DIFFERENTIAL GRADED BRAUER GROUPS OVER DG-RINGS

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Abstract. We define a Brauer group for differential graded algebras over differential graded graded-commutative or commutative base rings. Based on previous work we give an explicit classification of dg-fields, and compute the so-defined Brauer group in each case explicitly.

Introduction

Differential graded algebras (dg-algebras for short) were defined by Cartan in 1954 [9], and proved to be a most essential tool in homological algebra, algebraic topology, algebraic geometry, differential geometry and many other subjects. Most astonishing, a ring theory point of view was not taken in their studies, until very recently. Aldrich and Garcia-Rozas characterized acyclic dg-algebras completely [2]. Orlov [21] considered finite dimensional dg-algebras over a field from an algebraic geometry point of view. Goodbody [12], the second author [25, 27, 29, 30] and Orlov [22] studied the ring theory of dg-algebras with respect to several point of views and directions. A general observation is that the property needed is asked for the graded ring formed by the cycles of the dg-algebra. An important achievement for the present paper is that in [28] dg-division algebras were defined as those dg-algebras without non trivial left or right dg-ideal, and that it could be shown that this is equivalent to the property that the cycles of the dg-algebras form a graded-division algebra (cf [24], see also [19, 20]). Dg-division algebras are either acyclic or the differential is 0 and then the algebra is a graded-division ring.

In [26] the second author defined a dg-Brauer group for ordinary fields whose elements are equivalence classes of central simple algebras, which happen to be a differential graded algebra. The equivalence relations takes into account the dg-structure, and the so-defined dg-Brauer group is isomorphic to the Brauer group of the base field. Brauer groups of rings proved to be a powerful invariant and are subject to intensive study (cf Grothendieck [13, 14, 15], Caenepeel [7], Caenepeel and van Oystaeyen [8], Colliot-Thélènes [10]). For the most classical case we refer to [23].

In the present paper we answer the question how to define and study a Brauer group of a commutative or graded-commutative dg-algebra. We need to answer first the question what should be a dgAzumaya algebra. We propose two different dg-Brauer groups, denoted $\mathrm{dgBr}^I(K, d_K)$ and $\mathrm{dgBr}^{II}(K, d_K)$ below. They are related to each other, and in order to prove our main result, we need both concepts.

More precisely, a dg-algebra (A, d_A) over a (graded)-commutative dg-base ring (K, d_K) is dg-Azumaya of the first kind if $\ker(d_A)$ is graded-Azumaya over $\ker(d_K)$ (cf [1, 8]). Equivalence classes are those given by the cycles being equivalent in the graded setting. The product of two such algebras is the equivalence class of the tensor product of the cycles over $\ker(d_K)$ and then tensored over $\ker(d_K)$ with K, giving by this the differential obtained from K. This turns out that his defines a well-defined dgBrauer group (cf Proposition 3.5), called $\mathrm{dgBr}^I(K, d_K)$. It is not too hard to show that this group is isomorphic to the graded Brauer group of $\ker(d_K)$ (cf Lemma 4.4 and the Appendix).

Then, a second, and in its structure richer, dg-Brauer group is obtained by the following construction. A dg-algebra (A, d_A) is called dg-Azumaya of the second kind if A is graded-Azumaya over K. We prove in Proposition 3.2 that this gives a well-defined dgBrauer group denoted $\mathrm{dgBr}^{II}(K, d_K)$.

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We always have a homomorphism

$$\mathrm{dgBr}^{II}(K,d_K)\longrightarrow \mathrm{dgBr}^{I}(K,d_K)$$

which proves to be an isomorphism in case of acyclic (K,d_K) (cf Theorem 4.8), and proves to be a split epimorphism in case of $d_K=0$ (cf Theorem 4.2). If (K,d_K) is more general, the group $\mathrm{dgBr}^{II}(K,d_K)$ remains quite mysterious and is related to bad gradings on matrix algebras.

We are able to compute the dgBrauer groups of both types in case of dg-fields (K,d_K) in each case completely (cf Section 5.1).

Our paper is organized as follows. In Section 1 we first recall some basic facts about dg-algebras and dg-modules, also in order to fix our conventions and notations. Then, we add some concepts for the constructions over (graded) commutative dg-algebras. Further, we recall our results on dg-division algebras. As a complement to our results in [28] as our first main result Theorem 2.1, we classify completely in Section 2 commutative and graded-commutative dg-division algebras. In Section 3 we define our two different concepts of dgBrauer groups, and show that these provide groups in each case. In Section 4 we show in Theorem 4.2 that the dgBrauer group of the second kind maps always to the dgBrauer group of the first kind and this map is split epic if the differential of the base ring is 0. This is our second main result. Further, for acyclic base ring we prove that there is an isomorphism of the two concepts, which presents our second main result Theorem 4.8. Finally, we provide Examples in Section 5. In particular, we compute the dg-Brauer groups of each type explicitly and completely for each of the cases of base dg-fields from Theorem 2.1. In Appendix A we show that for a graded algebra over a graded commutative ring the equivalence classes of graded Azumaya algebras form a graded Brauer group. This may be well-known to the experts, but we could not find an explicit treatment in the literature.

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1. Notations, conventions, and generalities on differential graded algebras

1.1. **Elementary definitions of dg-algebras and modules.** Let K be a commutative base ring. A differential graded algebra (dg-algebra for short) over K is a $\mathbb{Z}$-graded K-algebra A which is equipped with a K-linear map $d:A\longrightarrow A$ which is homogeneous of degree 1, with $d^2=0$ and such that for all homogeneous $a,b\in A$ we get

$$d(a\cdot b)=d(a)\cdot b+(-1)^{|a|}a\cdot b,$$

where we denote by $|b|$ the degree of $b\in A$. The opposite algebra A^{op} is A with the same additive structure as A and multiplication denote by $\cdot_{op}$ and defined as

$$b\cdot_{op}a:=(-1)^{|a||b|}a\cdot b$$

for all homogeneous $a,b\in A$. Then (A^{op},d) is again a dg-algebra, with the same differential d.

A left dg-module (M,δ) over a dg-algebra (A,d) over K is a $\mathbb{Z}$-graded left A-module with a K-linear endomorphism $\delta:M\longrightarrow M$ of degree 1 with $\delta^2=0$ and

$$\delta(a\cdot m)=d(a)\cdot m+(-1)^{|a|}n\cdot\delta(m)$$

for all homogeneous $a\in A$ and $m\in M$. A right dg-module (M,δ) over (A,d) is a left dg-module over (A^{op},d). A homomorphism of dg-algebras $\varphi:(A,d_A)\longrightarrow(B,d_B)$ is a graded degree 0 homomorphism of algebras $\varphi:A\longrightarrow B$ such that $\varphi\circ d_A=d_B\circ\varphi$.

If (M,δ_M) is a left dg-module over (A,d_A), and (N,δ_N) is a right dg-module over (A,d_A), then $N\otimes_A M$ allows a differential $\delta_{N\otimes_A M}$ given by

$$\delta_{N\otimes_A M}(n\otimes m):=\delta_N(n)\otimes m+(-1)^{|n|}n\otimes\delta_M(m)$$

for all homogeneous $n\in N$ and $m\in M$.

If (M,δ_M) and (N,δ_N) are left dg-modules over (A,d_A). Then

$$\mathrm{Hom}^{\bullet}_A(M,N):=\bigoplus_{k\in\mathbb{Z}}\mathrm{Hom}^k_A(M,N)$$

and

$$\mathrm{Hom}^k_A(M,N) := \left\{ f : M \longrightarrow N \mid \begin{array}{c} \forall_{\ell\in\mathbb{Z}} f(M_\ell) \subseteq N_{\ell+k} \text{ and} \\ \forall_{a\in A, m\in M \text{ homogeneous}},\ f(am) = (-1)^{|a|k} a f(m) \ \text{ and} \\ f \text{ additive} \end{array} \right\}$$

$\mathrm{Hom}^\bullet_A(M,N)$ is equipped with a differential

$$d_{\mathrm{Hom}}(f) := \delta_N \circ f - (-1)^{|f|} f \circ \delta_M.$$

Denote $\mathrm{End}^\bullet_A(M,\delta_M) := \mathrm{Hom}^\bullet_A(M,M)$, and likewise for N, then $(\mathrm{End}^\bullet_A(M,\delta_M), d_{\mathrm{Hom}})$ is a dg-algebra and likewise for (N,δ_N). Further, $(\mathrm{Hom}^\bullet_A(M,N), d_{\mathrm{Hom}})$ is a differential graded bimodule over $(\mathrm{End}^\bullet_A(N,\delta_N), d_{\mathrm{Hom}}) - (\mathrm{End}^\bullet_A(M,\delta_M), d_{\mathrm{Hom}})$.

A homomorphism of dg-modules $(M,\delta_M) \longrightarrow (N,\delta_N)$ over the same dg-algebra (A,d_A) is an element in the degree 0 cycles of $\mathrm{Hom}^\bullet_A((M,\delta_M),(N,\delta_N)), d_{\mathrm{Hom}})$.

Let (A,d_A) be a dg-algebra. Then the left regular module is a left dg-module over (A,d_A), and likewise the right regular module is a right dg-module over (A,d_A). Also, (A,d_A) is a dg-bimodule over $(A,d_A) - (A,d_A)$. A dg left ideal of (A,d_A) is a left dg-submodule of the left regular module, and likewise for right and twosided dg-ideals.

For a dg-algebra (A,d) we denote by $dg - mod(A,d)$ the category of finitely generated left dg-modules over A, and by $gr - mod(A)$ the category of finitely generated graded left modules over A.

1.2. **dg-algebras over (graded) commutative dg-algebras.** If (R,d_R) is a graded-commutative dg-algebra over K, then we shall define the structure of a dg-algebra (A,d_A) over the dg-algebra (R,d_R).

