ORE LOCALISATION FOR DIFFERENTIAL GRADED RINGS; TOWARDS GOLDIE'S THEOREM FOR DIFFERENTIAL GRADED ALGEBRAS

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Abstract. We study Ore localisation of differential graded algebras. Further we define dg-prime rings, dg-semiprime rings, and study the dg-nil radical of dg-rings. Then, we define dg-essential submodules, dg-uniform dimension, and apply all this to a dg-version of Goldie's theorem on prime dg-rings.

Introduction

Differential graded algebras first appeared in a paper by Cartan [3] and were then developed mainly in the context of algebraic topology, algebraic geometry and differential geometry. For a modern treatment we refer to Yekutieli [15]. For a commutative base ring K a differential graded K-algebra is an associative unital $\mathbb{Z}$-graded K-algebra together with a degree 1 endomorphism d of square 0 satisfying Leibniz formula

$$d(a\cdot b)=d(a)\cdot b+(-1)^{|a|}a\cdot d(b)$$

for all $a,b\in A$ such that a is homogeneous of degree $|a|$. A differential graded left (resp. right) module (M,δ) over a differential graded K-algebra (A,d) is a $\mathbb{Z}$-graded left (resp. right) A-module with a K-linear endomorphism $\delta:M\to M$ of degree 1 such that

$$\delta(a\cdot m)=d(a)\cdot m+(-1)^{|a|}a\cdot\delta(m)\ \text{(respectively } \delta(m\cdot a)=\delta(m)\cdot a+(-1)^{|m|}m\cdot d(a))$$

for all homogeneous $a\in A$ and $m\in M$. So, a differential graded algebra is an algebra at first, and this fact remained largely unexplored until very recently. In a recent sequel of papers by the author [16, 17], by Orlov [13, 14], by Aldrich and Garcia-Rozas [1], and by Goodbody [6] differential graded rings are considered from a ring theoretic point of view. Still Orlov's papers have an algebraic geometric perspective in mind.

Orlov [13, 14], and later but independently [16], defined semisimple dg-algebras from an algebra point of view, namely that in a dg-simple dg-algebra there is no non-trivial two-sided differential graded ideal, and a dg-semisimple dg-algebra is a direct product of dg-simple dg-algebras. In contrast to the classical case, this definition leads to a concept different from when we consider dg-algebras whose dg-module category is semisimple. This latter point of view has been completely settled by Aldrich and Garcia-Rozas [1]. Following Orlov's approach, a definition of a dg-radical is given by Goodbody [6] for finite-dimensional algebras with separable radical quotient, including a dg-version of Nakayama's lemma. In [16] there is a different definition of a dg-radical and Nakayama's lemma, avoiding hypotheses on finite dimensionality and separability. In [17] a dg-version of the Brauer group is given. The present paper continues these investigations.

In classical non-commutative ring theory Ore localisation is a very important tool. Using it, an important result is Goldie's theorem for prime rings satisfying the ascending chain condition on right annihilator ideals and satisfying that there is no infinite direct sum of ideals in the ring. The maximal number of ideals in a direct sum is well-defined and is called the uniform dimension $\mathrm{udim}(R)$ of the ring. The direct sum of $\mathrm{udim}(R)$ ideals is then an essential ideal. Rings that satisfy these conditions are called Goldie rings. Goldie's theorem [4, 5] states that the Ore localisation at the regular elements of a prime Goldie rings is a simple Artinian ring. A group-graded version for an abelian group of this theorem was given in its most accessible form by Goodearl and Stafford [7].

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In the present note, we first consider Ore localisation and show that the Ore localisation at homogeneous regular elements of a differential graded algebra still is a differential graded algebra, extending the differential graded structure of the initial algebra. A different approach of Ore localisation focusing on the homology of the dg-algebra was given by Braun, Chuang and Lazarev [2]. Our extension of the differential provides an explicit definition (cf Proposition 2.7) but also generalises Braun-Chuang-Lazarev's result considerably.

Along the way we consider the dg-nil radical of an algebra and see that the dg-prime radical, which is the intersection of dg-prime ideals, in general strictly includes the dg-nil radical. We show that if (A, d) satisfies the artinian condition on twosided differential graded ideals, then the dg-nil radical coincides with the dg-Jacobson radical in its twosided version and with the dg-prime radical.

Furthermore, we define and study elementary properties of dg-essential ideals, and the dg-uniform dimension. In the classical theory the dg-right singular ideal is the set of left annihilators of essential right ideals. In contrast to the classical or the graded case the differential graded version is not a two-sided ideal in general, nor is it nilpotent. We study the relation with the classical and with the graded case, in particular related to these concepts for the homology algebra and the cycles.

Finally, we prove a dg-version of Goldie's theorem for a dg-algebra (A, d), basically under the hypotheses that the cycles $\ker(d)$ satisfy the hypotheses of the graded version [7] of Goldie's theorem, where simplicity is defined as the absence of non-trivial two-sided dg-ideals. We show that this hypothesis is stronger than asking the dg-version of the corresponding Goldie's hypotheses. However, since the dg-singular ideal is badly behaved, it is unlikely that a direct generalisation of Goldie's theorem to the dg-world is possible.

The paper is structured as follows. In Section 1 we recall the necessary definitions and concepts of the classical and of the graded theory around Goldie's theorem. In Section 2 we revise existing results around Ore localisation.We also prove our first main result Theorem 2.5, which extends the differential of a dg-algebra to the Ore localisation at homogeneous regular elements. Section 3 then studies questions on a version of prime and semiprime dg-algebras in the differential graded sense, including properties of the dg-nil radical and dg-prime radical. In Section 4 we define and study elementary properties of a dg-version of a module, such as being dg-essential, used in Section 5 to define and study dg-uniform dimension, and the dg-singular ideal. In Section 6 we consider differential graded left or right annihilator ideals and study the connections with the dg-singular ideal and dg-essential ideals. Finally, Section 7 we prove our second main result Theorem 7.4, namely a dg-version of Goldie's theorem.

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1. The classical situation: Goldie's theorem and its graded version

We refer to McConnell and Robson [10, Chapter 2] and Nastacescu-van Oystaen [11] for the treatment of the present section.

We shall use a number of standard definitions in ring theory. We shall only give the graded versions below, since they reduce to the standard versions by considering a trivial grading.

- For a group G we call a *G-graded ring* R is a ring R such that $R = \bigoplus_{g\in G} R_g$ such that $R_g \cdot R_h \subseteq R_{gh}$ for all $g, h \in G$. Then a *G-graded module* M is an R-module such that $M = \bigoplus_{g\in G} M_g$ and such that $R_g \cdot M_h \subseteq M_{gh}$ for all $g, h \in G$. A G-graded module is *gr-simple* if it does not have any G-graded submodule. It is *gr-Artinian (resp. gr-Noetherian)* if any descending (resp. ascending) chain of G-graded submodules is finite.
- A G-graded ring is called *gr-prime* if for any two non zero G-graded two-sided ideals I and J we have $IJ \neq 0$.
- For a G-graded module M a G-graded submodule N is *gr-essential* if for any non zero G-graded submodule X of M we have $N \cap X \neq 0$. Further, M is called *gr-uniform* if it is non zero and any G-graded submodule is gr-essential. M has *finite gr-uniform dimension* if it does not contain an infinite direct sum of G-graded non zero submodules. The *gr-singular ideal* of R is the set of homogeneous elements of R such that there is a gr-essential right ideal I with $aI = 0$.

- A G-graded ring is a *right gr-Goldie* ring if it has finite right gr-uniform dimension on G-graded right ideals and it satisfies the ascending chain condition on right annihilators of homogeneous elements.

For the trivial grading, a prime ring is gr-prime, an essential submodule is gr-essential submodule, etc.

With these preparations Goldie's celebrated theorem states as follows (cf e.g. [10, 2.3.6]).

Theorem 1.1. *(Goldie) The following are equivalent.*

- *R is semiprime right Goldie.*
- *R is semiprime, $\zeta(R) = 0$ and the right uniform dimension of R is finite.*
- *The Ore localisation Q of R at the regular elements of R is semisimple Artinian.*

Further, in the last item R is prime if and only if Q is simple.

Goldie's theorem was generalised to rings graded by an abelian group by Goodearl and Stafford (see also [11, 8.4.5 Theorem]).

Theorem 1.2. [7, Lemma 2, Theorem 1] *Let R be a G-graded ring, where G is an abelian group. Suppose that R is a gr-prime right gr-Goldie ring.*

1. *Then any non zero graded twosided ideal of R contains a non-nilpotent homogeneous element,*
2. *the right gr-singular ideal of R is nilpotent, whence 0,*
3. *and the localisation of R at homogeneous regular elements is a gr-simple, gr-Artinian ring.*

Note that this is an almost faithful transposal of Goldie's theorem to the graded situation, graded by an abelian group.

Example 1.3. (cf [7]) The attentive reader may have observed that in Goldie's Theorem 1.1 the ring is assumed to be semiprime, whereas in Theorem 1.2 the graded ring R is assumed to be gr-prime. Goodearl and Stafford mention in [7] the example $R = K[X] \oplus K[Y]$ with X in degree 1, Y in degree -1, and $XY = 0$. Then R is graded semiprime, is not gr-semisimple, but does not have any homogeneous regular element other than in degree 0. Hence Theorem 1.2 does not generalise to gr-semiprime rings.

2. Ore localisation and differential graded rings

2.1. **Classical Ore localisation revisited.** We recall the theory of Ore localisation from classical ring theory.

Let S be a non empty multiplicatively closed subset of R and define (the right version)

$$\text{ass}(S) := \{r \in R \mid \exists s \in S \,:\, rs = 0\}.$$

A *right quotient ring of R with respect to S* is a ring Q together with a ring homomorphism $\theta : R \longrightarrow Q$ such that

1. $\theta(S) \subseteq Q^\times$, the group of invertible elements in Q.
2. $\forall q \in Q\ \exists s \in S\ \exists r \in R \,:\, q \cdot \theta(s) = \theta(r)$
3. $\ker(\theta) = \text{ass}(S)$.

Similarly one defines the left quotient ring by modifying the second condition and the definition of $\text{ass}(S)$ to the left version accordingly.

A multiplicatively closed subset S satisfies the *right Ore condition* if

$$\forall r \in R \forall s \in S \exists r' \in R \exists s' \in S \,:\, rs' = sr'$$

Dually one defines the left Ore condition. It is easy to see that if a right quotient ring exists, then S satisfies the right Ore condition. Further, by e.g. [10, Chapter 2.1.12], if the mutiplicatively closed set S satisfies the right Ore condition, then $\text{ass}(S)$ is a two-sided ideal of R and the right quotient ring R_S with respect to S exists if and only if the image of S in $R/\text{ass}(S)$ consists of regular elements.

2.2. **Extending the differential to the Ore localisation.** Let S be a multiplicative system, and suppose that the left (resp. right) quotient ring R_S exists with respect to some multiplicative set S. We shall see in the proof of Theorem 2.5 below that, if $\widehat{d}$ is an extension of d to R_S $\widehat{d}$, if it exists, is uniquely defined by the above formula.

We still need to show that this formula is well-defined. This is the subject of Theorem 2.5 below.

Example 2.1. Let K be a field. Consider the polynomial algebra $K[X]$ in one variable and X in degree -1. Then $d(X) := 1$ extends to a dg-algebra structure on $K[X]$. By the Leibnitz formula we get $d(X^{2n}) = 0$ and $d(X^{2n+1}) = X^{2n}$ for all $n \in \mathbb{N}$. Further, this also gives a well-defined differential, since if $X^{2n} = X^k \cdot X^\ell$ implies that k and ℓ are either both even or both odd, and likewise for X^{2n+1}. Then $K[X]$ is $\mathbb{Z}$-graded, integral, and hence all non zero elements are regular. Its field of fractions is $K(X)$, the field of rational functions, and the grading on $K[X]$ does not extend to a grading on $K(X)$.

