda(Q,Q) \otimes_R Hom_\Lambda(C,C) \\ Hom_\Lambda(P,Q) \otimes_R Hom_\Lambda(C,C) \\ Hom_\Lambda(\tilde{P},Q) \otimes_R Hom_\Lambda(C,C) \end{pmatrix}$$

$$\phi \longrightarrow \phi \otimes id_C$$

and $Q \longrightarrow L = End_\Lambda(L)$ in the obvious way. Since the two mappings coincide in $\underline{End}_\Lambda(L)$, this defines a mapping $Q \longrightarrow X_0$. The mapping $P \longrightarrow \widehat{\Omega L}$ yields the diagram

$$\begin{array}{ccc} P & \stackrel{\varphi}{\longrightarrow} & Q \\ \downarrow \chi & & \downarrow \\ X_1 & \longrightarrow & X_0 \end{array}$$

commutative. Hence, we get a morphism of complexes of left Λ-modules. By construction this mapping of complexes induces an isomorphism on the level of the degree 0 homology, namely the identity on L.

We shall prove that $\chi|_C$ is an isomorphism. This also is almost already done by the construction. In fact, $\chi|_C$ has image in $ker(X_1 \longrightarrow X_0)$ which is $H_1(X)$. In turn, the first column there is C.

$$\chi|_C : \begin{pmatrix} Hom_\Lambda(Q,C) \\ Hom_\Lambda(P,C) \\ Hom_\Lambda(\tilde{P},C) \end{pmatrix} \longrightarrow \begin{pmatrix} 0 \\ Hom_\Lambda(P,C) \\ Hom_\Lambda(\tilde{P},C) \end{pmatrix}$$

which is the identity mapping.

This completes the proof of the first part of the Lemma.

We come to the statement on the right Γ structure. We shall define first a complex morphism

$$T_r \longrightarrow X_r$$

which we then prove to be a quasi-isomorphism.

Multiplying by $e_{\Omega L}$ gives a morphism

$$Hom_\Lambda(P, C \oplus P \oplus \tilde{P}) \longrightarrow (0 \ \ Hom_\Lambda(P,\Omega L) \ \ Hom_\Lambda(P, e_{\Omega L}\tilde{P}) \ \ ) \subseteq \overline{X_1}.$$

Moreover

$$id_Q \otimes Hom_\Lambda(\iota, C \oplus P \oplus \tilde{P}) : \ Hom_\Lambda(P, C \oplus P \oplus \tilde{P}) \longrightarrow End_\Lambda(Q) \otimes_R Hom_\Lambda(C, C \oplus P \oplus \tilde{P})$$

gives a morphism to $\overline{X_0}$. Since

$$Hom_\Lambda(P, C \oplus P \oplus \tilde{P}) \longrightarrow Hom_\Lambda(C, C \oplus P \oplus \tilde{P}) \longrightarrow Ext^1_\Lambda(L, C)$$

is exact,

$$im(\ id_Q \otimes Hom_\Lambda(\iota, C \oplus P \oplus \tilde{P})\ ) \subseteq \widehat{\Omega L}\ .$$

Since the two morphisms coincide in $\overline{\Omega}$, we define this way a morphism

$$Hom_\Lambda(P, C \oplus P \oplus \tilde{P}) \longrightarrow X_1\ .$$

Moreover, $id_Q \otimes Hom_\Lambda(\iota, C \oplus P \oplus \tilde{P})$ factorizes via Q^* in the following way. We define the morphism

$$\begin{array}{ccc} Q^* & \longrightarrow & \begin{pmatrix} Hom_\Lambda(Q,Q) \\ Hom_\Lambda(P,Q) \\ Hom_\Lambda(\tilde{P},Q) \end{pmatrix} \otimes_R Hom_\Lambda(C, C \oplus P \oplus \tilde{P}) \\ q & \longrightarrow & \begin{pmatrix} id_Q \\ 0 \\ 0 \end{pmatrix} \otimes \hat{q} \end{array}$$

where $Q^* \ni q \longrightarrow \hat{q} \in Hom_\Lambda(C, C \oplus P \oplus \tilde{P})$ is the mapping given by the defining property of Q^* as pullback. We have a natural map $Q^* \longrightarrow End_\Lambda(L)$ also by the defining property of Q^* as pullback. Since these two mappings coincide in $\overline{End}_\Lambda(L)$, this defines a mapping $Q^* \longrightarrow X_0$.

which makes the diagram

$$\begin{array}{ccc} Hom_\Lambda(P, C \oplus P \oplus \tilde{P}) & \longrightarrow & Q^* \\ \downarrow \xi & & \downarrow \\ X_1 & \longrightarrow & X_0 \end{array}$$

commutative. By construction this mapping induces an isomorphism on the level of the degree 0 homology. Again, we shall show that $\xi|_{Hom_\Lambda(\Omega L, C \oplus P \oplus \tilde{P})}$ is an isomorphism. The argument there is completely analogous to the one we discussed for the Λ-structure.