Recall that for a commutative ring K a K-algebra A is a ring A together with a ring homomorphism

$$\lambda : K \longrightarrow Z(A).$$

If K is graded commutative, this definition does not make sense anymore. However we can obtain a meaningful correction easily. Let K be a graded-commutative ring. Then a graded K-algebra A is a graded ring A together with a graded ring homomorphism

$$\lambda : K \longrightarrow Z_{gr}(A).$$

For dg-algebras, the second condition is much more natural for the following reason. If (K,d_K) is a dg-algebra itself, then we have

$$d(xy) = d(x)y + (-1)^{|x|} x d(y).$$

If K is commutative, then

$$(-1)^{|y|} d(x)y + xd(y) = d(y)x + (-1)^{|y|} y d(x) = d(yx) = d(xy) = d(x)y + (-1)^{|x|} x d(y).$$

Hence

$$d(x)y(1-(-1)^{|y|}) = xd(y)(1-(-1)^{|x|})$$

for all homogeneous x and y. If K is graded commutative, then

$$d(xy) = d(x)y + (-1)^{|x|} x d(y) = (-1)^{|x||y|} d(y)x + (-1)^{|y|(|x|+1)} y d(x) = (-1)^{|x||y|} d(yx).$$

We see that dg-algebras over a commutative dg-ring impose conditions on the base ring, whereas dg-algebras A over graded-commutative dg-rings do not have to satisfy any additional condition.

If (R,d_R) is graded-commutative, and (A,δ_A) as well as (B,δ_B) are dg-algebras over (R,d_R), then $(A\otimes_R B, d_{A\otimes_R B})$ is a dg-algebra again by

$$(a_1 \otimes b_1)\cdot(a_2 \otimes b_2) = (-1)^{|b_1||a_2|}(a_1a_2 \otimes b_1b_2)$$

for all homogeneous $a_1, a_2 \in A$ and $b_1, b_2 \in B$. This is well-defined since (A,d_A) is a dg-algebra over (R,d_R) and hence satisfies the Leibniz rule

$$d_A(a\cdot r) = d_A(a)\cdot r + (-1)^{|a|} a \cdot d_R(r)$$

for all homogeneous $a \in A$ and $r \in R$, and likewise for (B,d_B). In particular, we cannot forget the differential on R, and in general (R,d_R) is not a dg-algebra over $(R,0)$.

Since we could not find a complete enough treatment of the theory of graded algebras over a graded-commutative algebra (cf [4, Remark page 1535 - fourth paragraph]), we reprove in Appendix A the necessary ring theoretical background in particular the aspects necessary to prove that equivalence classes of graded Azumaya algebras form a group, the graded Brauer group.

1.3. **Various versions of Brauer groups.** The most classical Brauer group of a field is defined as follows (cf e.g. [23]). A finite dimensional K-algebra is central simple if it is a simple algebra with centre K. A consequence of the property of being central simple is that $A\otimes_K A^{op}$ is Morita equivalent to K. The Morita equivalence classes of central simple K-algebras then form an abelian group, the Brauer group $\mathrm{Br}(K)$, with group law given by $-\otimes_K -$.

Following Azumaya [3], for general commutative rings R one defines an R-algebra to be an Azumaya algebra if A is a finitely generated projective generator in the category of R-modules, and

$$\begin{array}{ccc} A\otimes_R A^{op} & \longrightarrow & \mathrm{End}_R(A) \\ a\otimes b & \mapsto & (c\mapsto acb) \end{array}$$

is an isomorphism. Again, the Morita equivalence classes of Azumaya R-algebras form a group under tensor product over R, the Azumaya Brauer group over R. Azumaya algebras are closely related to separable extension (cf [1]).

For a graded commutative $\mathbb{Z}$-graded ring R one considers $\mathbb{Z}$-graded R-algebras A. We call these rings $\mathbb{Z}$-graded-Azumaya if A is a finitely generated projective generator in the category of graded R-modules, and

$$\begin{array}{ccc} A\otimes_R A^{op} & \longrightarrow & \mathrm{End}_R(A) \\ a\otimes b & \mapsto & (c\mapsto (-1)^{|b||c|}acb) \end{array}$$

is a graded isomorphism (recall the Koszul sign rule). Again, two graded-Azumaya algebras A and B are equivalent if there is a faithfully projective graded R-module C_A and faithfully projective graded R-module C_B such that

$$A\otimes_R \mathrm{End}_{R,graded}(C_A)\simeq B\otimes_R \mathrm{End}_{R,graded}(C_B).$$

Again, the equivalence classes form a group, the graded Brauer group $\mathrm{grBr}(R)$ with group law the tensor product over R. Since we do need details on these groups, and since we did not find a treatment of the graded Brauer group for graded algebras over a graded commutative base ring, we develop the theory in more detail in the Appendix A.

In [26] the second author defined a dg-Brauer group over a field K as equivalence classes of central simple K-algebras which happen to be differential graded algebras. Equivalence was given analogous to the case of graded Brauer groups. Our $\mathrm{dgBr}^{II}(K,d_K)$ generalizes this case.

We finally mention, though we do not need this approach, that a very general definition of a Brauer group of a properly enriched category is given by Borceux and Vitale in [6].

1.4. **Dg-division algebras.** A dg-module (M,δ) is dg-simple if it does not contain a dg-submodule other than 0 and itself.

Definition 1.1. A dg-algebra (A,d) is

- *dg-simple* if it does not contain a twosided dg-ideal other than 0 or A.
- a *dg-division algebra* if it does neither contain a left dg-ideal and nor a right dg-ideal other than 0 or A.
- a *dg-field* if it is commutative or graded commutative dg-division algebra.

A dg-module is dg-Noetherian (resp. dg-artinian) if it satisfies the ascending (resp. descending) chain condition on dg-submodules.

We shall use frequently the following basic result.

Theorem 1.2. [28, 29] *Let (A,d) be a dg-algebra. Then*

- *(A,d) is a dg-division algebra if and only if $\ker(d)$ is a $\mathbb{Z}$-gr-division algebra (cf* [19]*).*
- *If (A,d) is a dg-division algebra,*
  - *then,*
    - *either $d=0$ and $\ker(d)$ is a graded-division algebra,*

  * *or* $H(A,d)=0$ *and there is a skew field* R_0 *such that* $\ker(d)\simeq R_0[X,X^{-1};\phi]$ *for an automorphism* ϕ *of* R_0 *and* $Xr=\phi(r)X$ *for any* $r\in R_0$.
- *If* $H(A,d)=0$*, then there is a homogeneous element* $y\in A$ *with* $d(y)=1$ *and* $y^2\in\ker(d)$*, and there is a map* $D:\ker(d)\longrightarrow\ker(d)$ *of degree* 1 *defined by*
$$D(a)=-(-1)^{|a|}d(yay)=ya-(-1)^{|a|}ay$$
for any homogeneous $a\in\ker(d)$*, such that* A *is isomorphic with the quotient of the twisted polynomial ring*
$$A\simeq\ker(d)[T;D]/(T^2-y^2).$$
Moreover, the algebra structure on the twisted group ring is given by $D(a)=Ta-(-1)^{|a|}aT$ *for any homogeneous* $a\in\ker(d)$*. Furthermore,* $A=\ker(d)\oplus y\ker(d)$*, and the isomorphism is*
$$\begin{array}{rcl}\Phi:\ker(d)[T;D] & \longrightarrow & A\\ b+Ta & \mapsto & b+ya\end{array}$$
for any homogeneous $a,b\in\ker(d)$*. Further, for any homogeneous* $a,b\in\ker(d)$ *we get* $d(b+ya)=a$.

2. Classsification of commutative and graded-commutative dg-fields

As a complement to Theorem 1.2, as our first main result we give an explicit classification of dg-fields.

Theorem 2.1. *Let* K *be a commutative ring and let* $(A,d))$ *be a dg-field (commutative or graded commutative). Then precisely one of the situations occur, and all these provide dg-fields.*

1. *A is concentrated in degree* 0 *and is a field.*
2. $d=0$ *and* $A=L[T,T^{-1}]$ *is a graded-field, where* T *is in non zero degree and* L *is a field (if* A *is graded-commutative then* $|T|\in 2\mathbb{Z}$*).*
3. $\ker(d)_0=:L$ *is a field,* $A=L[T,T^{-1}]$ *and* $d(T^{2n+1})=T^{2n}$*,* $d(T^{2n})=0$ *for all* n.
4. $\ker(d)_0=:L$ *is a field,* $A=\ker(d)\otimes_K K[Y]/Y^2$*,* $d(Y)=1$*, and*
   (a) *either* $\ker(d)=L$
   (b) *or* $\ker(d)=L[T,T^{-1}]$ *for some* T *in non zero even degree.*
5. $\ker(d)_0=:L$ *is a field,* $A=L[T,T^{-1},U]/(T^2-U^2)$ *is commutative,* $d(U)=1$*,* $d(T^n)=0$ *for all* n*, and*
   (a) *either* $U^2\neq 0$ *and* T *is of degree* -1*,*
   (b) *or else* T *is in some odd degree and* $U^2=0$.

Proof. We know that $\ker(d)$ is a graded-field, and hence either $\ker(d)$ is concentrated in degree 0, or else $\ker(d)=L[T,T^{-1}]$ for some field L, and some variable T in non zero degree. The case $d=0$ then provides the cases (1) and (2) in the statement.

In the sequel we may suppose that $d\neq 0$, and hence (A,d) is acyclic, which implies directly that $A_{-1}\neq 0$.

Suppose first that $\ker(d)$ *is concentrated in even degrees only.* We have two cases.
(i) In the first case $\ker(d)=\ker(d)_0$ is a field,
(ii) and in the second case $\ker(d)=\ker(d)_0[T,T^{-1}]$ for some T of degree $n\in 2\mathbb{Z}$.
Consider an element $Y'\in A$ of degree -1. It cannot be in $\ker(d)$, since $\ker(d)$ is concentrated in even degrees only. Whence $d(Y')=u\in A_0\smallsetminus\{0\}$, and actually $u\in\ker(d)_0$. Hence, u is invertible, and for $Y:=u^{-1}Y'$ we get $d(Y)=1$. If there is another $\tilde{Y}$ in degree -1, with Y, then $d(\widetilde{Y})=v\neq 0$ and $vY-\widetilde{Y}\in\ker(d)$. Since $\ker(d)$ does not allow any element of odd degree, $A_{-1}=\ker(d)_0\cdot Y$.
Suppose $Y^2=0$. Now, in the first case (i) we get $A=\ker(d)_0[Y]/Y^2=\ker(d)[Y]/Y^2$ with $d(Y)=1$.