Proposition 2.2. *Let (R,d) be a differential graded ring and let S be a multiplicative subset of $\ker(d)$ satisfying the right Ore condition in R. Then $\mathrm{ass}(R) := \{r \in R \mid \exists s \in S \,:\, rs = 0\}$ is a two-sided dg-ideal of (R,d). Similar statements hold for the left version of $\mathrm{ass}(S)$ and the left Ore condition.*

Proof. By McConnell and Robson [10, Section 2.1.9] we get that $\mathrm{ass}(S)$ is a two-sided ideal. We need to show that it is a dg-ideal. For this, let $r \in \mathrm{ass}(S)$ and let $rs = 0$ for some $s \in S$. Then for all homogeneous $r \in R$ we get

$$0 = d(0) = d(rs) = d(r)s + (-1)^{|r|} r \cdot d(s) = d(r)s$$

since $S \subseteq \ker(d)$. Hence $\mathrm{ass}(S)$ is a two-sided dg ideal. ∎

Remark 2.3. Let S be a multiplicative system of homogeneous regular elements of (R,d). Then (R,d) is actually either unbounded or S is concentrated in degree 0. Indeed, the k-th power of an element x in degree $2n$ is in degree $2nk$. If $\overline{x} \neq 0$, then $\overline{x}$ regular implies that $x^k \neq 0$ and therefore also the degree $2nk$ of R is non zero.

Remark 2.4. Recall from [10, end of 2.1.16] that in a left Ore localisation R_S we have $(a,s) = (b,t)$ for homogeneous elements a,b,s,t if and only if there are $c_1 \in S$ and $a_2 \in R$ such that $c_1 b = a_2 a \in R$ and $c_1 t = a_2 s \in S$. Hence, if S only contains regular elements, then the natural ring homomorphism $R \longrightarrow R_S$ is injective.

2.3. **Localisation of dg-rings at homogeneous elements.** We prove the first main result of the paper.

Theorem 2.5. *Let (R,d) be a dg-ring, and let S be a multiplicative set of homogeneous elements. Assume that either S consists of regular elements, or else $S \subseteq \ker(d)$ is a left Ore set and the image of S in $R/\mathrm{ass}(S)$ consists of regular elements of $R/\mathrm{ass}(S)$. Then*

$$d(b,s) := (-1)^{|s|+1}(d(s),s)\cdot(b,s) + (-1)^{|s|}(d(b),s)$$

defines a differential graded structure on R_S, and the natural homomorphism is a dg ring homomorphism $\lambda : (R,d) \longrightarrow (R_S, d_S)$ such that $\lambda(S) \in R_S^\times$, the group of invertible elements of R_S, and such that for any $q \in R_S$ there exists $s \in S$ with $\lambda(s)\cdot q \in \mathrm{im}(\lambda)$. Similar statements hold for the right version.

Proof. We deal with the left Ore case, the other being dual. We also deal with the left version of $\mathrm{ass}(S)$. If $\mathrm{ass}(S) \neq 0$, then by the hypothesis, $S \subseteq \ker(d)$, and using Proposition 2.2 and Remark 2.4, we may replace R by $R/\mathrm{ass}(S)$, and then assume that $\mathrm{ass}(S) = 0$.

Let $t \in S$ be homogeneous. Then by the left Ore condition there are $g_1 \in S$ and $a_1 \in R$ such that $g_1 a = a_1 t$ and for $s,t \in S$ there is $s_1, t_1 \in S$ with $t_1 s = s_1 t$. Define then

$$(a,s)\cdot(b,t) = (a_1 b, g_1 s) \text{ and } (a,s) + (b,t) = (t_1 a + s_1 b, t_1 s).$$

As we have seen in the proof of Proposition 2.2 we get that $d(1) = 0$. Assume that we may find a dg-ring as in the statement of the theorem in which s is invertible. Then

$$0 = d(s\cdot s^{-1}) = d(s)\cdot s^{-1} + (-1)^{|s|} s \cdot d(s^{-1})$$

and hence we need to define

$$d(s^{-1}) = (-1)^{|s|+1} s^{-1} \cdot d(s) \cdot s^{-1}.$$

Therefore, we can determine a general formula for the differential.

$$\begin{aligned} d(b,s) =& d((1,s)\cdot(b,1)) \\ =& d(1,s)\cdot(b,1) + (-1)^{|s|}(1,s)\cdot d(b,1) \\ =& (-1)^{|s|+1}(1,s)\cdot d(s,1)\cdot(1,s)\cdot(b,1) + (-1)^{|s|}(1,s)\cdot(d(b),1) \\ =& (-1)^{|s|+1}(d(s),s)\cdot(b,s) + (-1)^{|s|}(d(b),s) \end{aligned}$$

Let $b \in R$ and $s \in S$. Note that this is hence the unique possible extension of the differential d to the quotient ring. Consider now an element $(sb,s) = (b,1)$. Then, since d should extend the differential on R, we get $d(b,1) = (d(b),1)$. Then

$$\begin{aligned} d(sb,s) =& (-1)^{|s|+1}(d(s),s)\cdot(sb,s) + (-1)^{|s|}(d(sb),s)) \\ =& (-1)^{|s|+1}(d(s),s)\cdot(b,1) + (-1)^{|s|}(d(s)b + (-1)^{|s|}sd(b),s) \\ =& (-1)^{|s|+1}(d(s)b,s) + (-1)^{|s|}(d(s)b,s) + (d(b),1) \\ =& (d(b),1) \end{aligned}$$

We assume that $(a,s) = (b,t)$ for homogeneous elements a,b,s,t. Then there are $c_1 \in S$ and $a_2 \in R$ such that $c_1 b = a_2 a \in R$ and $c_1 t = a_2 s \in S$. Then for any $t \in R$ with $ts \in S$ we get

$$\begin{aligned} d(ta,ts)-&d(a,s) = \\ =& (-1)^{|ts|+1}(d(ts),ts)\cdot(ta,ts) + (-1)^{|ts|}(d(ta),ts) - d(a,s) \\ =& (-1)^{|ts|+1}(d(t)s + (-1)^{|t|}td(s),ts)\cdot(ta,ts) + (-1)^{|ts|}(d(t)a + (-1)^{|t|}td(a),ts) - d(a,s) \\ =& (-1)^{|ts|+1}(d(t)s,ts)\cdot(ta,ts) + \\ & (-1)^{|ts|}(d(t)a,ts) + (-1)^{|s|+1}(d(s),s)\cdot(a,s) + (-1)^{|s|}(d(a),s) - d(a,s) \\ =& (-1)^{|ts|+1}(d(t)s,ts)\cdot(ta,ts) + (-1)^{|ts|}(d(t)a,ts) \\ =& (-1)^{|ts|}((d(t)a,ts) - ((d(t),ts)\cdot(s,1)\cdot(a,s))) \\ =& (-1)^{|ts|}((d(t)a,ts) - ((d(t),ts)\cdot(a,1))) \\ =& 0 \end{aligned}$$

Hence

$$d(a,s) = d(a_2 a, a_2 s) = d(c_1 b, c_1 t) = d(b,t)$$

This shows that the above definition is well-defined.

We need to verify the Leibniz formula. We need to verify that

$$d((a,s)\cdot(b,t)) = d(a,s)\cdot(b,t) + (-1)^{|a|-|s|}(a,s)\cdot d(b,t)$$

for homogeneous elements $a,b \in R$ and $s,t \in S$.

Let $a_1 \in R$ and $g \in S$ such that $a_1 t = ga$. Let us compute the left hand term

$$\begin{aligned} d((a,s)\cdot(b,t) =& d((a_1 b, gs)) \\ =& (-1)^{|gs|+1}(d(gs),gs)(a_1 b,gs) + (-1)^{|gs|}(d(a_1 b),gs) \\ =& (-1)^{|gs|+1}\left[(d(g)s,gs)(a_1 b,gs) + (-1)^{|g|}(gd(s),gs)(a_1 b,gs)\right] \\ & + (-1)^{|gs|}\left[(d(a_1)b,gs) + (-1)^{a_1|}(a_1 d(b),gs)\right] \\ =& (-1)^{|gs|+1}(d(g)s,gs)(a_1 b,gs) + (-1)^{|s|+1}(d(s),s)(a_1 b,gs) \\ & + (-1)^{|gs|}(d(a_1)b,gs) + (-1)^{|gs|+|a_1|}(a_1 d(b),gs) \end{aligned}$$

The right hand term reads as

$$\begin{aligned} d((a,s))(b,t)+(-1)^{|(a,s)|}(a,s)d(b,t) =&(-1)^{|s|+1}(d(s),s)(a,s)(b,t)+(-1)^{|s|}(d(a),s)(b,t)\\ &+(-1)^{|(a,s)|}(a,s)\left[(-1)^{|t|+1}(d(t),t)(b,t)+(-1)^{|t|}(d(b),t)\right]\\ =&(-1)^{|s|+1}(d(s),s)(a_1b,gs)+(-1)^{|s|}(d(a),s)(b,t)\\ &+(-1)^{|(a,s|)+|t|+1}(a_1d(t),gs)(b,t)+(-1)^{|(a,s)|+|t|}(a,s)(d(b),t) \end{aligned}$$

The second term of the left hand side equals the first term of the right hand side. We hence need to verify

$$\begin{aligned} &(-1)^{|gs|+1}(d(g)s,gs)(a_1b,gs)+(-1)^{|gs|}(d(a_1)b,gs)+(-1)^{|gs|+|a_1|}(a_1d(b),gs)\overset{!}{=}\\ &\overset{!}{=}(-1)^{|s|}(d(a),s)(b,t)+(-1)^{|(a,s|)+|t|+1}(a_1d(t),gs)(b,t)+(-1)^{|(a,s)|+|t|}(a,s)(d(b),t) \end{aligned}$$

This is equivalent to

$$\begin{aligned} &(-1)^{|gs|+1}(d(g)s,gs)(a_1b,gs)+(-1)^{|gs|}(d(a_1)b,gs)+(-1)^{|gs|+|a_1|}(a_1d(b),gs)\overset{!}{=}\\ &\overset{!}{=}(-1)^{|s|}(d(a),s)(b,t)+(-1)^{|(a,s|)+|t|+1}(a_1d(t),gs)(b,t)+(-1)^{|(a,s)|+|t|}(a_1d(b),gs). \end{aligned}$$

However, by [11, 8.1.1 Lemma] we see that in the equation $ga=a_1t$, we may assume that also a_1 is homogeneous. Further we get $|ga_1|=|a_1t|$ and therefore $(-1)^{|a_1|+|g|}=(-1)^{|a|+|t|}$ and hence the last terms of the left hand side and the right hand side coincide. Therefore the equation we need to verify is equivalent to

$$\begin{aligned} &(-1)^{|gs|+1}(d(g)s,gs)(a_1b,gs)+(-1)^{|gs|}(d(a_1)b,gs)\overset{!}{=}\\ &\overset{!}{=}(-1)^{|s|}(d(a),s)(b,t)+(-1)^{|(a,s|)+|t|+1}(a_1d(t),gs)(b,t). \end{aligned}$$

Further, both sides are right multiples of $(b,1)$, and hence we are done once we proved

$$\begin{aligned} &(-1)^{|gs|+1}(d(g)s,gs)(a_1,gs)+(-1)^{|gs|}(d(a_1),gs)\overset{!}{=}\\ &\overset{!}{=}(-1)^{|s|}(d(a),s)(1,t)+(-1)^{|(a,s|)+|t|+1}(a_1d(t),gs)(1,t). \end{aligned}$$

