ADDENDUM TO "DIFFERENTIAL GRADED DIVISION ALGEBRAS, THEIR MODULES, AND DG-SIMPLE ALGEBRAS"

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Theorem 2.4 in the paper "Differential graded division algebras, their modules, and dg-simple algebras" allows a simple alternative proof, different from the one used in Lemma 2.2, not using the strange condition on the set of homogeneous left, resp. right regular elements.

Indeed, suppose that (A,d) is a dg-division ring. We need to see that $\ker(d)$ is a $\mathbb{Z}-gr$-division ring.

Suppose this is not true. Then there is an in $\ker(d)$ non invertible homogeneous element $0 \neq x \in \ker(d)$. Hence, x is not left invertible, or not right invertible in $\ker(d)$, or both, since else x would be invertible, contradicting the hypothesis.

Suppose first that $x \cdot \ker(d) \neq \ker(d)$. If $xA \neq A$, then, since $x \in \ker(d)$, the ideal xA would be a non trivial dg-right ideal of A. This is impossible since (A,d) is assumed to be a dg-division algebra. Hence $xA = A$, and therefore there is a homogeneous $y \in A$ with $xy = 1$. But then

$$0 = d(1) = d(xy) = d(x)\cdot y + (-1)^{|x|}x \cdot d(y) = (-1)^{|x|}x \cdot d(y).$$

Since $x \in \ker(d)$, the left ideal Ax is a dg-left ideal of A. Hence, since (A,d) is a dg-division ring, $Ax = A$. But then

$$A \cdot d(y) = A \cdot x \cdot d(y) = A \cdot 0 = 0,$$

which implies $d(y) = 0$. Hence $y \in \ker(d)$, which implies in turn

$$\ker(d) \supseteq x \cdot \ker(d) \supseteq x \cdot y \cdot \ker(d) = \ker(d).$$

This was excluded.

If $\ker(d) \cdot x \neq \ker(d)$, the analogous argument gives a contradiction as well. Hence $\ker(d)$ is a $\mathbb{Z}-gr$-division ring.

It is easy to give rings with elements having only a one-sided regular element. I thank Manuel Saorín for examples of this kind. Let K be a field, and consider the free algebra $K\langle X,Y,Z\rangle$ in the three variables X,Y,Z. Let I be the two-sided ideal generated by $XY - XZ$. Then in $A := K\langle X,Y,Z\rangle/I$ the element X is left regular (multiplication by X from the right is injective), but not right regular, since $XY = XZ$ whereas $Y \neq Z$. Further, as the relation is homogeneous of degree 2, this algebra is graded, even quadratic, when X,Y,Z are assumed to be in degree 1.

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