This completes the proof of Lemma 13. ■

7 The main theorem

We are now ready to formulate the principal theorem.

Theorem 2 *Under the hypotheses of Theorem 1 suppose in addition that* $Hom_\Lambda(C, \Omega L) = 0$ *and that* Γ *is a Gorenstein order. Then the complex*

$$\ldots \longrightarrow 0 \longrightarrow X_1 \longrightarrow X_0 \longrightarrow 0 \longrightarrow \ldots$$

is a two-sided tilting complex in $D^b(\Lambda \otimes \Gamma^{op})$*. The restrictions of* X *to* Λ *and to* Γ^{op} *are both one sided tilting complexes of the type described in Theorem 1.*

Proof. By Lemma 13 and Lemma 12 the complex X restricts as complex of Λ-modules to T and as complex of Γ-modules to T^*. Lemma 6 assures that T^* is a tilting complex over Γ^{op} and Corollary 1 proves that it has endomorphism ring Λ^{op}. Theorem 1 assures that T is a tilting complex over Λ with endomorphism ring Γ. Lemma 7 proves that X then is a two-sided tilting complex as claimed. ■

As a corollary it is possible to compute the perfect isometry ([1, 8]) induced by the complex X. In fact, the only thing we have to do is to separate $K \otimes_R P \simeq K \otimes_R \Omega L \oplus K \otimes_R C$. To simplify the notation we write $K\Lambda$ instead of $K \otimes_R \Lambda$ etc. Then,

$$\begin{array}{rcl} K \otimes_R \Gamma & \simeq & End_{K\Lambda}(KL \oplus KC \oplus K\Omega L \oplus C \oplus K\tilde{P}) \\ & \simeq & End_{K\Lambda}(KL \oplus (KC)^2 \oplus K\Omega L \oplus K\tilde{P}) \end{array}$$

and

$$K \otimes_R \Lambda \quad \simeq \quad End_{K\Lambda}(KL \oplus KC \oplus (K\Omega L)^2 \oplus K\tilde{P})$$

Both expressions simplify considerably using that

$$Hom_\Lambda(L, \Omega L \oplus C \oplus \tilde{P}) = Hom_\Lambda(C, \Omega L) = 0.$$

Corollary 2 *The perfect isometry induced by the complex X in theorem 2 is up to a global sign change*

$$-[KL] + \left[\left(\begin{array}{ccc} Hom_{K\Lambda}((K\Omega L)^2, K\Omega L) & 0 & Hom_{K\Lambda}((K\Omega L)^2, K\tilde{P}) \\ 0 & Hom_{K\Lambda}(KC, KC^2) & Hom_{K\Lambda}(KC, K\tilde{P}) \\ Hom_{K\Lambda}(K\tilde{P}, K\Omega L) & Hom_{K\Lambda}(K\tilde{P}, KC^2) & End_{K\Lambda}(K\tilde{P}) \end{array}\right)\right]$$

where the rings $K\Lambda$ and $K\Gamma$ act via the above matrices. Moreover, any perfect isometry induced by a derived equivalence whose one-sided tilting complex equals T of theorem 1 differs from the above only by an automorphism of Γ.

In fact, the perfect isometry is just the character induced by $-[H_0(X)] + [H_1(X)]$ in the Grothendieck group of $K \otimes_R \Lambda \otimes_R \Gamma^{op}$. The second part of the corollary follows from joint work with R. Rouquier [14] (see also [17, Theorem 4])

Remark The perfect isometry changes the characters coming from the Λ-module ΩL to those coming from the Γ-module $Hom_\Lambda(C \oplus P \oplus \tilde{P}, \Omega L)$, the characters coming from the Λ-module C to the Γ-module $Hom_\Lambda(C \oplus P \oplus \tilde{P}, C)$, associates to the character coming from L to the virtual character associated to the Γ-module $-(End_\Lambda(L))$ and leaves the rest of the characters unchanged.