Since we assumed that $\mathrm{ass}(S)=0$, we may multiply with $(t,1)$ from the right and use $ga=a_1t$ so that we need to show

$$\begin{aligned} &(-1)^{|gs|+1}(d(g)s,gs)(ga,gs)+(-1)^{|gs|}(d(a_1)t,gs)\overset{!}{=}\\ &\overset{!}{=}(-1)^{|s|}(d(a),s)+(-1)^{|(a,s|)+|t|+1}(a_1d(t),gs). \end{aligned}$$

which is equivalent to

$$\begin{aligned} &(-1)^{|gs|+1}(d(g)s,gs)(a,s)+(-1)^{|gs|}(d(a_1)t,gs)\overset{!}{=}\\ &\overset{!}{=}(-1)^{|s|}(d(a),s)+(-1)^{|(a,s|)+|t|+1}(a_1d(t),gs). \end{aligned}$$

Since in the multiplication rule for the left most product, $gd(g)s=a_1s$, whence $a_1=gd(g)$, the multiplication rule gives $(d(g)s,gs)\cdot(a,s)=(gd(g)a,g^2s)=(d(g)a,gs)$. The above equation in turn is hence equivalent to

$$\begin{aligned} &(-1)^{|gs|+1}(d(g)a,gs)-(-1)^{|s|}(gd(a),gs)\overset{!}{=}\\ &\overset{!}{=}(-1)^{|(a,s|)+|t|+1}(a_1d(t),gs)-(-1)^{|gs|}(d(a_1)t,gs). \end{aligned}$$

However, $ga=a_1t$ implies

$$d(g)a+(-1)^{|g|}gd(a)=d(ga)=d(a_1t)=d(a_1)t+(-1)^{|a_1|}a_1d(t).$$

Now, multiplying this by $(1,gs)$ from the left, we obtain precisely what we need, signs are as they should be, and the equation we need to verify is true. ■

2.4. **Application: Goldie's theorem.** We have seen that Goldie's theorem 1.1 makes use of Ore localisations. We want to find a version of Goldie's theorem for differential graded rings and take a few lines about some considerations in this direction. We consider the case of a differential graded ring (R,d).

We have an easy first consequence of Theorem 2.5 and Theorem 1.2 in this direction.

Corollary 2.6. *Let (R,d) be a differential graded ring. If R is a $\mathbb{Z}$-graded $\mathbb{Z}$-prime right $\mathbb{Z}$-gr-Goldie ring then the localisation of R at homogeneous regular elements is a differential graded dg-simple, dg-Artinian ring.*

Proof. Apply Theorem 1.2 to the case $G = \mathbb{Z}$ and use that by Theorem 2.5 the localisation at regular homogeneous elements is differential graded. ∎

2.5. **Comparing homology localisation with Ore localisation.** Let (R,d) be a differential graded ring.

Let S be a multiplicative system of homogeneous elements of even degree in $\ker(d)$ such that the image of S in $R/\mathrm{ass}(S)$ contains only regular elements, then, following Proposition 2.2 we may form the Ore localisation R_S at S. However, since $S \subseteq \ker(d)$, we may form the image $\overline{S}$ in $H(R,d)$. Since S is a multiplicative system, $\overline{S}$ is a multiplicative system in $H(R,d)$. Suppose for the moment that (R,d) is a differential graded k-algebra for a field k. Then, by [2, Theorem 3.10] we see that

$$R *^{\mathbb{L}}_{k(\overline{S})} k(\overline{S},\overline{S}^{-1})$$

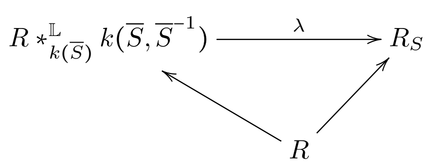
is the universal ring inverting all elements of $\overline{S}$. Here, $*^{\mathbb{L}}_{k(\overline{S})} k(\overline{S},\overline{S}^{-1})$ denotes the derived coproduct in the category of dg-rings. It can be computed by replacing the left hand argument with a cofibrant dg-algebra under $k(\overline{S})$. More precisely (cf [2, Definition 2.4]), consider the under category $A \downarrow dgAlg$ formed by objects being dg-algebra homomorphisms $A \longrightarrow C$ and morphisms being commutative triangles. For any dg-algebra homomorphism $A \longrightarrow B$ we obtain the restriction functor

$$B \downarrow dgAlg \longrightarrow A \downarrow dgAlg$$

and see that this has a left adjoint denoted by $B *_A -$. The derived functor, replacing A and B by cofibrant replacements, is then denoted by $B *^{\mathbb{L}}_A -$. Hence, there is a unique homomorphism of dg k-algebras

$$R *^{\mathbb{L}}_{k(\overline{S})} k(\overline{S},\overline{S}^{-1}) \xrightarrow{\lambda} R_S$$

such that the diagram



is commutative. However, since $R *^{\mathbb{L}}_{k(\overline{S})} k(\overline{S},\overline{S}^{-1})$ does not necessarily invert all elements of S, but only those in $\overline{S}$, we do not necessarily get that λ is invertible. An example of this kind occurs if (R,d) is acyclic.

Proposition 2.7. *We consider left Ore sets and the left version of* $\mathrm{ass}(S)$. *Suppose that $S \subseteq \ker(d)$ is a multiplicative left Ore set of homogeneous elements, and the image of S in $R/\mathrm{ass}(S)$ consists of regular elements in $R/\mathrm{ass}(R)$. Then $H(\lambda)$ is an isomorphism and hence $R *^{\mathbb{L}}_{k(\overline{S})} k(\overline{S},\overline{S}^{-1})$ is quasi-isomorphic to R_S. In particular, if $\overline{S}$ is the image of S in $H(R)$, then*

$$H(R_S) \simeq H(R)_{\overline{S}}.$$

Proof. We consider $H(R_S)$. By [2, Proposition 5.14] we get $H(R_S) = H(R)_{\overline{S}}$. This then shows that $H(\lambda)$ is an isomorphism and hence λ is a quasi-isomorphism since both $H(R *^{\mathbb{L}}_{k(\overline{S})} k(\overline{S},\overline{S}^{-1}))$ and $H(R)_{\overline{S}}$ have the same universal property being initial with respect to inverting $\overline{S}$. ∎

Remark 2.8. Note that the authors of [2] mention that their construction of $R *^{\mathbb{L}}_{k(\overline{S})} k(\overline{S},\overline{S}^{-1})$ is hard to perform explicitly. Our Proposition 2.7 provides such a construction up to quasi-isomorphism and our localisation is a lot more general.

3. Semiprime differential graded rings

We want to prove a dg-version of the results of Section 1.

3.1. dg-prime, dg-semiprime.

Definition 3.1. Let (R,d) be a differential graded ring. We define $\mathrm{dgnil}(R,d)$ to be the sum of all nilpotent differential graded two-sided ideals.

Definition 3.2. A dg-ring (R,d) is called *dg-Noetherian* if any ascending chain of dg-ideals of (R,d) is finite. A dg-ring (R,d) is *dg-Artinian* if any descending chain of dg-ideals of (R,d) is finite.

Lemma 3.3. *If (R,d) is a dg-ring, then $\mathrm{dgnil}(R,d)$ is a twosided dg-ideal of (R,d), which we call the* differential graded nil radical. *If (R,d) is dg-Noetherian, then $\mathrm{dgnil}(R,d)$ is nilpotent.*

Proof. Since the sum of ideals is an ideal, and since the differential is additive, and elements of a sum of ideals is a finite sum of elements of the constituents, the sum of dg-ideals is a dg-ideal.

Since the sum of two nilpotent ideals is a nilpotent ideal, since the sum of two differential graded ideals is a differential graded ideal, at least if (R,d) is dg-Noetherian, using the ascending chain condition, the sum of all nilpotent differential graded two-sided ideals is actually a finite sum of nilpotent differential graded two-sided ideals. Hence if (R,d) is dg-Noetherian, then $\mathrm{dgnil}(R,d)$ is a nilpotent differential graded ideal. ∎

Lemma 3.4. *Let (R,d) be a dg-Noetherian differential graded ring. Then* $\mathrm{dgnil}(R/\mathrm{dgnil}(R,d)) = 0$.

Proof. Since Lemma 3.3 shows that $\mathrm{dgnil}(R,d)$ is a dg-ideal, and since any quotient R/I of a differential graded algebra by a differential graded ideal (I,d) is again a differential graded ring, d induces a differential $\overline{d}$ on $R/\mathrm{dgnil}(R)$. Also, since preimages of dg-ideals under dg-homomorphisms are dg-ideals, $(R/\mathrm{dgnil}(R,d),\overline{d})$ is dg-Noetherian. Hence, if $(\overline{I},\overline{d})$ is a nilpotent differential graded ideal of $(R/\mathrm{dgnil}(R,d),\overline{d})$, its preimage I is a differential graded ideal (I,d). Since there is an integer n such that $\overline{I}^n = 0$, and hence $(I,d)^n \subseteq \mathrm{dgnil}(R,d)$. Now, $\mathrm{dgnil}(R,d)^m = 0$ for some m, since (R,d) is dg-Noetherian, and therefore $I^{n+m} = 0$. This then shows $I \subseteq \mathrm{dgnil}(R,d)$ and hence $\overline{I} = 0$. ∎

Definition 3.5. Let (R,d) be a differential graded ring.

- A two-sided differential graded ideal (P,d) is called *dg-prime* if whenever (S,d) and (T,d) are two-sided differential graded ideals with $ST \subseteq P$, then $S \subseteq P$ or $T \subseteq P$.
- (R,d) is called *dg-semiprime* if $\mathrm{dgnil}(R) = 0$.
- (R,d) is called *dg-prime* if for all non zero two-sided dg-ideals (I,d) and (J,d) we get $IJ \neq 0$.

Again, if R is concentrated in degree 0 (and $d = 0$), then the concept of dg-(semi-)prime coincides with the concept of (semi-)prime.

Lemma 3.6. *dg-prime rings (R,d) are dg-semiprime.*

Proof. Indeed, let (I,d) be a nilpotent dg-ideal. say $I^k = 0$. Then

$$I \cdot I^{k-1} = 0.$$

Since (R,d) is assumed to be dg-prime, we get $I^{k-1} = 0$ (or $I = 0$ which implies the former) and by induction on k we get $I = 0$. Therefore $\mathrm{dgnil}(R,d) = 0$, whence (R,d) is dg-semiprime. ∎

Let R be a Noetherian ring. Recall that the nil radical $\mathrm{Nil}(R)$ of a ring R is the sum of all nilpotent two-sided ideals of R. It is, by definition, the largest nilpotent ideal of R. Further, it is a classical result that for Noetherian rings $\mathrm{Nil}(R)$ is the intersection of all prime ideals of R.

Example 3.7. Let K be a field and let $A = K[X]/X^2$. Then A is graded when we declare X to be in degree -1. Further, $d(X) = 1$ and $d(1) = 0$ gives a structure of differential graded algebra on A. The only ideals are 0, $XK[X]/X^2$, and A. The ideal $XK[X]/X^2$ is not differential graded and hence the intersection of dg-prime ideals of A is 0, as well as $\mathrm{dgnil}(A,d)=0$. Note that the classical prime radical is XA, hence larger. Recall (cf e.g. [8, Chapter VIII]) the classical result that for Noetherian rings $\mathrm{Nil}(R)$ is the intersection of all prime ideals.

Definition 3.8. Let (R,d) be a differential graded ring. Then the *dg-prime radical* $\mathrm{Prad}(R,d)$ is the intersection of all dg-prime ideals of (R,d).

In classical ring theory the analog of dg-$\mathrm{Prad}(R)$ is sometimes called the Baer radical and the analog of $\mathrm{dgNil}(R)$ is occasionally called the Levitzky radical. We believe that our notion is more suggestive.

Lemma 3.9. *Let (R,d) be a differential graded ring. Then* $\text{dg-Prad}((R/\text{dg-Prad}(R,d),\overline{d}))=0$.

Proof. Let $\overline{I}:=\text{dg-Prad}((R/\text{dg-Prad}(R,d),\overline{d}))$ and denote by I the preimage of $\overline{I}$ in R. Then clearly $\text{dg-Prad}(R,d)\subseteq I$. We show that the inclusion in the opposite sense also holds.