In the second case (ii) we have that T is invertible. Let $Z\in A_d\smallsetminus\ker(d)$ be an element of degree m. Then $d(Z)\in\ker(d)$, and hence $d(Z)=uT^n$ for some n and some $0\neq u\in\ker(d)_0$. Then,

$T^{-n}Z \in A_{-1} = \ker(d)_0 \cdot Y$ and $m = n-1$. Therefore,

$$A = \ker(d)_0[Y]/Y^2 \otimes_{\ker(d)_0} \ker(d) = \ker(d)[Y]/Y^2$$

in this case as well, and hence in both cases. This gives the item (4) in the statement of the theorem.

Now suppose that $Y^2 \neq 0$. Note that $Y^2 \in \ker(d)$. Then, by Aldrich-Garcia Rozas' theorem we get that

$$A = \ker(d)[T]/(T^2 - Y^2),$$

taking into account that A was supposed to be commutative or graded commutative, and hence the derivation D in the theorem is 0. Since $Y^2 \neq 0$, and therefore Y^2 is invertible in $\ker(d)$, we observe that $L[Y^2, Y^{-2}] = \ker(d)$, and we may replace T^2 by Y^2. This then yields the situation (3) of the statement of the theorem.

Suppose now that $\ker(d)$ *has a non zero component in odd degrees.* Since $\ker(d)$ is a graded-field, and for any $Z \in \ker(d)$ of odd degrees, we would get that $Z^2 = 0$ in case (A,d) is graded-commutative, Z cannot be invertible in this case. Hence, in this case we have that (A,d) needs to be commutative actually. Still $\ker(d)$ is a graded-field, and hence $\ker(d) = \ker(d)_0[U, U^{-1}]$ for some U of degree $2m+1$. Since the algebra was assumed to be commutative, the derivation D in Aldrich-Garcia Rozas' theorem is 0. Hence, by Aldrich-Garcia Rozas' theorem there is an element $Y \in A_{-1}$ with $d(Y) = 1$, satisfying $Y^2 \in \ker(d)$, and

$$A = \ker(d)[T]/(T^2 - Y^2)$$

for some $T \in A_{-1}$. Since $T^2 \in \ker(d)$, either $T^2 = 0$, or else T^2 is in degree -2. If $T^2 \neq 0$, we have that $\ker(d)_{-2} \neq 0$, and therefore U is of degree -1. This then implies then $\ker(d)_{-1} = \ker(d)_0 \cdot U$ and hence this theorem then gives the situation (5) of the statement of the theorem. The case $T^2 = 0$ is one case of the situation (5).

Altogether this then proves the theorem. ■

Remark 2.2. For a dg-field (K, d_K) we have that $\ker(d_K)$ is a graded-field (cf [28] and [29]). Any graded-module over a graded-field allows a graded-basis (cf [24, Lemma 1.7]), and hence is graded-flat.

Remark 2.3. Note that the classification of Proposition 2.1 shows that any graded-commutative dg-division algebra is actually commutative.

3. Defining dg-Brauer groups

3.1. **Definition of dg-Azumaya algebras.** Let K be a commutative ring. Recall that an Azumaya algebra over K is a K-algebra A which is faithfully projective over $K = Z(A)$, and such that the map

$$\begin{array}{ccc} A \otimes_K A^{op} & \longrightarrow & \mathrm{End}_K(A) \\ a \otimes b & \mapsto & (x \mapsto axb) \end{array}$$

is an isomorphism.

Similarly a graded-Azuyama algebra is a graded algebra A which is faithfully projective over $K = Z_{gr}(A)$ and such that the map

$$\begin{array}{ccc} A \otimes_K A^{op} & \longrightarrow & \mathrm{End}_{K,graded}(A) \\ a \otimes b & \mapsto & (x \mapsto (-1)^{|b||x|} axb) \end{array}$$

is an isomorphism. Note that we use Koszul signs in order to be able to cope with the various signs appearing in the construction of dg-algebras. By [8, Proposition III.4.1] we see that a graded algebra A over a commutative R is graded-Azumaya if and only if A is Azumaya.

Definition 3.1.
- Let (K, d_K) be a commutative or graded commutative dg-ring and let (A,d) be a dg-algebra. We say that (A,d) is *dg-Azumaya of the first kind* if $\ker(d)$ is graded-Azumaya algebra over $\ker(d_K)$.

  We say that two dg-Azumaya algebras (A, d_A) and (B, d_B) of the first kind are equivalent if $\ker(d_A)$ and $\ker(d_B)$ represent the same class in the graded-Brauer group $\mathrm{grBr}(\ker(d_K))$.

- Let (K, d_K) be a graded commutative dg-ring and let (A, d) be a dg-algebra. We say that (A, d) is *dg-Azumaya of the second kind* if A is graded-Azumaya algebra over K.

  We say that two dg-Azumaya algebras (A, d_A) and (B, d_B) of the second kind are equivalent if there are dg-modules (C_A, ∂_A) and (C_B, ∂_B), which are finitely generated faithful projective as graded K-modules, and such that
$$(A, d_A) \otimes_K (\mathrm{End}^\bullet_K(C_A), d_{\mathrm{Hom}}) \simeq (B, d_B) \otimes_K (\mathrm{End}^\bullet_K(C_B), d_{\mathrm{Hom}})$$
  as dg-algebras.

3.2. **The group structure of dg-Brauer groups of the first and second kind.**

Proposition 3.2. *Let (K, d_K) be a graded commutative dg-ring. Then the set of equivalence classes of dg-Azumaya algebras of the second kind over (K, d_K) form a group $\mathrm{dgBr}^{II}(K, d)$ with group law being given by*
$$[(A, d_A)] \cdot [(B, d_B)] := [(A \otimes_K B, d_{A \otimes_K B})].$$

Proof. The group law is clearly well-defined. We have that the map
$$\begin{array}{ccc} A \otimes_K A^{op} & \xrightarrow{\mu} & \mathrm{End}_{K,graded}(A) \\ (a \otimes b) & \mapsto & (x \mapsto (-1)^{|b||x|} axb) \end{array}$$
is an isomorphism. We need to check that the isomorphism maps the differential on the tensor product to d_{Hom} on the complex (A, d_A).
$$\begin{array}{rcl} d_{\mathrm{Hom}}(\mu(a \otimes b))(x) & = & (d_A \circ \mu(a \otimes b) - (-1)^{|a|+|b|} \mu(a \otimes b) \circ d_A)(x) \\ & = & d_A((-1)^{|b||x|} axb) - (-1)^{|a|+|b|+(|b|(|x|+1))} a d_A(x) b \\ & = & (-1)^{|b||x|} \left(d_A(axb) - (-1)^{|a|} a d_A(x) b \right) \\ & = & (-1)^{|b||x|} \left(d_A(a) xb + (-1)^{|a|} a d_A(xb) - (-1)^{|a|} a d_A(x) b \right) \\ & = & (-1)^{|b||x|} \left(d_A(a) xb + (-1)^{|a|} a d_A(x) b + (-1)^{|a|+|x|} a x d_A(b) - (-1)^{|a|} a d_A(x) b \right) \\ & = & (-1)^{|b||x|} \left(d_A(a) xb + (-1)^{|a|+|x|} a x d_A(b) \right) \\ & = & (-1)^{|b||x|} d_A(a) xb + (-1)^{|a|+(|b|+1)|x|} a x d_A(b) \\ & = & \mu(d_A(a) \otimes b + (-1)^{|a|} a \otimes d_A(b))(x) \\ & = & \mu((d_A \otimes 1 + 1 \otimes d_A)(a \otimes b))(x) \end{array}$$
Hence,
$$[(A, d_A)] \cdot [(A^{op}, d_A)] = [(\mathrm{End}^\bullet_K(A), d_{\mathrm{Hom}})] = [(K, d_K)]$$
by the definition of a graded-Azumaya algebra and the equivalence relation. Associativity of the tensor product is clear. This shows the lemma. ∎

Remark 3.3. Note that in [26] we gave precisely the definition of a dg-Brauer group in case of a dg-field K with differential 0 and concentrated in degree 0.

Lemma 3.4. *Let (K, d_K) be a commutative or graded-commutative dg-algebra. Let (A, d_A) and (B, d_B) be dg-Azumaya algebras of the first kind over (K, d_K). Then*
$$(\ker(d_A) \otimes_{\ker(d_K)} \otimes \ker(d_B)) \otimes_{\ker(d_K)} K$$
is a dg-Azumaya algebra of the first kind.

Proof. By [8, III.4.5] we see that $(\ker(d_A) \otimes_{\ker(d_K)} \otimes \ker(d_B))$ is graded-Azumaya $\ker(d_K)$-algebra. By definition, a graded-Azumaya algebra over $\ker(d_K)$ is faithfully projective as a graded $\ker(d_K)$-module. Hence, $(\ker(d_A) \otimes_{\ker(d_K)} \otimes \ker(d_B)) =: X$ is projective, and therefore flat as a $\ker(d_K)$-module. This implies that
$$0 \longrightarrow X \otimes_{\ker(d_K)} \ker(d_K) \longrightarrow X \otimes_{\ker(d_K)} K \longrightarrow X \otimes_{\ker(d_K)} d_K(K)[1] \longrightarrow 0$$
is exact.
$$\begin{array}{rcl} \ker(\mathrm{id}_{\ker(d_A) \otimes_{\ker(d_K)} \otimes \ker(d_B)} \otimes_{\ker(d_K)} d_K) & = & \ker(d_A) \otimes_{\ker(d_K)} \otimes \ker(d_B) \otimes_{\ker(d_K)} \ker(d_K) \\ & = & \ker(d_A) \otimes_{\ker(d_K)} \otimes \ker(d_B) \end{array}$$

using that X is graded-flat as a $\ker(d_K)$ module. This shows the lemma. ■

Proposition 3.5. *Let (K, d_K) be a commutative or graded-commutative dg-algebra. Then the equivalence classes of dg-Azumaya of the first kind form a group $\mathrm{dgBr}^I(K, d_K)$ by the group law*

$$[(A, d_A)] \cdot [(B, d_B)] := [(\ker(d_A) \otimes_{\ker(d_K)} \ker(d_B) \otimes_{\ker(d_K)} K, \mathrm{id} \otimes d_K)]$$

Proof. Lemma 3.4 shows that $\ker(d_A) \otimes_{\ker(d_K)} \ker(d_B) \otimes_{\ker(d_K)} K$ is a dg-Azumaya algebra of the first kind.