8 Examples

8.1 Blocs with cyclic defect groups:

A block B of a group ring RG for a finite group G over a complete discrete valuation ring R with cyclic defect group is a Green order. A construction which is very similar to the above was carried out by the author in this special case in [18] (see also [8, section 6.4]). The complex carried out there is isomorphic to the one constructed above if one specializes to this special case.

8.2 Graph orders:

M. Kauer informed the author that a suitable generalization of the concept of Green orders, so called 'Graph orders' [13], are discussed in his doctoral dissertation [5] and the classification of the derived equivalence classes of these is done by tilting complexes satisfying our hypotheses.

8.3 Algebras of semidihedral type:

We shall illustrate now, how one can apply theorem 2 even in case of finite dimensional algebras. Let R be a complete discrete valuation ring with field of fractions K of characteristic 0 and with algebraically closed residue field k of characteristic 2.

Type $SD(3\mathcal{H})^s$ is equivalent to $SD(3\mathcal{C}_{2,I})^s$: Th. Holm gave a tilting complex T over a finite dimensional k-algebra, named $A := SD(3\mathcal{H})^s$ in [4] with endomorphism ring being a k-algebra $B := SD(3\mathcal{C}_{2,I})^s$. The algebra A has three projective indecomposable modules P_0, P_1 and P_2 and K. Erdmann shows [2, 3] that a block with semidihedral defect group has a restricted structure as algebra and two of the possible algebras are A and B for certain parameters s.

Suppose that A is the image of a symmetric Gorenstein order, e.g. a block of a group ring kG.

Then, in this case, there is a symmetric R-order Λ, e.g. the block of RG, with $R/rad\ R = k$, such that $\Lambda \otimes_R k \simeq A$. In [2] it is proved that then, the decomposition matrix of A is of the form

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}_{(*)}$$

where $(*)$ means that the second lowest line of the matrix has to be taken $2^{n-2}-1$ times; where the parameter s equals 2^{n-2} which is one fourth the order of the defect group. Let K be the field of fractions of R. Following [4] we have

$$T: \quad \ldots \longrightarrow P_0 \oplus P_1 \oplus P_1 \stackrel{(0,0,\delta)}{\longrightarrow} P_2 \longrightarrow 0 \longrightarrow \ldots$$

for a certain mapping δ. This is a complex as discussed in [7]. Then, by the composition series of the module P_2 as described in [3] or directly by the quiver in [2], one gets $coker\ \delta \simeq S_2$ is a simple module.

Let $\hat{T}$ be the unique lifting of T to Λ (the existence and unicity is proved in [12]); i.e. the unique tilting complex $\hat{T}$ in $D^b(\Lambda)$ with $\hat{T} \otimes_R k \simeq T$. Set $L := H_0(\hat{T})$. Since $K \otimes_R T$ is again a tilting complex over $K \otimes_R \Lambda$, expressing the images of the projective indecomposable Λ-modules in $K \otimes_R \Lambda$ by means of their columns in the decomposition matrix, one gets that the complex is spliced together from the two exact sequences of $K \otimes_R \Lambda$-modules

$$0 \longrightarrow \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}_{(*)} \longrightarrow \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}_{(*)} \longrightarrow \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}_{(*)} \longrightarrow 0$$

and

$$0 \longrightarrow \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}_{(*)} \longrightarrow \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}_{(*)} \longrightarrow \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}_{(*)} \longrightarrow 0 \quad .$$

One sees that

$$K \otimes_R C \simeq \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}_{(*)} \quad \text{and } K \otimes_R \Omega L \simeq \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}_{(*)}$$

which obviously do not have any common simple direct summand. Hence,

$$Hom_\Lambda(C, \Omega L) = 0.$$

We have to prove that $H_0(T)$ is torsion free. We have seen that $coker\ \ \delta$ is simple. Let L_t be the torsion submodule of L. Then, by vanishing of $Tor_1^R(k, R^{dim_K(K \otimes_R L)})$, we get that $k \otimes_R L_t$ is a non zero submodule of $k \otimes_R L$. Now, $k \otimes_R H_0(T) \simeq coker\ \ \delta$. Therefore, either $L_t = L$ or $L_t = 0$. Since

$$K \otimes_R L \simeq \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}_{(*)}$$

which is non zero, hence, $L_t = 0$ and L is torsion free. Therefore, the hypotheses of theorem 2 are satisfied. Theorem 2 gives us a two-sided tilting complex X in $D^b(\Lambda \otimes_R \Gamma^{op})$ with Γ being the endomorphism ring of $\hat{T}$. Moreover, one knows by [12] that $k \otimes_R \Gamma \simeq B$. By [11], one gets that $k \otimes_R X$ is a two-sided tilting complex in $D^b(A \otimes_R B^{op})$.