To do so, we need to show that I is contained in every dg-prime ideal P of R. Let Q be such a dg-prime ideal. Then the image $\overline{Q}$ of Q in $R/\text{dg-Prad}(R,d)$ is again a dg-prime ideal of $R/\text{dg-Prad}(R,d)$. Hence $\overline{I}\subseteq\overline{Q}$, and therefore $I\subseteq Q$. This shows the lemma. ∎

Lemma 3.10. *Let (R,d) be a differential graded ring. Then* $\mathrm{dgnil}(R,d)\subseteq\text{dg-Prad}(R,d)$.

Proof. Indeed, let L be a nilpotent differential graded ideal of R. Then $L^k=0$ for some k. Let Q be a dg-prime ideal of (R,d). Then $L\cdot L^{k-1}=L^k=0\subseteq Q$ and hence, since Q is dg-prime, $L^{k-1}\subseteq Q$. By induction on k we get that $L\subseteq Q$. Hence $L\subseteq\text{dg-Prad}(R,d)$ since Q is arbitrary dg-prime. Since $\mathrm{dgnil}(R,d)$ is the sum of all nilpotent differential graded ideals, we also get $\mathrm{dgnil}(R,d)\subseteq\text{dg-Prad}(R,d)$. This shows the lemma. ∎

Lemma 3.11. *Let (R,d) be a dg-Noetherian differential graded ring. Then $(R/\mathrm{dgnil}(R,d),\overline{d})$ is dg-semiprime.*

Proof. This is an immediate consequence of Lemma 3.4. ∎

Lemma 3.12. *Let (R,d) be a dg-Noetherian differential graded ring. Then $(R/\text{dg-Prad}(R,d),\overline{d})$ is dg-semiprime.*

Proof. By Lemma 3.10 we get a surjective homomorphism of differential graded rings

$$(R/\mathrm{dgnil}(R,d),\overline{d}_1)\longrightarrow(R/\text{dg-Prad}(R,d),\overline{d}_2)$$

given by the natural inclusion from Lemma 3.10.

Let $\overline{I}:=\mathrm{dgnil}(R/\text{dg-Prad}(R,d),\overline{d}_2)$, and let I be the preimage of $\overline{I}$ in $R/\mathrm{dgnil}(R,d)$. Since R is dg-Noetherian, also $R/\text{dg-Prad}(R,d)$ is dg-Noetherian, and hence $\overline{I}$ is nilpotent. Therefore there is $k\in\mathbb{N}$ such that $I^k\subseteq\text{dg-Prad}(R,d)/\mathrm{dgnil}(R,d)$. But this implies that $I\subseteq\text{dg-Prad}(R,d)/\mathrm{dgnil}(R,d)$ by the defining property of a dg-prime ideal, and a standard induction on k. Hence $\overline{I}=0$ and we proved the lemma. ∎

Definition 3.13. Let (R,d) be a differential graded ring. We say that (R,d) is *strongly dg-semiprime* if $\text{dg-Prad}(R,d)=0$.

Proposition 3.14. *Let (A,d) be a differential graded algebra and suppose that (A,d) is left dg-artinian and dg-Noetherian. Then* $\text{dg-Prad}((A,d)=\mathrm{dgnil}(A,d)=\mathrm{dgrad}_2(A,d)$.

Proof. In a first step we show that (A,d) only contains a finite number of maximal twosided differential graded ideals. Indeed, if $I_1,I_2,\dots$ is a sequence of maximal twosided differential graded ideals, then

$$I_1\supsetneq I_1\cap I_2\supsetneq\dots$$

is a strictly descending sequence of twosided differential graded ideals of (A,d). Hence this has to stop, by the dg-artinian property on twosided dg-ideals. This is a first step.

In a second step we show that $\mathrm{dgrad}_2(A,d)$ is nilpotent. Indeed, this is a direct consequence of the dg-Nakayama Lemma [16, Lemma 4.28] and the dg-Artinianity on twosided dg-ideals.

The third step shows that any dg-prime ideal is twosided maximal. Indeed, if $\wp$ is a dg-prime ideal, then since $\mathrm{dgrad}_2(A,d)$ is nilpotent, also $\mathrm{dgrad}_2(A,d)\subseteq\wp$. But if $I_1,\dots,I_n$ are the (finite number!) maximal twosided differential graded ideals, then

$$I_1\cdot I_2\cdot\dots\cdot I_n\subseteq \mathrm{dgrad}_2(A,d)\subseteq\wp$$

and since $\wp$ is dg-prime, there is j such that $I_J\subseteq\wp$. Since I_j is maximal, we have equality.

We proved the proposition. ∎

4. DG-ESSENTIAL DG-SUBMODULES

Recall from Section 1 the notion of a gr-essential submodule.

Definition 4.1. A non zero submodule M of a differential graded module (X,δ) over a differential graded ring (R,d) is called *dg-essential* if for any differential graded submodule (N,δ) of (X,δ) one has $M\cap N\neq 0$. A *dg-essential ideal* is a dg-essential submodule of the regular module (R,d). A *dg-essential two-sided ideal* is a dg-essential submodule of the $(R\otimes_{\mathbb{Z}}R^{op},d\otimes d^{op})$-module (R,d).

Note that for a submodule M of (X,δ) we did not assume that M is differential graded. We shall need this subtlety later. However, most of the time we shall assume that M is differential graded.

If X is ungraded and $\delta=0$, we get back the usual concept of an essential submodule. Similarly, if the grading is non zero, but the differential is zero, we get the concept of a gr-essential submodule.

Lemma 4.2. *Let (R,d) be a dg-ring, and let (M,δ) be a differential graded (R,d)-module. If (N,δ) is a dg-submodule, and if the $\mathbb{Z}$-graded submodule N is gr-essential in M, then (N,δ) is dg-essential.*

Proof. Indeed, if (X,δ) is a dg-submodule of (M,δ), then forgetting the differential, X is a $\mathbb{Z}$-graded submodule, and hence $X\cap N\neq 0$. Hence (N,δ) is dg-essential in (M,δ). ∎

Lemma 4.3. *Let (R,d) be a differential graded ring.*

(1) *If (R,d) is dg-prime, then any non zero two-sided dg-ideal is dg-essential.*
(2) *The relation of being a dg-essential submodule is transitive.*
(3) *The intersection of two dg-essential submodules is dg-essential.*
(4) *If U is a dg-module, and if N is a dg-essential dg-module in the dg-module M, then $U\oplus N$ is dg-essential in $U\oplus M$.*
(5) *If (N_i,δ_i) is a dg-essential submodule of (M_i,δ_i) for all $i\in\{1,\dots,n\}$, then $(\bigoplus N_i,\bigoplus\delta_i)$ is a dg-essential submodule of $(\bigoplus M_i,\bigoplus\delta_i)$.*
(6) *If (N,δ) is a differential graded submodule of (M,δ). Then there is a differential graded submodule (X,δ) of M with $N\cap X=0$ and $N\oplus X$ is dg-essential in M.*

Proof. The proof of the first three items are trivial.

(4) If N is dg-essential in M, then $U\oplus N$ is dg-essential in $U\oplus M$. Indeed, We denote by $\pi:U\oplus M\longrightarrow M$ the canonical projection. Let X be a dg-submodule of $U\oplus M$. Then $\pi(X)$ is a submodule of $\pi(U\oplus M)=M$. Either, $\pi(X)=0$ or, using that N is essential in M, we get $\pi(X)\cap\pi(U\oplus N)\neq 0$. If $\pi(X)=0$, then $X\subseteq U$ and hence $X\cap(U\oplus N)=X\oplus 0\neq 0$. If $\pi(X)\cap\pi(U\oplus N)\neq 0$, let $0\neq x\in\pi(X)\cap\pi(U\oplus N)$. Then there is $u\in U$ such that $(u,x)\in X$ But then $(u,x)\in U\oplus N$, and therefore $(U\oplus N)\cap X\neq 0$.
(5) We proceed by induction on n. Since N_1 is dg-essential in M_1, by the previous statement we have $N_1\oplus N_2$ is dg-essential in $M_1\oplus N_2$. Again by the previous statement $M_1\oplus N_2$ is dg-essential in $M_1\oplus M_2$. By the second statement $N_1\oplus N_2$ is dg-essential in $M_1\oplus M_2$. We may assume that $N_1\oplus\dots\oplus N_{n-1}$ is dg-essential in $M_1\oplus\dots\oplus M_{n-1}$. By the case of two factors we see that $(N_1\oplus\dots\oplus N_{n-1})\oplus N_n$ is dg-essential in $(M_1\oplus\dots\oplus M_{n-1})\oplus M_n$.
(6) Let $\mathcal{X}$ be the set of dg-submodules (Z,δ) of (M,δ) such that $(N\cap Z)=0$. Since $(0,\delta)$ is in $\mathcal{X}$, we get that $\mathcal{X}\neq\varnothing$. Clearly $\mathcal{X}$ is partially ordered by inclusion. If $\mathcal{Y}$ is a totally ordered subset of $\mathcal{X}$, we get
$$\widehat{Y}:=\bigcup_{Y\in\mathcal{Y}}Y$$
is a dg-submodule of (M,δ). Further, $\widehat{Y}\cap N=0$, since else there is $0\neq y\in\widehat{Y}\cap N$. Then $y\in\widehat{Y}$ implies that there is $Y\in\mathcal{Y}$ with $y\in Y$. But this contradicts $Y\cap N=0$. Hence, by

Zorn's lemma there is a maximal element (X,δ) of $\mathcal{X}$. By definition $N\cap X=0$. Let Y be a differential graded submodule of M with $Y\cap(N\oplus X)=0$. Then $X\oplus Y$ still is a differential graded submodule satisfying $N\cap(X\oplus Y)=0$. By maximality of X we get $Y=0$. Therefore $N\oplus X$ is dg-essential in M.

This proves the lemma. ∎

Lemma 4.3.(6) suggests the following definition.

Definition 4.4. Let (R,d) be a differential graded ring and let (M,δ) be a differential graded (R,d)-module. For a differential graded (R,d)-submodule (N,δ) of (M,δ) we say that a differential graded (R,d)-submodule (L,δ) is a *dg-complement* to (N,δ) if the following two conditions hold: $N\cap L=0$, and (L,δ) is maximal with respect to this property.

Remark 4.5. As a consequence, if (N,δ) is a dg-submodule of (M,δ), and if (L,δ) is a dg-complement to (N,δ) in (M,δ), then $N\oplus L$ is dg-essential. By Lemma 4.3.(6) such a complement always exists.

Remark 4.6. Note that a dg-complement need not be a gr-complement. Moreover, a dg-essential submodule need not be a gr-essential submodule.

Analogous to the classical case we get

Corollary 4.7. *Let (R,d) be a differential graded ring. Then a differential graded (R,d)-module (M,δ) is a direct sum of simple differential graded (R,d)-modules if and only if 0 and M are the only dg-essential submodule of (M,δ).*

Proof. If (M,δ) is a direct sum of dg-simple submodules, say $M=\bigoplus_{i\in I}M_i$, and let (N,δ) be a dg-essential submodule of M. Then $N\cap M_i\neq 0$ for all $i\in I$ since N is dg-essential and (M_i,δ) is a dg submodule of M. Since M_i is dg-simple, and since $N\cap M_i\leq M_i$, we get that $N\cap M_i=M_i$, whence $M_i\subseteq N$. This holds for all i, and therefore $N=M$.

Conversely, suppose that 0 and M are the only dg-essential submodules of M. Then, Lemma 4.3.(6) implies that any differential graded submodule is complemented by a differential graded submodule, and therefore, by [16, Lemma 4.17], which is formulated for finite sums only, but which can be generalised to arbitrary sums by the usual application of Zorn's lemma, (M,δ) is a direct sum of simple dg-submodules as differential graded module. ∎

5. DG-UNIFORM DIMENSION, THE DG-SINGULAR IDEAL

Recall from Section 1 the notions of uniform modules and uniform dimension. We can easily transpose this concept to the differential graded situation.