We shall first verify that the group law is well-defined. If $[(A, d_A)] = [(A', d_{A'})]$, then, by definition, $[\ker(d_A)]_{graded} = [\ker(d_{A'})]_{graded}$. But as is shown in [8, III.4.5], the group law in the graded Brauer group is given by $- \otimes_{\ker(d_K)} -$, and this is well-defined with respect to equivalence of graded algebras. Similarly, we argue for the second component.

Further, $[(A^{op}, d_A)]$ is the inverse to $[(A, d_A)]$, since the cycles of $[(A^{op}, d_A)]$ are precisely $\ker(d_A)^{op}$, and its graded equivalence class is the inverse of $[\ker(d_A)]_{graded}$. Associativity is clear, and we showed the result. ■

4. Computing dgBrauer groups of the first and second kind

4.1. dgBrauer group of the second kind and graded Brauer groups, the differential zero case.

Remark 4.1. Commutative or graded commutative graded-fields R behave very much like ordinary fields. All graded modules over such rings are free and allow bases (cf [24, Lemma 1.7]). Further, by [5] a Chevalley-Jacobson density theorem holds. These are the only ingredients in the proof of [11, Theorem 1.4]. Hence, the proof of this result can be copied literally to see that any matrix algebra over a commutative or graded commutative graded-fields R is graded-isomorphic to one with a good grading, in the sense that the elementary matrices are homogeneous. This in turn is the hypothesis for our proof from [26, Propositiion 3.3] that any differential on $\mathrm{End}_{graded,K}(P)$ is actually equivalent to d_{Hom} for some differential ∂ on P, where K is a field. This last statement is therefore true as well for commutative or graded commutative graded-fields R, with literally the same proof as for ungraded fields.

Theorem 4.2. *Let (K, d_K) be a graded commutative dg-ring. Then there is a group homomorphism*

$$\Psi : \mathrm{dgBr}^{II}(K, d_K) \longrightarrow \mathrm{grBr}(K).$$

If $d_K = 0$, then

$$\mathrm{dgBr}^{II}(K, 0) \simeq L \times \mathrm{grBr}(K)$$

for the subgroup L of $\mathrm{dgBr}^{II}(K, 0)$ formed by the class of dg-algebras obtained by putting a differential on the graded endomorphism algebra of some faithfully graded projective K-module.

If K is a graded-field, then $L = 0$.

Proof. Let $[(A, d_A)]$ be an equivalence class in $\mathrm{dgBr}^{II}(K, d_K)$. Then A is graded-Azumaya, by definition. Further, if $[(A, d_A)] = [(B, d_B)]$ in $\mathrm{dgBr}^{II}(K, d_K)$, then there are dg-modules (C_A, ∂_A) and (C_B, ∂_B) over (K, d_K), faithfully projective over K, such that

$$(A, d_A) \otimes_K \mathrm{End}^{\bullet}_K(C_A, \partial_A) \simeq (B, d_B) \otimes_K \mathrm{End}^{\bullet}_K(C_B, \partial_B)$$

as dg-algebras. Forgetting the differential this implies

$$A \otimes_K \mathrm{End}_{K,graded}(C_A) \simeq B \otimes_K \mathrm{End}_{K,graded}(C_B)$$

as graded algebras, and hence $[A] = [B]$ in $\mathrm{grBr}(K)$. Hence, forgetting the differential yields a group homomorphism

$$\Psi : \mathrm{dgBr}^{II}(K, d_K) \longrightarrow \mathrm{grBr}(K).$$

Suppose now $d_K = 0$. Then Ψ is a split epimorphism. Indeed, let A be a graded-Azumaya algebra over K. Then $(A, 0)$ is a dg-algebra, since K is equipped with differential 0. Moreover, $(A, 0)$ is dg-Azumaya of the second kind over K. Further, if $[A] = [B]$, then there are faithfully projective graded modules C_A and C_B over K, such that

$$A \otimes_K \mathrm{End}_{K,graded}(C_A) \simeq B \otimes_K \mathrm{End}_{K,graded}(C_B)$$

as graded algebras. Since the differential on K is 0, the graded modules $(C_A, 0)$ and $(C_B, 0)$ are dg-modules over $(K, 0)$. Hence, we get an isomorphism

$$(A,0) \otimes_K \mathrm{End}^\bullet_K(C_A, 0) \simeq (B,0) \otimes_K \mathrm{End}^\bullet_K(C_B, 0)$$

of dg-algebras. Hence,

$$\begin{array}{ccc} \mathrm{grBr}(K) & \xrightarrow{\Phi} & \mathrm{dgBr}^{II}(K,0) \\ [A] & \mapsto & [(A,0)] \end{array}$$

is a group homomorphism satisfying $\Psi \circ \Phi = id$. In particular, Ψ est surjective.

Let (A, d) be a dg-algebra such that A is graded-Azumaya over K in the kernel of Ψ. Hence, A as a graded algebra is equivalent to a graded endomorphism algebra of a faithful graded projective graded K-module P. The differential d is then a differential on the graded endomorphism algebra of a faithfully projective K-module. But this precisely means that (A, d) belongs to the subgroup defined by L. Hence $\mathrm{dgBr}^{II}(K, 0) \simeq L \rtimes \mathrm{grBr}(K)$ for $L = \ker(\Phi)$. However, $\mathrm{dgBr}^{II}(K, 0)$ is an abelian group (cf [26, Lemma 2.6]). Therefore the semidirect product is a direct product and

$$\mathrm{dgBr}^{II}(K,0) \simeq L \times \mathrm{grBr}(K)$$

for $L = \ker(\Phi)$.

Now, suppose that L is a graded-field. Then by [8, Proposition IV.1.3] we get that A is graded-simple central. By Remark 4.1 we obtain that (A, d) is equivalent to $(K, 0)$.

This shows the theorem. ■

Remark 4.3. It would be nice to get a necessary and sufficient criterion when $L = 0$.

4.2. **dgBrauer groups of the first kind and graded Brauer groups.** dgBrauer groups of the first kind are designed to be actually isomorphic to the graded Brauer group of the cycles, as is observed by the following easy lemma.

Lemma 4.4. *If (K, d_K) is a commutative or graded commutative dg-algebra, then* $\mathrm{dgBr}^I(K, d_K) \simeq \mathrm{grBr}(\ker(d))$.

Proof. We shall use Proposition 3.5.

Let (A, d_A) be a dg-Azumaya algebra of the first kind. Then by definition $\ker(d_A)$ is a graded-Azuyama algebra over $\ker(d_K)$. Further, by definition, equivalent dg-Azumaya algebras of the first kind yield equivalent graded algebras of cycles. Hence, taking cycles yields a well-defined group homomorphism

$$\mathrm{dgBr}^I(K, d_K) \longrightarrow \mathrm{grBr}(\ker(d)).$$

This map is injective since two dg-Azumaya algebras of the first kind with the same image in $\mathrm{grBr}(\ker(d_K))$ yield equivalent classes in $\mathrm{dgBr}^I(K, d_K)$, by definition.

Given a graded-Azumaya algebra C over $\ker(d_K)$, then $C \otimes_{\ker(d_K)} K$ is a dg-algebra with differential $\partial_D := \mathrm{id}_C \otimes d_D$. Since C is graded Azumaya over $\ker(d_K)$, it is finitely generated graded-projective over $\ker(d_K)$, and hence flat over $\ker(d_K)$. This implies that $\ker(\partial_C) = C$. Hence, the above group homomorphism is surjective as well. This proves the lemma. ■

4.3. **Linking dgBrauer groups of the first and the second kind; the acyclic case.** In this section we shall consider the case of (K, d_K) being acyclic. Then, by the main theorem in [2] we have that every dg-algebra over (K, d_K) is acyclic as well. We shall use the following result frequently.

Lemma 4.5. *Let (K, d_K) be an acyclic graded commutative dg-algebra, and let (A, d_A) be a dg-algebra over (K, d_K). Then there is an isomorphism of dg-algebras*

$$\ker(d_A) \otimes_{\ker(d_K)} (K, d_K) \xrightarrow{\alpha} (A, d_A)$$

given by $a \otimes x \mapsto ax$.

Proof. We get

$$\begin{array}{rcl}\alpha((a\otimes x)\cdot(b\otimes y)) &=& \alpha((-1)^{|x||b|}ab\otimes xy)\\ &=& (-1)^{|x||b|}abxy\\ &=& axby\\ &=& \alpha(a\otimes x)\cdot\alpha(b\otimes y)\end{array}$$

Hence, α is an algebra homomorphism. Further, denote by $\partial_{\ker(d_A)\otimes K}=\mathrm{id}\otimes d_K$ the differential on $\ker(d_A)\otimes_{\ker(d_K)}K$,

$$\begin{array}{rcl}\alpha(\partial_{\ker(d_A)\otimes K}(a\otimes x)) &=& \alpha((-1)^{|a|}a\otimes d_K(x))\\ &=& (-1)^{|a|}a\cdot d_K(x)\\ &=& d_A(a\cdot x)\\ &=& d_A(\alpha(a\otimes x))\end{array}$$

using that $a\in\ker(d_A)$.