Type $SD(3\mathcal{H})^k$ is equivalent to $SD(3\mathcal{C}_{2,II})^k$: The very same argument applies to the tilting complex T given by TH. Holm between the forenamed types of algebras. The tilting complex he uses is

$$T: \quad \ldots \longrightarrow 0 \longrightarrow P_1 \oplus P_0 \oplus P_0 \longrightarrow P_2 \longrightarrow 0 \longrightarrow \ldots$$

with simple homology in degree 0 over the algebra $C := SD(3\mathcal{C}_{2,II})^k$. For the decomposition matrix Erdmann give two possibilities depending on the parameters, namely:

$$D = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}_{(*)} \quad \text{or } D = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}_{(*)}$$

If there is a symmetric order Λ with $k \otimes_R \Lambda \simeq C$, then our theorem applies. In fact, then there is a unique tilting complex $\hat{T}$ with $k \otimes_R \hat{T} \simeq T$ and endomorphism ring being an R-order reducing to B. Tensored with K the tilting complex is a splicing of the short exact sequences

$$0 \longrightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}_{(*)} \longrightarrow \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}_{(*)} \longrightarrow \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}_{(*)} \longrightarrow 0$$

and

$$0 \longrightarrow \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}_{(*)} \longrightarrow \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}_{(*)} \longrightarrow \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}_{(*)} \longrightarrow 0$$

or in the second case

$$0 \longrightarrow \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}_{(*)} \longrightarrow \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}_{(*)} \longrightarrow \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}_{(*)} \longrightarrow 0$$

and

$$0 \longrightarrow \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}_{(*)} \longrightarrow \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}_{(*)} \longrightarrow \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}_{(*)} \longrightarrow 0$$

One observes that the second case does not produce a tilting complex. Hence, the second decomposition matrix does not occur in the case $SD(3\mathcal{C}_{2,II})$. The rest of the argument is completely analogous to the afore discussed case.

Existence of a lifting to an order: The question if there is an order Λ such that $\Lambda \otimes_R k$ is Morita equivalent to A can be treated using theorem 1.

The algebras $A = SD(3\mathcal{H})^s$ and $E := SD(3\mathcal{D})^s$ of Erdmann's list in [2] are derived equivalent, choosing parameters for E such that blocks theoretically could occur, as shown in [4] by giving a tilting complex over A with endomorphism ring E. We know, that the algebra E is Morita equivalent to a principal block of a group ring, namely $B_0(PSL_3(q))$ over an algebraically closed field of characteristic 2, the principal block of the projective special linear group of degree 3 over $\mathbb{F}_q$, the field with q elements with $q \equiv 3 \pmod 4$ (see the remark at the end of [3]). Then, $s = 2^{n-2}$ where the Sylow-2 subgroup of $PSL_3(q)$ is semidihedral and has order $4s$.

We can give a tilting complex T over E with endomorphism ring being isomorphic to A.

Let P_0, P_1, P_2 be the three projective indecomposable modules of D. We apply theorem 1 to L being the top of P_0. Identify P_1 with the projective indecomposable corresponding to the vertex 2 in A and P_2 with the projective indecomposable corresponding to the vertex 1 in A. Then, the main result in [7] gives that $End_{D^b(E)}(T) \simeq A$. Moreover L is simple. Using that the decomposition matrix is, according to [2]

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}_{(*)}$$

and hence the lifting $\hat{T}$ of T to the corresponding order has a non torsion part in the degree 0 homology, we see that $H_0(\hat{T})$ is a lattice. The hypothesis of theorem 2 is not satisfied. Nevertheless, theorem 1 shows that there is an order Λ with $\Lambda \otimes_R k \simeq A$ for $s = 2^n$.

We even do not have to verify that $End_{D^b(E)}(T) \simeq A$. The complex Th. Holm gives is a two term complex S with $H_0(S)$ being simple. We may assume that A and E are basic. Then, the k-dimension of $H_0(S)$ is one. Hence, the existence of a twosided tilting complex X being isomorphic to S if restricted to the left implies the existence of a two term tilting complex T over E with endomorphism ring being A and simple homology in degree 0. Since homology being the simple with projective cover P_1 or P_2 do not lead to a T with sufficiently big k-dimension in degree 1, the complex we give above is the correct one.

Remark It is not clear to the author if there is *always* an R-order Λ reducing to A. As mentioned in [15] this seems to be an open problem in general.

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