Definition 5.1. Let (A,d) be a differential graded algebra.

- A non zero differential graded (A,d)-module (M,δ) is called *dg-uniform* if all non zero differential graded (A,d)-submodules (N,δ) of (M,δ) are dg-essential.
- A dg-module (M,δ) is said to have *finite dg-uniform dimension* if (M,δ) does not contain an infinite direct sum of differential graded submodules.
- If (M,δ) contains a dg-essential dg-submodule (N,δ) which is the direct sum of dg-uniform submodules $N_1,\dots,N_n$, then we say that n is the *dg-uniform dimension of* (M,δ) and write dg-udim(M,δ) for the dg-uniform dimension, or dg-udim$_{(A,d)}(M,\delta)$ in case we need to make precise the dg-ring which operates.

As for the classical case we shall need to show that the dg-uniform dimension is well-defined. But the proof of the classical case [10, 2.2.7, 2.2.8, 2.2.9] carries through verbatim. In particular,

Proposition 5.2. *Let (M,δ) be a differential graded (R,d)module with finite uniform dimension. Suppose that $\bigoplus_{i=1}^n(U_i,\delta)$ be a dg-essential submodule of (M,δ) such that each (U_i,δ) is uniform for each i, then any direct sum of dg-submodules of (M,δ) has at most n non zero terms, and a direct sum of non zero dg-uniform submodules of (M,δ) is dg-essential if and only if the sum has n terms.*

Proof. Indeed, the proof of [10, 2.2.9] carries through verbatim. ∎

Analogous to [10, 2.2.10] we get for the dg-situation

Lemma 5.3. *Let (R, d) be a differential graded ring and let (M, δ), (M_1, δ_1), (M_2, δ_2) be differential graded (R, d)-modules. Then*

(1) dg-udim$(M, \delta) = 1$ *if and only if* (M, δ) *is dg-uniform.*
(2) *If* (N, δ) *is a dg-submodule of* (M, δ) *and* dg-udim$(M, \delta) = n$, *then* dg-udim$(N, \delta) \leq n$ *and*
$$\text{dg-udim}(N, \delta) = n \Leftrightarrow N \text{ is dg-essential in } M$$
(3) dg-udim$(M_1 \oplus M_2, \delta_1 \oplus \delta_2)$ = dg-udim(M_1, δ_1) + dg-udim(M_1, δ_2)

Proof. The first item is simply the definition. For the second item let $N_1, \dots, N_t$ be dg-submodules of (N, δ) such that $N_1 + \dots + N_t = N_1 \oplus \dots \oplus N_t$. Since $(N, \delta) \leq (M, \delta)$, this direct sum $N_1 \oplus \dots \oplus N_t$ of dg-submodules of N is also a direct sum of dg-submodules of M. Hence
$$\text{dg-udim}(N, \delta) \leq \text{dg-udim}(M, \delta).$$
If
$$\text{dg-udim}(N, \delta) = \text{dg-udim}(M, \delta),$$
then by definition $N_1 \oplus \dots \oplus N_t$ is dg-essential in (M, δ), but then also (N, δ) is dg-essential in (M, δ) since it contains the direct sum. If (N, δ) is dg-essential in (M, δ), and let $N_1 \oplus \dots \oplus N_t$ be dg-essential, and each N_i uniform. Then by Lemma 4.3 the direct sum is dg-essential in (M, δ), and hence dg-udim$(N, \delta) = t =$ dg-udim$(M, \delta) = n$. The third item is trivial and follows by the definition. This proves the Lemma. ∎

Remark 5.4. Since any dg-submodule is a submodule, it is clear that dg-udim$(M, \delta) \leq$ udim(M) for any differential graded module (M, δ). In case M is concentrated in degree 0 (and as a consequence $\delta = 0$), then dg-udim(M, δ) = udim(M). The inequality may be strict for general $\delta \neq 0$. We shall see an instance in Example 5.8.(1). There, dg-udim$(A, d) = 1$ whereas udim$(A) = 2$ for the 2×2 matrix algebra A with the differential as given there.

Recall from e.g. [9, Chapter 3, §7] the classical notion of a singular module.

Definition 5.5. (cf e.g. [9, Chapter 3, §7]) Let M be an R-right module. Then

- $m \in M$ is *singular* if $\text{ann}(m) = \{r \in R \mid mr = 0\}$ is essential in R.
- The *singular submodule* of M is the set of singular elements in M.
- A *module is called singular* it all elements are singular. It is not difficult to show that this is indeed a submodule.
- The *right singular ideal* is the singular submodule of R_R, and the *left singular ideal* is the singular submodule of ${}_RR$.
- Accordingly, for a subset I of (R, d) we denote $\text{rann}_R(I) := \{r \in R \mid Ir = 0\}$ and $\text{lann}_R(I) := \{r \in R \mid rI = 0\}$.

When R is clear from the context we write rann(I) for $\text{rann}_R(I)$, and likewise for $\text{lann}_R(I)$.

Definition 5.6. Let (R, d) be a differential graded ring. Then the *right singular dg-ideal* $\zeta_{dg}(R, d)$ is formed by those $a \in R$ such that there is a dg-essential differential graded right ideal (E, d) of (R, d) with $a \cdot E = 0$.

As we see, we need to deal with annihilators in the dg-context. Let us give some elementary observations.

Lemma 5.7. *Let (R, d) be a differential graded ring and I a subset of R.*

(1) *Then* rann(I) *is a right ideal and* lann(I) *is a left ideal.*
(2) *If I is a left ideal, then* lann(I) *is a two-sided ideal.*
(3) *If I is a right ideal, then* rann(I) *is a two-sided ideal.*
(4) *If $I \subseteq \ker(d)$, and I is graded, then* rann(I) *is a dg-right ideal and* lann(I) *is a dg-left ideal.*
(5) *If (I, d) is a dg-left ideal, then* lann(I, d) *is a two-sided dg-ideal.*
(6) *If I is a dg-right ideal, then* rann(I, d) *is a two-sided dg-ideal.*

Proof. The proofs of the first three items are classical, and actually trivial. Item (5) and item (6) are dual. We hence only need to prove items (4) and (5).

(4) Since I is supposed to be graded, also $\mathrm{rann}(I)$ and $\mathrm{lann}(I)$ are graded. Let $x \in \mathrm{rann}(I)$ be homogeneous and $z \in I$. Then

$$0 = d(0) = d(zx) = d(z)x + (-1)^{|z|}zd(x) = (-1)^{|z|}zd(x)$$

and hence $d(x) \in \mathrm{rann}(I)$ as well. Therefore $\mathrm{rann}(I)$ is a dg-right ideal. The case of $\mathrm{lann}(I)$ is analogous.

(5) If (I,d) is a dg-right ideal, then, by the previous $\mathrm{rann}(I)$ is a two-sided ideal. Further, for all homogeneous $x \in I$ and $r \in R$ we get

$$d(xr) = d(x)r + (-1)^{|x|}xd(r)$$

which implies that $d(x)r \in I$ for all $x \in I$ and $r \in R$. Hence $\mathrm{rann}(I,d)$ is a two-sided dg-ideal.

This proves the lemma. ∎

Example 5.8. (1) We recall from [16] the following differential graded ring. For any two differential graded (A,d)-modules (M,δ_M) and (N,δ_N) we let $\mathrm{Hom}^n(M,N)$ be the abelian group of degree n homogeneous maps $f : M \to N$ such that $f(am) = (-1)^{|a|n}a \cdot f(m)$ for all homogeneous $a \in A$ and $m \in M$. The space $\mathrm{Hom}^\bullet(M,N) := \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}^n(M,N)$ allows a differential $d_{\mathrm{Hom}}(f) := \delta_N \circ f - (-1)^{|f|} f \circ \delta_M$. This way, for $M = N$, we obtain a differential graded ring $(\mathrm{Hom}^\bullet(M,M), d_{\mathrm{Hom}})$, denoted in the sequel $(\mathrm{End}^\bullet(M), d_{\mathrm{Hom}})$. Let R be any integral domain, considered as a trivial dg-ring concentrated in degree 0, let $R \xrightarrow{\lambda} R$ be the complex of R-modules concentrated in degree -1 and 0, and let

$$A = \mathrm{End}^\bullet(R \xrightarrow{\lambda} R).$$

Then

$$A = \begin{pmatrix} R & R \\ R & R \end{pmatrix}$$

where the main diagonal is the set of degree 0 elements, the lower diagonal is in degree -1 and the upper diagonal is in degree 1. The differential is

$$d(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}) := \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix},\ d(\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}) := \begin{pmatrix} 0 & \lambda(y-x) \\ 0 & 0 \end{pmatrix} \text{ and } d(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}) = 0.$$

Then $\ker(d) = R[X]/X^2$ where X is in degree 1. If $R = K$ is a field, then

$$\zeta(\ker(d)) = \zeta(K[X]/X^2) = \mathrm{soc}(K[X]/X^2) = XK[X]/X^2).$$

Suppose from now on that $R = K$ is a field. Then the algebra A is semisimple. Right ideals correspond to rows of the matrix ring and the only non trivial differential graded ideal is

$$\begin{pmatrix} K & K \\ 0 & 0 \end{pmatrix} =: I$$

since it needs to be stable by the differential and if the right lower coefficient is non zero, then also the left lower coefficient (since it is a right ideal) and by the differential also the upper two coefficients. Hence I is dg-essential, and it is the only dg-essential right ideal. Note that

$$\mathrm{rann}(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}) = \mathrm{rann}(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}) = I$$

and linear combinations of these two elements are the only ones with right annihilator I. Hence

$$\zeta_{dg}(A,d) = \begin{pmatrix} 0 & K \\ 0 & K \end{pmatrix}.$$

This ideal is a left ideal only, and in particular is not a twosided ideal. Further,

$$\zeta_{dg}(A) \cdot \zeta_{dg}(A) = \begin{pmatrix} K & K \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & K \\ 0 & K \end{pmatrix} = \begin{pmatrix} 0 & K \\ 0 & K \end{pmatrix} = \zeta_{dg}(A).$$

Hence, $\zeta_{dg}(A)$ is not nilpotent, unlike [10, Lemma 3.4] in the classical case. Since the algebra A is semisimple as an algebra, $\zeta(A) = 0$.

(2) Let K be a field. Then $A = K[X]/X^2$ is a dg-algebra with $d(X) = 1$. Then there is no non trivial dg-ideal of A. Hence, $\zeta_{dg}(A) = 0$. However, $XA = \mathrm{soc}(A)$ and hence any ideal intersects non trivially with $\mathrm{soc}(A)$. Furthermore $\mathrm{rann}(X) = \mathrm{soc}(A)$. Therefore $\zeta(A) = \mathrm{soc}(A)$.