Since (K,d_K) is acyclic, also (A,d_A) is acyclic, By [2] we get that

$$A=\ker(d_A)\oplus\ker(d_A)T_A=\ker(d_A)\oplus T_A\ker(d_A)$$

as left (resp. right) modules over $\ker(d_A)$, and also

$$K=\ker(d_K)\oplus T_K\ker(d_K)=\ker(d_K)\oplus\ker(d_K)T_K$$

for some $T_K\in K$ and $d_K(T_K)=1$. Observe that any homogeneous T_A with $d(T_A)=1_A$ does have this property. As (A,d_A) is a dg-module over (K,d_K), we obtain by definition a dg-ring homomorphism

$$(K,d_K)\xrightarrow{\nu}Z_{gr}(A,d_A)$$

since K is graded commutative. This implies

$$d_A(\nu(T_K))=\nu(d_K(T_K))=\nu(1)=1_A$$

and hence we may identify

$$\nu(T_K)=T_A\in Z_{gr}(A).$$

Hence, we pose $T:=T_A=\nu(T_K)$, the image of T_K in A. Therefore,

$$A\simeq\ker(d_A)\oplus\ker(d_A)T\simeq\ker(d_A)\otimes_{\ker(d_K)}K.$$

This shows that α is bijective and we proved the lemma. ∎

Proposition 4.6. *Let (K,d_K) be an acyclic and graded commutative dg-ring. Let (A,d_A) be a dg-Azumaya algebra of the second kind over (K,d_K). Then (A,d_A) is a dg-Azumaya algebra of the first kind.*

Proof. Suppose that the map

$$\begin{array}{rcl}A\otimes_K A^{op} &\xrightarrow{\phi}& \mathrm{End}_{K,graded}(A)\\ a\otimes b &\mapsto& (x\mapsto(-1)^{|b||x|}axb)\end{array}$$

is an isomorphism.

Now,

$$A\otimes_K A^{op}\simeq\mathrm{End}_{K,graded}(A)$$

induces a differential ∂_E on $\mathrm{End}_{K,graded}(A)$ by transport of structure, from the differential $d_A\otimes d_{A^{op}}$ on $A\otimes A^{op}$. Again, this way $(\mathrm{End}_{K,graded}(A),\partial_E)$ is a dg-module over (K,d_K), and is hence acyclic as well. By Lemma 4.5 and its proof,

$$\begin{array}{rcl}A\otimes_K A^{op} &\simeq& \left(\ker(d_A)\otimes_{\ker(d_K)}K\right)\otimes_K\left(K\otimes_{\ker(d_K)}\ker(d_A)\right)^{op}\\ &\simeq& \ker(d_A)\otimes_{\ker(d_K)}K\otimes_{\ker(d_K)}\ker(d_A)^{op}\\ &\simeq& \left(\ker(d_A)\otimes_{\ker(d_K)}\ker(d_A)^{op}\right)\oplus\left(\ker(d_A)\otimes_{\ker(d_K)}\ker(d_A)^{op}\cdot T\right)\\ &\xrightarrow{\chi}& \mathrm{End}_{\ker(d_K),graded}(\ker(d_A))\oplus\mathrm{End}_{\ker(d_K),graded}(\ker(d_A))T\end{array}$$

where χ is the homomorphism given by left and right multiplication. We know by hypothesis that χ is injective, and surjective to $\mathrm{End}_{K,graded}(A)$. However,

$$\begin{aligned}\mathrm{End}_{K,graded}(A) &= \mathrm{Hom}_{\ker(d_K)\otimes_{\ker(d_K)}K,graded}(\ker(d_A)\otimes_{\ker(d_K)}K, \ker(d_A)\otimes_{\ker(d_K)}K)\\ &= \mathrm{Hom}_{\ker(d_K),graded}(\ker(d_A), \ker(d_A)\otimes_{\ker(d_K)}K)\\ &= \mathrm{Hom}_{\ker(d_K),graded}(\ker(d_A), \ker(d_A)\oplus \mathrm{Hom}_{\ker(d_K),graded}(\ker(d_A),\ker(d_A))T\end{aligned}$$

and hence χ is an isomorphism. Now, the only non trivial action of the differential ∂_E is on T, and there we get $\partial_E(T)=1$. But then, taking cycles the isomorphism

$$\begin{aligned}&\left(\ker(d_A)\otimes_{\ker(d_K)}\ker(d_A)^{op}\right)\oplus\left(\ker(d_A)\otimes_{\ker(d_K)}\ker(d_A)^{op}\cdot T\right)\\ &\qquad\xrightarrow{\phi}\quad \mathrm{End}_{\ker(d_K),graded}(\ker(d_A))\oplus \mathrm{End}_{\ker(d_K),graded}(\ker(d_A))T\end{aligned}$$

yields an isomorphism of cycles, which gives

$$\ker(d_A)\otimes_{\ker(d_K)}\ker(d_A)^{op}\simeq \mathrm{End}_{\ker(d_K),graded}(\ker(d_A)).$$

This shows the proposition. ∎

By Aldrich and Garcia-Rozas' theorem we get for an acyclic dg-algebra (A,d_A) an equivalence of categories

$$F_A:(A,d_A)-dgmod\longrightarrow \ker(d_A)-grmod$$

given by taking cycles and a quasi-inverse G_A given by $A\otimes_{\ker(d_A)}-$.

Lemma 4.7. *Let (K,d_K) be an acyclic commutative or graded commutative dg-ring, and let (A,d_A) and (B,d_B) be dg-algebras over (K,d_K). Then*

$$F_{A\otimes_K B}(A\otimes_K B)\simeq F_A(A)\otimes_{\ker(d_K)}F_B(B).$$

Proof. By Aldrich and Garcia-Rozas' theorem we get that (A,d_A) and (B,d_B) are both acyclic, and

$$A\simeq \ker(d_A)\oplus\ker(d_A)T_A$$

as well as

$$B\simeq \ker(d_B)\oplus\ker(d_B)T_B$$

as modules over $\ker(d_A)$, resp. $\ker(d_B)$ for $d_A(T_A)=1_A$ and $d_B(T_B)=1_B$ for homogeneous elements T_A,T_B. Any such element has this property. Therefore we may take T_A the image of T_K in A, and T_B the image of T_K in B, and, slightly abusing the notation, abbreviate all these elements by T.

$$\begin{aligned}A\otimes_K B &\simeq (\ker(d_A)\oplus\ker(d_A)T)\otimes_K(\ker(d_B)\oplus\ker(d_B)T)\\ &\simeq (\ker(d_A)\otimes_{\ker(d_K)}K)\otimes_K(K\otimes_{\ker(d_K)}\ker(d_B))\\ &\simeq \ker(d_A)\otimes_{\ker(d_K)}K\otimes_{\ker(d_K)}\ker(d_B)\\ &\simeq \ker(d_A)\otimes_{\ker(d_K)}(\ker(d_K)\oplus\ker(d_K)T)\otimes_{\ker(d_K)}\ker(d_B)\\ &\simeq (\ker(d_A)\otimes_{\ker(d_K)}\ker(d_B))\oplus(\ker(d_A)\otimes_{\ker(d_K)}\ker(d_B)T)\end{aligned}$$

Now, as $A\otimes_K B$ is a dg-module over (K,d_K) again, $A\otimes_K B$ is acyclic, and the differential on the subalgebra $(\ker(d_A)\otimes_{\ker(d_K)}\ker(d_B))$ is 0, whereas $d_{A\otimes_K B}(T)=1$. This shows the lemma. ∎

Theorem 4.8. *Let (K,d_K) be an acyclic graded-commutative dg-ring. Then there an isomorphism*

$$\mathrm{dgBr}^{II}(K,d_K)\longrightarrow \mathrm{dgBr}^{I}(K,d_K)$$

given by the identity representatives of equivalence classes of objects and hence

$$\mathrm{dgBr}^{II}(K,d_K)\simeq \mathrm{grBr}(\ker(d_K)).$$

Proof. Let (A,d) be a dg-Azumaya algebra of the second kind, by Proposition 4.6 we get that (A,d) is a dg-Azumaya algebra of the first kind.

We need to see that the map

$$\begin{aligned}\mathrm{dgBr}^{II}(K,d_K) &\longrightarrow \mathrm{dgBr}^{I}(K,d_K)\\ [(A,d)] &\mapsto [(A,d)]\end{aligned}$$

is well defined and is a group homomorphism.

Let (P,∂_P) be a dg-(K,d_K)-module, which is faithfully projective as a graded K-module. Then any dg-endomorphism φ of (P,∂_P) induces a graded $\ker(d_K)$-linear endomorphism of $\ker(\partial_P)$. Indeed, by Aldrich and Garcia-Rozas theorem [2] there is an equivalence of categories

$$\begin{array}{ccc} gr-mod(\ker(d_K)) & \longrightarrow & dg-mod(K,d_K) \\ M & \mapsto & K\otimes_{\ker(d_K)}M \end{array}$$

with quasi-inverse

$$\begin{array}{ccc} dg-mod(K,d_K) & \longrightarrow & gr-mod(\ker(d_K)) \\ (N,\partial_N) & \mapsto & \ker(\partial_N). \end{array}$$

Hence, the above construction gives an isomorphism

$$\mathrm{End}^\bullet_{(K,d_K)}(P,\partial_P)\simeq \mathrm{End}_{graded,\ker(d_K)}(\ker(\partial_P))$$

Also, since P is faithfully projective as a K-module, $\ker(\partial_P)$ is faithfully projective as a $\ker(d_K)$-module.