Proposition 5.9. *Let (R,d) be a dg-ring. Then*

$$\zeta(R) \cap \ker(d) \subseteq \zeta_{dg}(R,d) \cap \ker(d) \text{ and } \zeta(\ker(d)) \subseteq \zeta_{dg}(R,d) \cap \ker(d).$$

Proof. Let us compare $\zeta(R)$ and $\zeta_{dg}(R,d)$. For $a \in \zeta(R)$ we need to have that $\mathrm{rann}_R(a)$ is essential in R. For $a \in \zeta_{dg}(R,d)$ we need to have that $\mathrm{rann}_R(a)$ is a dg-essential dg-ideal. If $a \in \ker(d)$, then $\mathrm{rann}_R(a)$ is a dg-ideal, and essential ideals are trivially dg-essential. Hence

$$\zeta(R) \cap \ker(d) \subseteq \zeta_{dg}(R,d) \cap \ker(d).$$

Further, if I is a right ideal of R, then $I \cap \ker(d)$ is a right ideal of $\ker(d)$. Hence, if $a \in \zeta(\ker(d))$, then $\mathrm{rann}_{\ker(d)}(a)$ intersects non trivially any non zero ideal of $\ker(d)$. Therefore, if $a \in \zeta(\ker(d))$, then $\mathrm{rann}_R(a)$ intersects non trivially all ideals I of R with $I \cap \ker(d) \neq 0$. However, all dg-ideals (I,d) of (R,d) do intersect with $\ker(d)$. Indeed, if $y \in I \neq 0$, then $d(y) \in I$ as well, and hence either $y \in \ker(d)$ or $d(y) \in \ker(d)$. Hence

$$\zeta(\ker(d)) \subseteq \zeta_{dg}(R,d) \cap \ker(d).$$

This shows the statement. ∎

In Proposition 5.9 we considered the (right) singular ideal of the subalgebra $\ker(d)$ of the dg-ring (R,d). Since we have a surjective ring homomorphism $\ker(d) \longrightarrow H(R,d)$, we can also consider the singular ideal of the ring $H(R,d)$, and the singular ideal of the $\ker(d)$-module $H(R,d)$.

Proposition 5.10. *Let (R,d) be a differential graded ring. Let $\pi : \ker(d) \longrightarrow H(R,d)$ be the natural homomorphism. Then*

(1) *$\zeta(H(R,d))$ coincides with the singular submodule of the $\ker(d)$-module $H(R,d)$.*
(2) *$\pi(\zeta(\ker(d))) \subseteq \zeta(H(R,d))$.*

Proof. Since π is surjective, π induces a bijection between the ideals of $H(R,d)$ and the ideals of $\ker(d)$ containing $\ker(\pi) = \mathrm{im}(d)$. Hence,

$$\begin{aligned} h \in \zeta(H(R,d)) &\Leftrightarrow \mathrm{rann}_{H(R,d)}(h) \text{ is essential in } H(R,d) \\ &\Leftrightarrow \forall_{0 \neq I \leq_r H(R,d)} : I \cap \mathrm{rann}_{H(R,d)}(h) \neq 0 \end{aligned}$$

Moreover, the annihilator $\mathrm{rann}_{H(R,d)}(h)$ of h in $H(R,d)$ coincides with the image under π of the annihilator of h as a $\ker(d)$-module. Since $\ker(\pi)$ certainly annihilates h, we get that $\zeta(H(R,d))$ coincides with the singular submodule of the $\ker(d)$-module $H(R,d)$.

Let $a \in \zeta(\ker(d))$. This is equivalent to $a \in \ker(d)$ and $\mathrm{rann}_{\ker(d)}(a)$ essential in $\ker(d)$. But if an element b in $\ker(d)$ annihilates a, then $\pi(0) = \pi(ab) = \pi(a)\pi(b)$, and hence $\pi(\mathrm{rann}_{\ker(d)}(a)) \subseteq \mathrm{rann}_{H(R,d)}(\pi(a))$. Therefore if $\mathrm{rann}_{\ker(d)}(a)$ is essential in $\ker(d)$, then $\pi(\mathrm{rann}_{\ker(d)}(a))$ is essential in $H(R,d)$. ∎

Remark 5.11. The proof of Proposition 5.10 shows that whenever $\pi : R \longrightarrow S$ is a surjective ring homomorphism, then $\zeta(S)$ coincides with the singular R-submodule of the R-module S.

Lemma 5.12. *Let (R,d) be a differential graded ring. Then $\zeta_{dg}(R,d)$ is a differential graded left ideal of (R,d).*

Proof. Let $a,b \in \zeta_{dg}(R,d)$. Then there are dg-essential dg-right ideals E_a and E_b such that $aE_a = 0 = bE_b$. By Lemma 4.3.1 also $E_a \cap E_b$ is an essential dg-right ideal of R. Then a and b annihilate $E_a \cap E_b$, and hence also $a - b$. Let $x \in R$. Then xa annihilates E_a as well, and hence $\zeta_{dg}(R,d)$ is stable by left multiplication with elements in R. Now, for any homogeneous $x \in E_a$, supposing that $a \in \zeta_{dg}(R)$ is homogeneous, we have

$$d(a) \cdot x = d(ax) - (-1)^{|a|} a \cdot d(x)$$

and since $x \in E_a$, we have $ax = 0$, whence also $d(ax) = 0$. Since E_a is a dg-ideal, also $d(x) \in E_a$ and hence $a \cdot d(x) = 0$. Therefore $d(a)$ annihilates E_a as well, and hence $d(a) \in \zeta_{dg}(R,d)$. ∎

Remark 5.13. We cannot show in general that $\zeta_{dg}(R,d) \subseteq \zeta(R)$. Indeed, if $a \in \zeta_{dg}(R,d)$, then there is a dg-essential dg-ideal (E,d) of (R,d) such that $aE = 0$. However, a dg-essential dg-ideal does not need to be essential (cf Example 5.15). Since an essential ideal does not need to be a dg-ideal, we cannot show $\zeta_{dg}(R,d) \supseteq \zeta(R)$ neither (Example 5.8.(2) provides an example).

Remark 5.14. Since $\zeta_{dg}(R,d)$ is a dg-left ideal of (R,d), it is tempting to consider its homology $H(\zeta_{dg}(R,d),d)$. An element in $H(\zeta_{dg}(R,d),d)$ is represented by $y \in \ker(d)$ such that $\mathrm{rann}(y)$ is dg-essential in (R,d). Since by Proposition 5.9 we have

$$\zeta(\ker(d)) \subseteq \zeta_{dg}(R,d) \cap \ker(d)$$

and since $H(\zeta_{dg}(R,d),d)$ is a quotient of $\zeta_{dg}(R,d) \cap \ker(d)$, there is a natural map

$$\zeta(\ker(d)) \longrightarrow H(\zeta_{dg}(R,d),d)$$

induced by the natural map

$$\ker(d|_{\zeta_{dg}(R,d)}) \longrightarrow H(\zeta_{dg}(R,d),d).$$

Likewise, since by Proposition 5.9 we have

$$\zeta(R) \cap \ker(d) \subseteq \zeta_{dg}(R,d) \cap \ker(d)$$

we also get a natural map

$$\zeta(R) \cap \ker(d) \longrightarrow H(\zeta_{dg}(R,d),d).$$

However, since for an element $a \in \ker(d)$ the property for the dg-right ideal $\mathrm{rann}(a)$ to be essential is a lot more restrictive than to be dg-essential, there is no hope to have surjectivity of either one of these maps. An example is given below in Example 5.15.

Example 5.15. Recall Example 5.8. For a field K we defined a structure of a dg-algebra on $A = \mathrm{Mat}_2(K)$. Then, $\zeta(A) = 0$ and

$$\ker(d) = K \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + K \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Moreover,

$$\zeta_{dg}(A) = \begin{pmatrix} 0 & K \\ 0 & K \end{pmatrix} \text{ and } H(\zeta_{dg}(A,d),d) = K \cdot \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \simeq K.$$

Therefore in this case the map

$$\zeta(A) \cap \ker(d) \longrightarrow H(\zeta_{dg}(A,d),d).$$

is not surjective.

Further, in this case the only non trivial ideal of $\ker(d)$ is $J := K \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Hence, this ideal is essential. Its right annihilator ideal is J itself, and hence $\zeta(\ker(d)) = J$. This shows that the map

$$\zeta(\ker(d)) \longrightarrow H(\zeta_{dg}(R,d),d)$$

is the zero map, which is neither surjective nor injective.

6. dg-Goldie rings; left and right dg-annihilators

Definition 6.1. A differential graded ring (R,d) is called *dg-left (resp. dg-right) Goldie* if (R,d) satisfies

- the ascending chain condition on dg-left (resp. dg-right) annihilators, and
- has finite left (resp. right) dg-uniform dimension.

The statements of [10, Proposition 2.14] are formal, except the last statement, and can be transposed to the differential graded situation. We shall detail the parts which do not follow from the classical arguments.

Proposition 6.2. *Let (R,d) be dg-semiprime and let (I,d) be a two-sided dg-ideal. Suppose $R \neq I$. Then*

(1) *We have* $\mathrm{lann}(I,d) = \mathrm{rann}(I,d) =: \mathrm{ann}(I,d)$ *and this is a two-sided dg-ideal.*

(2) $\text{ann}(I,d)$ *is a two-sided dg-ideal with* $I \cap \text{ann}(I,d) = 0$*, and it is the unique one which is maximal with respect to this property. In particular,* $I \oplus \text{ann}(I,d)$ *is dg-essential in* R.
(3) *If* $(R,d)/\text{ann}(I,d)$ *is strongly dg-semiprime, then* $\text{ann}(I,d)$ *is the intersection of those minimal dg-prime ideals which do not contain* I.
(4) *If the* $(R,d)-(R,d)$*-bimodule* (I,d) *is dg-uniform then* $\text{ann}(I,d)$ *is a minimal dg-prime ideal. If* $R/\text{ann}(I)$ *is strongly dg-semiprime, then the converse holds as well.*
(5) (I,d) *is a dg-essential two-sided dg-ideal if and only if* $\text{ann}(I,d) = 0$.
(6) *Suppose that* $R/\text{ann}(I)$ *is strongly dg-semiprime. If* (I,d) *is not contained in any minimal dg-prime ideal, then* $\text{ann}(I,d) = \text{dg-Prad}(I,d)$*. In particular, if* (R,d) *is strongly dg-semprime, then* (I,d) *is dg-essential if and only if* (I,d) *is not contained in any dg-prime ideal.*

Proof. In order to prove (1), observe that $X := \{r \in R \mid rI = 0\}$ is a two-sided dg-ideal by Lemma 5.7. The rest of the argument is a literal repetition of the argument in [10, Proposition 2.14].

(2): again follows the classical counterpart, except that we have that the algebra is dg-semiprime. But this is enough since our ideals are all dg-ideals. Lemma 4.3.(6) implies that $I \oplus \text{ann}(I)$ is dg-essential.

(3): If (P,d) is a minimal dg-prime ideal which does not contain I, then

$$I \cdot \text{ann}(I,d) = 0 \subseteq P.$$

Since P is dg-prime, either $I \subseteq P$ or $\text{ann}(I,d) \subseteq (P,d)$. The first case was excluded, and so the intersection D of all (minimal) dg-prime ideals not containing I contains $\text{ann}(I,d)$. Since $R/\text{ann}(I)$ is strongly dg-semiprime, the intersection of all dg-primes of $R/\text{ann}(I)$ is 0. Taking preimages under $R \longrightarrow R/\text{ann}(I)$ we get that the intersection of all dg-primes containing $\text{ann}(I)$ is $\text{ann}(I)$. Since $I \cap \text{ann}(I) = 0$, we have that D equals the intersection of all dg-primes containing $\text{ann}(I)$. Hence we get $I \cap D = 0$. Therefore, $D = \text{ann}(I,d)$ by (2), whence the statement of (3).

(4): Suppose that (I,d) is dg-uniform. Let S and T be two-sided dg-ideals with $ST \subseteq \text{ann}(I,d)$. Then $IST = 0 = STI$. If $IS = 0$, then $S \subseteq \text{ann}_r(I,d) = \text{ann}(I,d)$ and we are done. Likewise, if $TI = 0$, then $T \in \text{ann}_\ell(I,d) = \text{ann}(I,d)$, and we are done as well. If $IS \neq 0 \neq TI$, then $T \subseteq \text{ann}_r(IS) = \text{ann}(IS)$ and $S \subseteq \text{ann}_\ell(TI) = \text{ann}(TI)$. Further, $0 \neq IS \subseteq I \cap S$ and $0 \neq TI \subseteq T \cap I$. By item (2) $\text{ann}(J,d)$ is a two-sided ideal with $J \cap \text{ann}(J,d) = 0$, and it is maximal with this property, and furthermore the unique maximal one.