Now, if $[(A,d_A)]=[(B,d_B)]$ in $\mathrm{dgBr}^{II}(K,d_K)$, then there are dg-modules (C_A,∂_A) and (C_B,∂_B), faithfully projective over K, such that

$$(A,d_A)\otimes_K \mathrm{End}^\bullet_K((C_A,\partial_A),d^A_{\mathrm{Hom}})\simeq (B,d_B)\otimes_K \mathrm{End}^\bullet_K((C_B,\partial_B),d^B_{\mathrm{Hom}})$$

as dg-algebras. Using now Lemma 4.7, we get that, taking cycles yields an isomorphism of graded algebras

$$\ker(d_A)\otimes\ker(d^A_{\mathrm{Hom}})\simeq\ker(d_B)\otimes\ker(d^B_{\mathrm{Hom}}).$$

Now, $\ker(d^A_{\mathrm{Hom}})$ is simply the graded endomorphisms of C_A as complexes, and likewise for C_B. As (C_A,∂_A) is a dg-module over (K,d_K) as well, and hence acyclic, we get that $\ker(d^A_{\mathrm{Hom}})$ coincides with the graded endomorphism ring of $\ker(\partial_A)$ over $\ker(d_K)$. Analogous statements hold for C_B. As (C_A,∂_A) is faithfully projective as a K-module, we get that $\ker(\partial_A)$ is faithfully projective as a graded $\ker(d_K)$-module. Analogous statements hold for $\ker(\partial_B)$.

This shows that

$$[\ker(d_A)]=[\ker(d_B)]$$

in $\mathrm{grBr}(\ker(d_K))$.

We need to show that the homomorphism has an inverse. In order to do so, let $[(A,d_A)]$ be an element in $\mathrm{dgBr}^I(K,d_K)$. Therefore the natural homomorphism

$$\ker(d_A)\otimes_{\ker(d_K)}\ker(d_A)^{op}\longrightarrow \mathrm{End}_{\ker(d_K),graded}(\ker(d_A))$$

is an isomorphism of graded algebras. We also have the natural homomorphism

$$A\otimes_K A^{op}\longrightarrow \mathrm{End}^\bullet_K(A)$$

which we need to show to be an isomorphism. However, as (K,d_K) is acyclic, by Aldrich and Garcia-Rozas' theorem, we get that also (A,d_A) is acyclic, and again by the argument above,

$$K=\ker(d_K)\oplus\ker(d_K)T \text{ and } A=\ker(d_A)\oplus\ker(d_A)T$$

with the same T in the (graded) centre of A. Hence,

$$\begin{array}{rcl} A\otimes_K A^{op} & = & (\ker(d_A)\otimes_{\ker(d_K)}K)\otimes_K(\ker(d_A)\otimes_{\ker(d_K)}K)^{op} \\ & = & \left(\ker(d_A)\otimes_{\ker(d_K)}\ker(d_A)^{op}\right)\oplus\left(\ker(d_A)\otimes_{\ker(d_K)}\ker(d_A)^{op}\right)T \\ & = & \mathrm{End}^\bullet_{\ker(d_K)}(\ker(d_A))\oplus\mathrm{End}^\bullet_{\ker(d_K)}(\ker(d_A))T \\ & = & \mathrm{End}^\bullet_K(A) \end{array}$$

using again that also $\mathrm{End}^\bullet_K(A)$ is a (K,d_K)-dg module and hence acyclic as well. The map given by

$$\mathrm{dgBr}^I(K,d_K)\ni[(A,d_A)]\mapsto[(A,d_A)]\in\mathrm{dgBr}^{II}(K,d_K)$$

is well-defined, by an argument similar as above. Indeed, if $[(A,d_A)]=[(B,d_B)]$ in $\mathrm{dgBr}^I((K,d_K))$, then there are faithfully projective graded $\ker(d_K)$-modules P and Q such that

$$\ker(d_A)\otimes_{\ker(d_K)}\mathrm{End}_{\ker(d_K),graded}(P)\simeq\ker(d_B)\otimes_{\ker(d_K)}\mathrm{End}_{\ker(d_K),graded}(Q).$$

Put $(\widehat{P}, \partial_P) := P \otimes_{\ker(d_K)} (K, d_K)$ and $(\widehat{Q}, \partial_Q) := Q \otimes_{\ker(d_K)} (K, d_K)$. Then

$$\begin{array}{rcl}(A, d_A) \otimes_K \mathrm{End}^\bullet_K(\widehat{P}, \partial_P) & = & \ker(d_A) \otimes_{\ker(d_K)} (K, d_K) \otimes_K \mathrm{End}^\bullet_K(P \otimes_{\ker(d_K)} (K, d_K)) \\ & = & \ker(d_A) \otimes_{\ker(d_K)} K \otimes_K (\mathrm{End}_{\ker(d_K), graded}(P) \otimes_{\ker(d_K)} K) \\ & = & \ker(d_A) \otimes_{\ker(d_K)} \mathrm{End}_{\ker(d_K), graded}(P) \otimes_{\ker(d_K)} K \\ & = & \ker(d_B) \otimes_{\ker(d_K)} \mathrm{End}_{\ker(d_K), graded}(Q) \otimes_{\ker(d_K)} K \\ & = & \ker(d_B) \otimes_{\ker(d_K)} K \otimes_K (\mathrm{End}_{\ker(d_K), graded}(Q) \otimes_{\ker(d_K)} K) \\ & = & \ker(d_B) \otimes_{\ker(d_K)} (K, d_K) \otimes_K \mathrm{End}^\bullet_K(Q \otimes_{\ker(d_K)} (K, d_K)) \\ & = & (B, d_B) \otimes_K \mathrm{End}^\bullet_K(\widehat{Q}, \partial_Q)\end{array}$$

Also the fact that this map is a group homomorphism follows by Lemma 4.7. The fact that this map is left and right inverse to the map

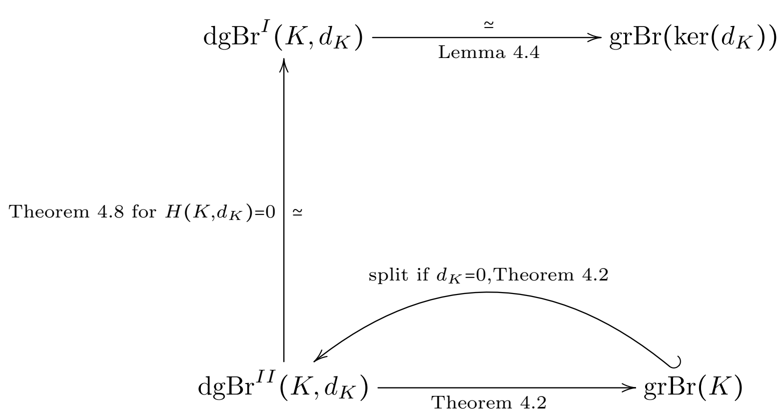
$$\begin{array}{ccc}\mathrm{dgBr}^{II}(K, d_K) & \longrightarrow & \mathrm{dgBr}^{I}(K, d_K) \\ [(A, d)] & \mapsto & [(A, d)]\end{array}$$

from above is clear by definition. Hence the Theorem follows. ∎

Remark 4.9. Lemma 4.4 together with Proposition 4.2 show a partial analogous statement for dg-base rings with differential 0. Here we treat the case of acyclic dg-base ring. In the case of differential 0 we only get a splitting, with kernel L being the subgroup generated by the dg-algebras A, where A is a graded algebra over $\ker(d_K)$ isomorphic to the graded endomorphisms of some faithfully projective $\ker(d_K)$-module, but with a non good grading.

We may hence consider L to be the group of bad gradings on matrix algebras (cf [11]), modulo the subgroup of good gradings in an appropriate sense.

We summarize our results up to now in the following scheme:



Remark 4.10. One may ask if there is a more natural definition of a dgBrauer group for a commutative or graded commutative dg-base ring (R, d_R). One could say that a dg-algebra (A, d_A) over (R, d_R) is dg-Azumaya of the third kind if

- (A, d_A) is a finitely generated projective object in the category of (R, d_R)-modules and
- the natural homomorphism $(A, d_A) \otimes_R (A, d_A)^{op} \longrightarrow (\mathrm{End}^\bullet_R(A, d_A), d_{\mathrm{Hom}})$ is an isomorphism of dg-algebras.

This definition parallels the classical definition of an Azumaya algebra naturally replacing each notion by the corresponding notion in the category of dg-modules, resp. dg-algebras. However, we will get back $\mathrm{dgBr}^{II}(R, d_R)$. Indeed, the proof of Lemma 3.2 shows that the graded isomorphism $A \otimes A^{op} \longrightarrow \mathrm{End}_{R, graded}(A)$ gives rise to a dg-isomorphism

$$(A \otimes A^{op}, d_{A \otimes A^{op}}) \longrightarrow (\mathrm{End}_{R, graded}(A, d_A), d_{\mathrm{Hom}}).$$

However, the hypothesis that (A, d_A) should be a projective object in the category of (R, d_R)-modules is more restrictive. Indeed, this implies that A is projective as a graded R-module, and in addition that (A, d_A) is acyclic (cf [2, Proposition 3.3; proof of Proposition 3.4]). Since the unit element in such a dg-Brauer group would has to be acyclic again, we would have to impose that (R, d_R) is acyclic, which then leads to the situation we already studied in Theorem 4.8.

Note however that the graded centre of an acyclic dg-algebra (A, δ_A) is not necessarily acyclic. An example is given by the algebra

$$\begin{pmatrix} K & K \\ K & K \end{pmatrix} =: A$$

with differential

$$\delta_A(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = \begin{pmatrix} c & d-a \\ 0 & c \end{pmatrix}$$

for a field K and the obvious grading. This algebra is acyclic and the graded centre is $K \cdot 1_A$. This latter algebra is not acyclic, and there is no acyclic dg-division algebra mapping onto $Z_{gr}(A)$.

5. Examples

5.1. **DgBrauer groups of dg-fields.** Recall the different classes of dg-fields from Theorem 2.1. We shall now apply our results to each of these cases and use the same numbering as in the theorem.

We first recall the following fact: By [8, IV.1.8 Theorem] we get

$$\mathrm{grBr}(L[T, T^{-1}]) = \mathrm{Br}(L) \oplus H^2_{gr}(Gal(L^{sep}/L), |T|\mathbb{Z}),$$

where, denoting by L^{sep} the separable closure of L,

$$H^2_{gr}(Gal(L^{sep}/L), |T|\mathbb{Z}) = \ker(H^2(Gal(L^{sep}/L), |T|\mathbb{Z}) \longrightarrow H^2(Gal(L^{sep}/L), \mathbb{Z})).$$

If the characteristic of L does not divide $|T|$, then

$$\mathrm{grBr}(L[T, T^{-1}]) = \mathrm{Br}(L) \oplus H^2_{gr}(\mathbb{Z}/|T|\mathbb{Z}, (L^{sep})^\times)$$

A more precise behaviour depends on the question if L is perfect or not [8, IV.1.10 Example].

(1) K an ordinary field with zero differential
Then Theorem 4.2 and Lemma 4.4 show that
$$\mathrm{dgBr}^{II}(K) = \mathrm{grBr}(K) = \mathrm{dgBr}^{I}(K).$$
(2) Laurent polynomials $K = L[T, T^{-1}]$ with zero differential
Again, if T is of even degree, Theorem 4.2 and Lemma 4.4 show that
$$\mathrm{dgBr}^{II}(L[T, T^{-1}]) = \mathrm{grBr}(L[T, T^{-1}]) = \mathrm{dgBr}^{I}(L[T, T^{-1}]).$$
If T is not of even degree, the second equality still holds.
(3) Laurent polynomials $K = L[T, T^{-1}]$ with $d(T) = 1$.
Then (K, d_K) is acyclic. Theorem 4.8 shows that
$$\mathrm{grBr}(L[T^2, T^{-2}]) = \mathrm{dgBr}^{I}(L[T, T^{-1}]).$$
If L is of characteristic 2, then commutativity and graded commutativity coincide, and $\mathrm{dgBr}^{I}(L[T, T^{-1}]) = \mathrm{dgBr}^{II}(L[T, T^{-1}])$ here as well.
(4) (a) acyclic algebras $K[Y]/Y^2$ for some field K and $d(Y) = 1$
As $K[Y]/Y^2$ is acyclic with $d(Y) = 1$, so are all dg-algebras over $K[Y]/Y^2$. Theorem 4.8 then shows that
$$\mathrm{dgBr}^{II}(K[Y]/Y^2, d(Y) = 1) = \mathrm{dgBr}^{I}(K[Y]/Y^2, d(Y) = 1) = \mathrm{grBr}(K).$$
(b) acyclic algebras $L[T, T^{-1}][Y]/Y^2$ for $d(Y) = 1$.
Again, as $L[T, T^{-1}][Y]/Y^2$ for $d(Y) = 1$ is acyclic, so are all dg-algebras over the algebra $L[T, T^{-1}][Y]/Y^2$ for $d(Y) = 1$. Theorem 4.8 then shows that
$$\begin{aligned} \mathrm{dgBr}^{II}((L[T, T^{-1}][Y]/Y^2), d(Y) = 1) &= \mathrm{dgBr}^{I}((L[T, T^{-1}][Y]/Y^2), d(Y) = 1) \\ &= \mathrm{grBr}(L[T, T^{-1}]). \end{aligned}$$

(5) (a) The acyclic case $L[T,T^{-1},U]/(T^2-U^2)$, $d(U)=1$, $d(T^n)=0$ for all n and $U^2\neq 0$
As this base ring is acyclic again, we may apply Theorem 4.8. Since $U^2\neq 0$, we get

$$\mathrm{dgBr}^I(L[T,T^{-1},U]/(T^2-U^2)) = \mathrm{grBr}(L[T,T^{-1}]).$$

Since in this case T is of degree -1, the characteristic of L does not divide $|T|$, and hence

$$H^2_{gr}(\mathbb{Z}/|T|\mathbb{Z},(L^{sep})^\times)=0.$$

Therefore

$$\mathrm{grBr}(L[T,T^{-1}])=\mathrm{Br}(L).$$

(b) The acyclic case $L[T,T^{-1},U]/(T^2-U^2)$, $d(U)=1$, $d(T^n)=0$ for all n and $U^2=0$
As this base ring is acyclic again, we may apply Theorem 4.8. Since $U^2=0$, we get again

$$\mathrm{dgBr}^I(L[T,T^{-1},U]/(T^2-U^2)) = \mathrm{grBr}(L[T,T^{-1}]).$$

In each of these subcases, if L is of characteristic 2, then we have equality with dgBr^{II} as above.

Appendix A. Defining the graded Brauer group of a graded commutative ring

Let R be a graded commutative graded ring and A a graded R-algebra. By definition, there is an injection from R into the graded center of A. In this case, if we ignore the grading, A is not necessarily an R-algebra. Hence, it is necessary to give a definition of a graded Azumaya algebra that does not rely on the ordinary notion of an Azumaya algebra. This is the purpose of this appendix.

Definition A.1. [29] A graded R-algebra A is called a *graded separable algebra* if it is projective as graded A-bimodule, or equivalently the multiplication map $A\otimes_R A^{op}\longrightarrow A$ is a split epimorphism of graded $A\otimes_R A^{op}$-modules.

Remark A.2. By [16, Proposition 1.2.15], a graded R-module P is a projective graded module if and only if P is a graded R-module and P is a projective R-module. Therefore, most properties are similar to those in the ungraded case.

For a graded A-bimodule M, denote by $M^A=\bigoplus_{n\in\mathbb{Z}}(M^A)_n$, where

$$(M^A)_n := \{m\in M_n | am=(-1)^{|a||m|}ma, \text{ for any homogenous element } a\in A\}.$$

It is easy to see that M^A isomorphic to the graded R-module $Hom_{A^e}(A,M)$ and $A^A=Z_{gr}(A)$. An R-derivation $\partial: A\to M$ is an R-linear graded map satisfying: $\partial(ab)=\partial(a)b+(-1)^{|a||\partial|}a\partial(b)$. Denote by $J(A)$ the kernel of the morphism $A^e\to A$ as A^e-module.

Proposition A.3. *For the graded R-algebra A. The following conditions are equivalent:*

1. *A is a graded separable R-algebra.*
2. *The functor $M\mapsto M^A$ is exact.*
3. *The R-derivation $\delta: A\longrightarrow A$ i.e. $\delta(a)=a\otimes 1-1\otimes a$ is an inner derivation.*
4. *All R-derivations are inner derivation.*

The proof similar to the ungraded case [17, III, Theorem 1.4]. Using the isomorphism $M^A\cong \mathrm{Hom}_{A^e}(A,M)$, $A\delta(A)=J(A)$ and the isomorphism $\mathrm{Hom}_{A^e}(J(A),M)\cong Der(A,M)$.

Remark A.4. Using this proposition, one can easily prove that a graded separable algebra still has the same properties as in [17, III, 1].

1. Let S_i for $i\in\{1,2\}$ be graded commutative graded R-algebras and A_i graded separable algebra over S_i for $i\in\{1,2\}$. Then $A_1\otimes_R A_2$ is graded separable algebra over $S_1\otimes_R S_2$, and $Z_{gr}(A_1\otimes_R A_2)=Z_{gr}(A_1)\otimes Z_{gr}(A_2)$. In particular, $S_i\otimes_R A_i$ is graded separable S_i-algebra, and $Z_{gr}(S_i\otimes_R A_i)=S_i\otimes Z_{gr}(A_i), i=1,2$.
2. The graded center $Z_{gr}(A)$ of a graded separable R-algebra A is a direct summand of A.
3. Let A and B be graded R-algebras. If B is faithfully projective R-module and $A\otimes_R B$ be graded separable R-algebra, then A is graded separable over R.

(4) Let $\phi : A \to B$ be an epimorphism of graded R-algebras. If A is a graded separable R-algebra, then so is B, and the graded center of B is the image under ϕ of the graded center of A.

Lemma A.5. *The graded center of a graded simple ring R is a graded field.*

Proof. Let $a \in Z_{gr}(R)$ be a nonzero homogenous element, then Ra is a graded two-sided ideal of R, thus $Ra = R$. There exists a homogenous element b such that $ba = 1$. Similarly, there exists a homogenous element c such that $ac = 1$. Clearly, $b = bac = c$. For any homogenous element $x \in R$,

$$bx = bxac = (-1)^{|x||a|}baxc = (-1)^{|x||b|}xb.$$

Hence, $Z_{gr}(R)$ is a graded commutative graded division ring. ∎

Proposition A.6. *Every graded separable algebra A over a graded field R is graded semisimple.*

Proof. Every graded A-module is projective graded R-module, since every graded module over a graded division ring is free [20, Proposition 4.6.1]. Moreover, for any graded A-modules M, N, we have an isomorphism $\mathrm{Hom}_A(M,N) \cong \mathrm{Hom}_R(M,N)^A$. It follows that every graded A-module is projective as graded A-module. In particular, every graded ideal of A is a direct summand of A. Therefore, it is easy to prove that A is graded semisimple. ∎

Corollary A.7. *Let A be a graded separable algebra over a graded field. If A is graded central, then A is graded simple algebra.*

Proof. In [20, Section 2.9], $A = A_1 \times \cdots \times A_n$, for some graded simple algebras A_i for $i \in \{1, \cdots, n\}$. Since

$$Z_{gr}(A_1 \times \cdots \times A_n) = Z_{gr}(A_1) \times \cdots \times Z_{gr}(A_n)$$

and from Lemma A.5 it follows that $n = 1$. Therefore A is graded simple. ∎

Recall the localization of graded rings [20, Chapter 8]. Suppose that R is graded commutative, it is clear that every multiplicative closed subset S of R consisting of homogenous elements satisfies the left and right Ore conditions. Therefore, $S^{-1}R$ is a graded ring.