Further, $IS \subseteq I$ and $TI \subseteq I$. If $0 \neq (J,d) \subseteq (I,d)$, then $\text{ann}(I,d) \subseteq \text{ann}(J,d)$ by definition. We claim that $\text{ann}(I,d) = \text{ann}(J,d)$. Indeed, (I,d) is dg-uniform, hence (J,d) is a dg-essential submodule of (I,d). Now, if $\text{ann}(I,d) \subsetneq \text{ann}(J,d)$, then $I \cap \text{ann}(J,d) \neq 0$, since else this would contradict item (2), namely the maximality of $\text{ann}(I,d)$ as being maximal with $I \cap \text{ann}(I,d) = 0$. But then $I \cap \text{ann}(J,d)$ is a non zero dg-submodule of I, and since (J,d) is dg-essential, $J \cap (I \cap \text{ann}(J,d)) \neq 0$. However, $\text{ann}(J,d)$ satisfies $J \cap \text{ann}(J,d) = 0$ by item (2). This contradiction shows that $\text{ann}(I,d) = \text{ann}(J,d)$. We can now consider $J = IS$ and this then implies $T \subseteq \text{ann}(IS) = \text{ann}(I)$. Hence $\text{ann}(I,d)$ is dg-prime.

Suppose that $\text{ann}(I,d)$ is dg-prime. We shall need to see that (I,d) is dg-uniform. Let (J,d) be a two-sided dg-ideal in (I,d). We shall need to see that (J,d) is dg-essential in (I,d). Let (K,d) be a dg-ideal in (I,d). Then $J \cdot K \subseteq J \cap K$. If $J \cdot K = 0$, then $J \cdot K \subseteq \text{ann}(I,d)$. Since $\text{ann}(I,d)$ is dg-prime, either $J \subseteq \text{ann}(I,d)$ or $K \subseteq \text{ann}(I,d)$. However, $J \subseteq I$ and $K \subseteq I$ implies $J = 0$ or $K = 0$ by item (2). Now, $\text{ann}(J,d)$ is an intersection of minimal dg-prime ideals, by item (3). By definition $\text{ann}(I,d) \subseteq \text{ann}(J,d)$ as $J \subseteq I$. But $\text{ann}(I,d)$ is a minimal dg-prime, which contributes to the intersection of minimal dg-primes giving $\text{ann}(J,d)$. Therefore $\text{ann}(J,d) \subseteq \text{ann}(I,d)$, and hence $\text{ann}(J,d) = \text{ann}(I,d)$ since the other inclusion was seen above. If now (I,d) contains a direct sum of two two-sided dg-ideals $(I_1,d) \oplus (I_2,d) \subseteq (I,d)$, then $I_1 \cdot I_2 \subseteq I_1 \cap I_2 = 0$, and hence $\text{ann}(I_1,d)$ contains (I_2,d). Therefore $\text{ann}(I_1,d) \supsetneq \text{ann}(I,d)$. Taking $J = I_1$ in the discussion above, we get $\text{ann}(I_1,d) = \text{ann}(I,d)$. This contradiction shows that the dg-uniform dimension is 1, and hence, by Proposition 5.2, (I,d) is dg-uniform. We proved item (4).

(5): If (I,d) is dg-essential, by item (2) we need to have $\text{ann}(I,d) = 0$. Let us prove the other direction. Suppose that $\text{ann}(I,d) = 0$. Let (J,d) be a two-sided dg-ideal of (R,d). Then $I \cdot J \subseteq I \cap J$.

If (I,d) is not dg-essential, then there is a non zero two-sided dg-ideal (J,d) with $I \cap J = 0$. Hence, by item (2) we have that $J \subseteq \mathrm{ann}(I,d) = 0$, and this contradiction gives item (5).

(6): By item (3) and the hypothesis, we get that $\mathrm{ann}(I,d)$ is the intersection of all dg-prime ideals, which is dg-Prad(R,d). In case (R,d) is strongly dg-semiprime, then dg-Prad$(R,d) = 0$. Hence (I,d) is not contained in any dg-prime ideal if and only if $\mathrm{ann}(I,d) = 0$ if and only if (I,d) is dg-essential. ■

Remark 6.3. Recall that in the non dg-situation the intersection of all prime ideals is the nil radical, which is 0 for semiprime rings. Hence I is essential in the situation of item (6). In the dg-situation we do not have this property (cf Example 3.7) and we need the stronger hypothesis of strongly dg-semiprime rings.

Let (R,d) be a dg ring. Let (I,d) be a two-sided dg-ideal. Trivially, if I is essential, then (I,d) is also dg-essential. The converse sometimes holds as well in a very specific situation.

Corollary 6.4. *If (R,d) is a differential graded algebra, and suppose that R is semiprime as a ring. Then a two-sided dg-ideal (I,d) is dg-essential if and only if $\mathrm{ann}(I,d) = 0$, if and only if $\mathrm{ann}(I) = 0$, if and only if I is essential.*

Proof. R is assumed to be semiprime, and hence R does not contain a non zero two-sided nilpotent ideal, whence neither a two-sided nilpotent dg-ideal. Hence (R,d) is dg-semiprime as well. We may now apply Proposition 6.2 item (5) and its classical counterpart [10, Proposition 2.14 item (5)]. If (I,d) is dg-essential, then $\mathrm{ann}(I,d) = 0$ and since by definition $\mathrm{ann}(I,d) = \mathrm{ann}(I)$, we have that [10, Proposition 2.14 item (5)] implies that I is essential. ■

As in the classical situation [10, 2.2.3 and 2.2.10] we may prove

Lemma 6.5. *Let (A,d) be a differential graded ring and let (M,δ) be a differential graded (A,d)-module.*

- *If (N,δ) is a dg-complement dg-submodule of (M,δ), then for all dg-submodules (L,δ) of (M,δ) with $N \subsetneq L$ there is a non zero dg-submodule (S,δ) of (L,δ) with $S \cap N = 0$.*
- *Then the following are equivalent:*
  - dg-udim$(M,\delta) < \infty$
  - *(M,δ) satisfies the ascending chain condition on dg-complement submodules. The dg-uniform dimension of (M,δ) is the maximal length of an ascending chain of dg-complement dg-submodules.*

Proof. As for the first item, suppose that (X,δ) is a dg-submodule of M, such that N is a dg-complement to X. Then $S := L \cap X$ is a dg-submodule of (M,δ). Further,

$$S \cap N = N \cap L \cap X = N \cap X = 0$$

since $N \subseteq L$ and since X is a dg-complement to N. If $S = 0$, then L would be the dg-complement to X, which is excluded since by hypothesis the dg-complement N of X is strictly smaller than L.

As for the second item, let $n = $ dg-udim(M,δ). Let

$$0 < S_1 < S_2 < S_3 < \cdots < S_t$$

be a maximal chain of dg-complement dg-submodules of (M,δ). We claim that $t \le n$. Indeed, applying the statement of the first item to $S_1 < S_2$, we obtain a dg-submodule S_2' of S_2 with $S_1 + S_2 = S_1 \oplus S_2'$. Similarly, $S_2 < S_3$ gives a dg-submodule $S_3' < S_3$ such that $S_2 + S_3' = S_2 \oplus S_3'$. Since S_2 contains $S_1 \oplus S_2'$, we obtain a direct sum $S_1 \oplus S_2' \oplus S_3'$ of three dg-submodules. By induction we get a direct sum of t non zero dg-submodules $S_i' \le S_i$ for all $i \in \{1,\dots,t\}$ such that $S_1 \oplus S_2' \oplus \cdots \oplus S_t'$ is a direct sum of dg-submodules of (M,δ). Since dg-udim$(M,\delta) = n$, we get $t \le n$.

If we have a direct sum $\bigoplus_{i=1}^t M_i$ of dg-submodules of (M,δ), then, for $s < t$ the dg-complement of $\bigoplus_{i=1}^s M_i$ contains $\bigoplus_{i=s+1}^t M_i$ but does not contain M_i for any $i \le s$. This direct sum hence induces a chain of dg-complement dg-submodules of length at least t. This then proves the second item. ■

Recall from Theorem 2.5 that a differential graded structure on a ring R can be extended to the Ore localisation at a set of homogeneous and regular Ore set. Moreover, since the Ore set is formed by regular elements, the natural homomorphism to the localisation is injective. Compare the following lemma with [10, 2.12]. We identify R with its image in R_S under the canonical dg-homomorphism $R \xrightarrow{\lambda} R_S$.

Lemma 6.6. *Let (R,d) be a differential graded ring and let S be an Ore set of homogeneous regular elements of R. Let (I,d) be a dg-right ideal of (R,d), and let (J,d) be a dg-right ideal of (R_S,d).*

1. *Then (I,d) is dg-essential in (R,d) if and only if $(I\cdot R_S,d_S)$ is dg-essential in (R_S,d_S).*
2. *Then (J,d_S) is dg-essential in (R_S,d_S) if and only if $(J\cap R,d)$ is dg-essential in (R,d).*
3. $\text{dg-udim}_{(R,d)}(I) = \text{dg-udim}_{(R_S,d_S)}(I\cdot R_S)$
4. $\text{dg-udim}_{(R,d)}(J\cap R) = \text{dg-udim}_{(R_S,d_S)}(J)$

Proof. **(1).** Suppose that (I,d) is dg-essential in (R,d). Let $(L(S),d_S)$ be a dg-ideal of R_S. Then $(L(S)\cap R,d)$ is a dg-ideal of (R,d) and hence $L(S)\cap R\cap I = L(S)\cap I \neq 0$. Then $(L(S)\cap I)\cdot R_S \subseteq L(S)\cap I\cdot R_S$, and since S is formed by regular elements, λ is injective, whence $L(S)\cap I\cdot R_S \neq 0$. This shows that $I\cdot R_S$ is dg-essential.

Suppose now that $I\cdot R_S$ is dg-essential and let L be a dg-ideal of R. If $I\cap L = 0$, then $I+L = I\oplus L$ is a dg-right ideal of R, and hence, using flatness of the localisation,

$$(I+L)R_S = (I\oplus L)R_S = (I\cdot R_S)\oplus(L\cdot R_S)$$

is a dg-right ideal of R_S. Since $L\cdot R_S$ is a dg-ideal of R_S, and since the intersection with $I\cdot R_S$ in 0, this contradicts the fact that $I\cdot R_S$ is dg-essential.

(2). Suppose that (J,d_S) is dg-essential in (R_S,d_S) and let (L,d) be a dg-ideal of (R,d). Then $J\cap R$ is a dg-ideal of R and if $(J\cap R)\cap L = 0$, then since $(J\cap R)\cap L = J\cap L$, we also get $J\cap(L\cdot R_S) = 0$. This implies $L\cdot R_S = 0$ since $L\cdot R_S$ is a dg-ideal of R_S and J is dg-essential. However, $L\cdot R_S = 0$ implies $L = 0$. Hence $J\cap R$ is dg-essential.

Suppose that $J\cap R$ is dg-essential in R and let L be a dg-ideal of R_S. If $J\cap L = 0$, then $J\cap L\cap R = (J\cap R)\cap(L\cap R)$. Since $J\cap R$ is dg-essential in R, we get $L\cap R = 0$. This implies $L = 0$ and we showed that J is dg-essential.

(3). The additive functor $-\otimes_R R_S : (R,d)-\text{dg-mod} \longrightarrow (R_S,d_S)-\text{dg-mod}$ preserves direct sums, and hence $\text{dg-udim}_{(R,d)}(I) \leq \text{dg-udim}_{(R_S,d_S)}(I\cdot R_S)$. If (I,d) is a dg-uniform ideal of (R,d), then $(I\cdot R_S,d_S)$ is dg-uniform. This follows from [10, (2.1.16) Proposition]. Using item (1) we have that a direct sum $\bigoplus I_i$ of uniform dg-ideals of (R,d) is dg-essential if and only if the direct sum $\bigoplus I_i\cdot R_S$ is dg-essential. Hence we proved the statement.

(4). Let $\bigoplus_{i=1}^n J_i \subseteq J$ be a direct sum of dg-uniform ideals of R_S. Then the dg-ideal $\sum_{i=1}^n (J_i\cap R)$ of R in $J\cap R$ is a direct sum, and hence $\text{dg-udim}_{(R,d)}(J\cap R) \geq \text{dg-udim}_{(R_S,d_S)}(J)$. Using [10, (2.1.16) Proposition] again we see that if J_i is dg-uniform, then also $J_i\cap R$ is dg-uniform. Further, by item (2) we have that $\bigoplus_{i=1}^n J_i$ is dg-essential in J if and only if $\sum_{i=1}^n (J_i\cap R)$ is essential in $J\cap R$. Hence we proved the statement. ∎

Remark 6.7. The fact that [10, (2.2.12) Lemma] follows from [10, (2.1.16) Proposition], and that these statements are independent of the presence of a dg-structure, also the generalisation of [10, (2.2.12) Lemma] to the differential graded situation follows from [10, (2.1.16) Proposition].

Remark 6.8. If I is a dg-ideal such that $\text{lann}(I)$ is maximal within the set of left annihilators, then $I = Ra$ for some $a\in\ker(d)$. Indeed, let $0\neq a\in I$, and if $d(a)\neq 0$, then replace a by $d(a)$, which is again in I since I is a dg-ideal. Hence, we may find $a\in I\cap\ker(d)$. Then Ra is a dg-ideal and $Ra\subseteq I$. This implies $\text{lann}(I)\subseteq\text{lann}(Ra)$. Maximality of $\text{lann}(I)$ shows that left annihilators which are maximal within the set of let annihilators are annihilators of dg-principal ideals.

Recall that in the classical ungraded case and differential 0 we have the following lemma.

Lemma 6.9. [10, Lemma 2.3.2] *Let R be a ring and suppose that R satisfies the ascending chain condition on left annihilators. Then*

1. *Each maximal left annihilator has the form* $\text{lann}(a)$ *for some* $a\in R$.
2. *For any $b\in R$ there is an integer m such that* $\text{lann}(b^n) = \text{lann}(b^m)$ *for any $n\geq m$. Then, for these $n\geq m$ we have* $\text{lann}(b^n)\cap Rb^n = 0$

We consider the differential graded case.

Corollary 6.10. *Let (R,d) be a differential graded left Goldie ring, and suppose that R satisfies the ascending chain condition on left annihilators. Let $b \in R$. Then if $\text{lann}(b^n) = \text{lann}(b^m)$ for any $n \geq m$, and if $b^n \in \ker(d)$, then $Rb^n \oplus \text{lann}(b^n)$ is an essential dg-ideal, whence in particular dg-essential.*

Proof. By [10, Lemma 2.3.3] we have that $Rb^n \oplus \text{lann}(b^n)$ is an essential left ideal. Since $b^n \in \ker(d)$, Rb^n is a dg-left ideal, and $\text{lann}(b^n)$ is a dg-left ideal by Lemma 5.7. This shows the statement. ∎

We generalise [10, Proposition 2.3.4] to the dg-situation.

Lemma 6.11. *Let (R,d) be a dg-ring*

- *If $c \in \ker(d)$ is right regular in R, then the dg-right ideal (cR,d) is dg-essential in (R,d)*
- *Suppose that (R,d) is dg-semiprime, suppose that it has finite left dg-uniform dimension, and suppose that $\zeta_{dg}(R,d) = 0$. Then if $c \in \ker(d)$ is right regular, c is regular.*

Proof. Since $c \in \ker(d)$ is right regular, we get $cx = 0$ implies $x = 0$, and also cR is a dg-ideal. Hence $R \ni x \mapsto cx \in R$ is an isomorphism of (R,d)-right dg-modules. Using the differential graded uniform dimension of right ideals, we get $\text{dg-udim}(R) = \text{dg-udim}(cR)$, and therefore using Lemma 5.3.(2) (transposed to right modules) we get (cR,d) is dg-essential in (R,d).

We need to see that c is left regular. Since $\zeta_{dg}(R,d) = 0$, there is no non zero element c and a dg-right ideal E such that $cE = 0$. Since cR is dg-essential, we need to have $\text{lann}(cR) = 0$. Since $\text{lann}(c) \subseteq \text{lann}(cR)$, we also obtain $\text{lann}(c) = 0$. But this shows that there is no non zero element $y \in R$ such that $yc = 0$. This is tantamount to saying that c is left regular. ∎

7. Differential graded Goldie-theorem

Recall from Section 1 the notions of dg-prime rings and gr-prime rings.

Lemma 7.1. *Let (A,d) be a dg-algebra. Then if $\ker(d)$ is gr-prime, we have that (A,d) is dg-prime.*

Proof. Let (A,d) be a dg-algebra and let $S := \ker(d)$. Let (I,d) and (J,d) be two two-sided dg-ideals. Then $IJ = 0$ implies $(I \cap S) \cdot (J \cap S) = 0$ in S, and since S is gr-prime, we get $I \cap S = 0$ or $J \cap S = 0$. However, for any $x \in I$ we get if $x \notin \ker(d)$, then $0 \neq d(x) \in \ker(d) \cap I$. Likewise for J. Hence $(I,d) = 0$ or $(J,d) = 0$. This shows the statement. ∎

Lemma 7.2. *Let (A,d) be a differential graded algebra. If $\ker(d)$ has finite gr-uniform dimension, then (A,d) has finite dg-uniform dimension.*

Proof. Put $S := \ker(d)$. If $I_1 \oplus I_2 \oplus \cdots \oplus I_n$ is a direct sum of two-sided dg-ideals of (A,d), then $(I_1 \cap S) \oplus (I_2 \cap S) \oplus \cdots \oplus (I_n \cap S)$ is a direct sum of two-sided graded ideals in S. Again, as in Lemma 7.1 we see that $I \cap S = 0$ implies $I = 0$. This shows the lemma. ∎

Corollary 7.3. *Let (A,d) be a differential graded algebra and suppose that (A,d) is dg-Noetherian as $(A,d) - A,d)$-bimodule. If $\ker(d)$ is left gr-Goldie, then (A,d) is left dg-Goldie.*

Proof. Lemma 7.2 and the hypothesis on the Noetherianity show the statement. ∎

Theorem 7.4. *Let R be a commutative ring and let (A,d) be a differential graded R-algebra. Suppose that $\ker(d)$ is a gr-prime ring and suppose that $\ker(d)$ is right gr-Goldie.*

- *If (A,d) is dg-Noetherian as bimodule, then the localisation of A at the homogeneous regular elements S_A of A exists and is dg-simple (in the sense that there is no non zero non trivial two-sided dg-ideal).*
- *If the homogeneous regular elements $S_{\ker(d)}$ of $\ker(d)$ form a right Ore set in A,*
  - *then the localisation of (A,d) at $S_{\ker(d)}$ is a dg-simple differential graded R-algebra (in the sense that there is no non zero non trivial two-sided dg-ideal).*
  - *Further, $S_{\ker(d)} \subseteq S_A$ and hence in case S_A is right Ore as well, A_{S_A} and $A_{S_{\ker(d)}}$ both exist, are dg-simple rings, and the natural homomorphism $A_{S_{\ker(d)}} \longrightarrow A_{S_A}$ is injective.*

Proof. Let (I,d) be a differential graded two-sided ideal of (A,d). Then by Lemma 4.3 and the fact that (A,d) is dg-prime (cf Lemma 7.1) we get that (I,d) is dg-essential. Further, since $\ker(d)$ is left gr-Goldie, and (A,d) is twosided dg-Noetherian, also (A,d) is dg-left Goldie (cf Lemma 7.3). By Theorem 1.2, referring to [12, Theorem C.1.1.6]) the homogeneous regular elements of A form a left Ore set.

Further, by Theorem 1.2 we know that $\overline{I} := I \cap \ker(d)$ is either 0 or, since $\ker(d)$ is gr-prime, $\overline{I}$ is gr-essential, and contains a homogeneous regular element. $\overline{I}$ cannot be 0 since for any $x \in I \setminus \{0\}$ we either have $x \in \ker(d)$, or else $d(x) \in I \cap \ker(d)$. Hence $\overline{I} \subseteq I$ contains a homogeneous regular element y. We claim that y is regular in A as well. Indeed, if $xy = 0$ for some $x \in A \setminus \ker(d)$. Considering each homogeneous component of x separately, using that y is homogeneous, we may assume that x is homogeneous as well. Given hence a homogeneous $x \in A$ with $xy = 0$, then

$$0 = d(x \cdot y) = d(x) \cdot y + (-1)^{|x|} x \cdot d(y) = d(x) \cdot y$$

since $y \in \ker(d)$. But then $d(x) \in \ker(d)$, and y regular in $\ker(d)$ implies that $d(x) = 0$. Since $x \notin \ker(d)$ by hypothesis we reach a contradiction. Likewise, y is right regular as well. By the above, in any of the two cases, $S \in \{S_A, S_{\ker(d)}\}$ the set S is a left Ore set, either by hypothesis, or else by the above considerations. Using Theorem 2.5 we may localise at the set S and then $A_S \cdot I = A_S$. Let now (L,d) be a two-sided dg-ideal of (A_S,d). Then $I := L \cap A$ is a two-sided dg-ideal of (A,d). By the above, it contains a regular element, and hence $L \supseteq A_S \cdot I = A_S$. Therefore (A_S,d) does not contain any proper non zero two-sided dg-ideal. We showed that (A_S,d) is dg-simple. ∎

By symmetry (or using the opposite algebra) we have the analogous statements for left localisation of left Goldie rings.

Example 7.5. (1) Recall Example 5.8, where $A = \mathrm{Mat}_2(K)$ with an appropriate non trivial differential d. There $\ker(d) = K[X]/X^2$ with X in degree 1 is not gr-prime. However, A is simple, and hence (A,d) is dg-simple.
(2) Recall Example 2.1, the polynomial ring in 1 variable X over some field K. There $\ker(d) = K[X^2]$, this ring is graded Goldie, it is Noetherian. The localisation of $K[X]$ at the homogeneous regular elements of $\ker(d)$ is the Laurent polynomial ring, such as the localisation at the homogeneous regular elements of A.

We close with a lemma which is an analogue of the 'lying over property' in commutative algebra. A graded ring is graded hereditary if any graded ideal is graded projective.

Lemma 7.6. *Let (A,d) be a differential graded algebra and let $S := \ker(d)$. Suppose that S is graded hereditary. Then for any graded ideal I of S we have that $A \cdot I =: J$ is a differential graded ideal of (A,d), and $S \cap J = I$.*

Proof. Since A, S and I are graded, and since A is a graded left S module, it is clear that $A \cdot I =: J$ is a graded ideal of A. Further, for any homogeneous $a \in A$ and $y \in I \subseteq \ker(d)$, we get

$$d(a \cdot y) = d(a) \cdot y + (-1)^{|a|} a \cdot d(y) = d(a) \cdot y.$$

Hence J is a dg-ideal of (A,d). Since S is supposed to be graded hereditary, I is projective, and we may suppose that $Y := \{y_i \mid i \in F\}$ is an S-basis of $I \oplus X$ for some graded S-module X.

Recall that the differential of $A \otimes_S (I \oplus X)$ is $d \otimes_S \mathrm{id}_{I \oplus X}$. Then let $y = \sum_{i=1}^n a_i \otimes_S y_i \in J \cap S$ for elements $a_i \in A$. Hence

$$0 = d(y) = \sum_{i=1}^n d(a_i) \otimes y_i$$

and since the set Y is S-free, we infer $d(a_i) \in \ker(d)$. Therefore we get that $d(a_i) = 0$ for every $i \in \{1, \ldots, n\}$ and hence $a_i \in S$. This shows that $y \in I$. ∎

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