Lemma A.8. *Let $\wp$ be a gr-prime of R and S a subset consisted of the homogenous elements of $R - \wp$. The graded ring $R_\wp := S^{-1}R$ is a gr-local ring.*

Proof. For any $a, b \in S$, we have $a, b \notin \wp$. By the definition of a gr-prime ideal, it follows that $ab \notin \wp$. Therefore, S is a multiplicative closed subset. Clearly, we can define a grading by

$$(\wp R_\wp)_n = \left\{ \frac{p}{s} | s \in S, p \in \wp, |p| - |s| = n \right\},$$

and $\wp R_\wp$ is the unique maximal graded ideal. Indeed, let I be a graded ideal containing $\wp R_\wp$. Suppose that there exists $\frac{r}{s} \in I \smallsetminus \wp R_\wp$. Then $r \in R - \wp$, and hence $\frac{s}{r} \in S^{-1}R$. Therefore, $1 = \frac{s}{r}\frac{r}{s} \in I$. On the other hand, suppose that I is a maximal graded ideal of $S^{-1}R$. If there exists $\frac{r}{s} \in I$ with $r \notin \wp$, then $1 = \frac{s}{r}\frac{r}{s} \in I$, which is absurd. Therefore, $I \subseteq \wp R_\wp$, and since I is maximal, it follows that $I = \wp R_\wp$. ∎

Lemma A.9. *Let A be a graded R-algebra and M a graded A-module, then the following conditions are equivalent:*

(1) *M is graded flat A-module;*
(2) *$M_m = A_m \otimes_A M = R_m \otimes_R A \otimes_A M$ is graded flat A_m-module for any maximal graded ideal m of R.*

Proof. (1) $\Rightarrow$ (2) is trivial.
(2) $\Rightarrow$ (1) Claim 1: $M = 0$ if and only if $M_m = 0$ for any maximal graded ideal m.
$'' \Rightarrow ''$ is trivial.
$'' \Leftarrow ''$ let $x \in M$ be a homogenous element, $ann(x) = \{r \in R | rx = 0\}$. It is easy to check that $ann(x)$ is a graded ideal of R. For any maximal graded ideal m we get by hypothesis $\frac{x}{1} = 0$ in M_m, i.e. $\exists s \in R \smallsetminus m$ such that $sx = 0$. Therefore, $s \in ann(x) \smallsetminus m$, it follows that $ann(x)$ not contained in any maximal graded ideal. Hence, $ann(x) = R$, which shows $M = 0$.

Claim 2: Suppose $L_1 \xrightarrow{f} L_2 \xrightarrow{g} L_3$ is a sequence in the category of graded R-module, then it is exact

if and only if $(L_1)_m \xrightarrow{f_m} (L_2)_m \xrightarrow{g_m} (L_3)_m$ is exact for any maximal graded ideal m.
$'' \Rightarrow''$ since R_m is a graded flat R-module.
$'' \Leftarrow''$ $g_m f_m = (1 \otimes g)(1 \otimes f) = 1 \otimes gf = 0$, which implies that $\mathrm{im}(f) \subseteq \ker(g)$ since m is was arbitrary.

$$(\ker(g)/\mathrm{im}(f))_m \cong (\ker(g))_m/(\mathrm{im}(f))_m \cong \ker(g)_m/\mathrm{im}(f)_m = 0,$$

for any maximal graded ideal m. Therefore, $\ker(g)/\mathrm{im}(f) = 0$, which implies $\mathrm{im}(f) = \ker(g)$.

We now finish the proof of (2) $\Rightarrow$ (1). Given a short exact sequence of graded right A-modules

$$0 \to L_1 \to L_2 \to L_3 \to 0,$$

since R_m is a graded flat R-module, we have the exact sequence

$$0 \to (L_1)_m \to (L_2)_m \to (L_3)_m \to 0.$$

By hypothesis, M_m is graded flat A_m-module, thus we have the exact sequence

$$0 \to (L_1)_m \otimes_{A_m} M_m \to (L_2)_m \otimes_{A_m} M_m \to (L_3)_m \otimes_{A_m} M_m \to 0$$

It is clear that $(L_i)_m \otimes_{A_m} M_m \cong (L_i \otimes_A M)_m$ for every $i \in \{1,2,3\}$. Therefore, for any maximal graded ideal m, we have the exact sequence

$$0 \to (L_1 \otimes_A M)_m \to (L_2 \otimes_A M)_m \to (L_3 \otimes_A M)_m \to 0$$

By claim 2,

$$0 \to L_1 \otimes_A M \to L_2 \otimes_A M \to L_3 \otimes_A M \to 0$$

is exact. ∎

Proposition A.10. *Let A be a graded R-algebra, and it is finite generated as R-module. Then the following conditions are equivalent:*

(1) *A is graded separable R-algebra.*
(2) *$A_\wp$ is graded separable $R_\wp$-algebra, for any gr-prime ideal $\wp$.*
(3) *A_m is graded separable R_m-algebra, for any maximal graded ideal m.*

Proof. (1) $\Rightarrow$ (2) $A_\wp = R_\wp \otimes_R A$, by remark A.4, $A_\wp$ is graded separable algebra over $R_\wp$.
(2) $\Rightarrow$ (3) is trivial.
(3) $\Rightarrow$ (1) Since A_m is graded separable over R_m, that is A_m is graded projective $(A_m)^e$-module. Thus, A_m is graded flat $(A_m)^e$-module. Therefore, by the lemma A.9, A is graded flat. Since A is finite generated R-module and $J(A) = A\delta(A)$, then it is clear that $J(A)$ is also a finite generated R-module. It follows that A has a finite presentation as A^e-module.
Now use the following two facts:
1. A module of finite presentation is flat if and only if it is a finitely generated projective;
2. A graded module M is a graded flat module over a graded ring R if and only if M is a graded and flat R-module [16, Proposition 3.4.5].
Therefore, A is a projective A^e-module. By [16, Proposition 1.2.15], A is graded projective A^e-module, whence graded-separable over R. ∎

Proposition A.11. *Let A be a graded R-algebra which is finitely generated as R-module. Then A is graded separable if and only if A/mA is graded separable over R/m, for any maximal graded ideals m.*

Proof. $'' \Rightarrow''$ By the remark A.4, $A/mA = A \otimes_R R/m$ is graded separable over R/m, for any maximal graded ideal m.
$'' \Leftarrow''$ It is easy to prove that $R_m/mR_m \cong R/m$. Then we have

$$A_m/mA_m = (A \otimes_R R_m)/m(A \otimes_R R_m) \cong A \otimes_R (R_m/mR_m) \cong A \otimes_R R/m \cong A/mA.$$

Therefore, we can prove the proposition for a graded local ring R first. Here the proof is similar as the ungraded case [17, Proposition 2.6]. Using proposition A.3 and the graded Nakayama's lemma then gives the first step. Then, use Proposition A.10. ∎

Definition A.12. Let R be a graded commutative graded ring. A graded R-algebra A is called a *graded Azumaya algebra* if it is both graded central and graded separable.

Proposition A.13. *Let R be a graded commutative graded ring and A a graded R-algebra. The following conditions are equivalent:*

(1) *A is graded Azumaya algebra over R.*
(2) *A is faithfully projective graded R-module, $a\otimes b \mapsto (x \mapsto (-1)^{|x||b|}axb)$ induces an isomorphism of graded algebras $A \otimes_K A^{op} \longrightarrow \mathrm{End}_{K,graded}(A)$.*
(3) *The functor $N \mapsto A \otimes N$ and $M \mapsto M^A$ establish a graded equivalence of categories of categories of graded R-modules and graded A^e-modules.*
(4) *A is finite generated graded R-module, A/mA is a graded Azumaya algebra over R/m, for any maximal graded ideal m of R.*
(5) *There exists a graded R-algebra B and a faithfully projective graded R-module P such that $A \otimes_K B \longrightarrow \mathrm{End}_{K,graded}(P)$ as graded R-algebras.*

Proof. (1)⇒ (2) By [16, Proposition 1.2.15] and [16, Proposition 2.2.5], there exists a Morita theory analogous to that in [17, I.7]. Using these propositions and lemmas, together with the graded version of the dual basis lemma [16, Theorem 1.2.17], one can follow the proof in the ungraded case given in [17].

(2)⇒ (3), (3)⇒ (1), (2)⇒(5) and (5)⇒(1) are proved similarly to the ungraded case [17], using general graded Morita theory [16, 2.3].

(1)⇒ (4) follows the proof of (1)⇒ (2).

(4)⇒ (1) By the proposition A.11, A is a graded separable R-algebra. By the proposition A.4, $Z_{gr}(A/mA) = Z_{gr}(A)/Z_{gr}(A) \cap mA = Z_{gr}(A)/mZ_{gr}(A)$. Since A/mA is graded central over R/m, thus $Z_{gr}(A/mA) = R/m$, it follows that $R/m \cong Z_{gr}(A)/mZ_{gr}(A)$, for any maximal graded ideal m. Then, similar as the proof of lemma A.9, one can prove that $R \cong Z_{gr}(A)$. ■

Consider an equivalence relation $'' \sim''$ on graded R-algebras: $A \sim B$ if and only if there exist faithfully projective graded R-modules P and Q such that the graded ring $A\otimes_R \mathrm{End}(P)$ is isomorphic to the graded ring $B \otimes_R \mathrm{End}(Q)$. One can prove that this is indeed an equivalence relation.
Let $\mathrm{grBr}(R)$ denote the set of equivalence class of all graded Azumaya R-algebras with respect to the equivalence relation $'' \sim''$. By the remark A.4, the operation of tensor product over R is compatible with the equivalence relation so that there is induced an associative and commutative multiplication in $\mathrm{grBr}(R)$. The equivalence class which contains R itself is clearly an identity for this multiplication. If A is a graded Azumaya R-algebra, then clearly A^{op} is also a graded Azumaya R-algebra. And by the proposition A.13, we have that A is a faithfully projective graded R-module and that $A \otimes_R A^{op} = \mathrm{Hom}_R(A, A)$. Therefore it follows that $A \otimes_R A^{op} \sim R$ so that the equivalence class of A^{op} is an inverse to that of A in $\mathrm{grBr}(R)$. Thus, we have proved that $\mathrm{grBr}(R)$ is a group.

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