DG-SEPARABLE DG-EXTENSIONS

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Abstract. We define and characterise completely dg-separable dg-extensions $\varphi : (A, d_A) \to (B, d_B)$. We completely characterise the case of graded commutative dg-division algebras in characteristic different from 2. We prove that for a dg-separable extension a short exact sequence of dg-modules over (B, d_B) splits if and only if the restriction to (A, d_A) splits, giving that (B, d_B) is acyclic and $\ker(d_B)$ graded-semisimple in case (A, d_A) is a graded commutative dg-division algebra with $d_A = 0$.

Introduction

Let K be a commutative ring. A differential graded K-algebra (A, d) is a $\mathbb{Z}$-graded algebra A together with a K-linear graded endomorphism $d : A \to A$ of degree 1 such that $d^2 = 0$ and

$$d(a \cdot b) = d(a) \cdot b + (-1)^{|a|} a \cdot d(b)$$

for all homogeneous $a, b \in A$, where we denote by $|a|$ the degree of $a \in A$. Differential graded algebras (or dg-algebras for short) were defined by Cartan [2] in 1954 and proved to be highly successful in many subjects, such as homological algebra, algebraic topology, differential and algebraic geometry, and alike. However, the ring theory of differential graded algebras remained largely unexplored until quite recently. The first of the results in this direction was a characterisation of Aldrich and Garcia Rozas [1] of acyclic dg-algebras. [14] then studied general ring theoretic properties, such as a dg-Nakayama lemma, and independently in a parallel development Orlov [9] studied finite dimensional dg-K-algebras over a field K. Goodbody [4] proved a version of Nakayama's lemma in the dg-setting following Orlov's approach. In the sequel [15] defined and studied a dg-Brauer group, and in [16] Ore localisation and a Goldie theorem was studied in the context of dg-algebras. Further, in [17] a concept of a dg-division algebra was developed, and a complete classification was given, under a technical condition. In this case we showed that a dg-division algebra is either acyclic or has differential $d = 0$.

In general, a K-algebra B is called separable over a K-subalgebra A if the multiplication map $B \otimes_A B \longrightarrow B$ is split as morphism of $B-B$-bimodules. A graded version was given by Nastasescu-van Oystaen [7] asking for a split in the category of graded bimodules.

We define in this paper a differential graded separability, asking simply that the splitting of the multiplication map is a map of differential graded bimodules. We use the classification from [17] to show that a field extension between two graded-commutative acyclic dg-division rings is separable if the extension of cycles is separable graded-separable. In characteristic different from 2, the converse also holds. Further, we show that in characteristic different from 2 an extension of dg-division algebras from an algebra with differential 0 to an acyclic algebra is never separable. We finally note that a dg-extension $(A, d_A) \to (B, d_B)$ where (A, d_A) is acyclic implies that (B, d_B) is acyclic as well. This gives a complete picture of separability of extensions of graded commutative dg-division algebras in characteristic different from 2. The results are displayed in Theorem 3.5.

In general, we show that a dg-extension $\varphi : (A, d_A) \to (B, d_B)$ is dg-separable if and only if there is an element $\omega \in \ker(d_{B \otimes_A B})$ with $b\omega = \omega b$ for all $b \in B$ and mapping to 1 under the multiplication map $B \otimes_A B \to B$. We show in Theorem 4.6 that this then implies that a short exact sequence of dg-modules over (B, d_B) if and only if the restriction to (A, d_A) splits. A special case is when $d_A = 0$ and A is a graded-division ring. This then implies that (B, d_B) is acyclic and $\ker(d_B)$ is

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graded-semisimple. We further mention that our concept of dg-separability gives that the restriction functor is a separable functor in the sense of Nastasescu, van den Bergh and van Oystaen [8].

The paper is organised as follows. In Section 1 we recall results from [17] concerning dg-division rings as far as they are relevant for this work. Section 2 then gives the definition of a dg-separable extension. In Section 3 we completely classify dg-separable dg-extension of graded-commutative dg-division rings, which includes our first main result Theorem 3.5. Finally, Section 4 shows the second main result Theorem 4.6.

1. Dg-division algebras revisited

First recall some notations. As a reference one may take [12] or [14, 16, 17] Let (A,d) be a dg-K-algebra. Then a left dg-module over (A,d) is a $\mathbb{Z}$-graded A-module M together with an endomorphism $\delta : M \to M$ of degree 1 with $\delta^2 = 0$ and $\delta(a\cdot m) = d(a)\cdot m + (-1)^{|a|}a\cdot\delta(m)$ for all homogeneous $a \in A$ and $m \in M$. If (A,d) is a dg-algebra, then (A^{op},d) is a dg-algebra as well, where A^{op} coincides with A as K-module, and where $a\cdot_{op} b := (-1)^{|a||b|}b\cdot a$ for all homogeneous $a,b \in A$. Further, a right dg-module over (A,d) is a left dg-module over (A^{op},d). For two dg-modules (M,δ_M) and (N,δ_N) over (A,d) we set

$$\mathrm{Hom}_A^k((M,\delta_M),(N,\delta_N)) := \{f \in \mathrm{Hom}_{K,\text{ graded}}(M,N) \mid f(am) = (-1)^{|a|k}af(m)\}$$

and put $d_{\mathrm{Hom}}(f) := \delta_M \circ f - (-1)^{|f|} \circ \delta_M$.

We abbreviate

$$\mathrm{Hom}_A^\bullet((M,\delta_M),(N,\delta_N)) := \bigoplus_{k\in\mathbb{Z}} \mathrm{Hom}_A^k((M,\delta_M),(N,\delta_N)).$$

and

$$\mathrm{End}_A^\bullet((M,\delta_M)) := \mathrm{Hom}_A^\bullet((M,\delta_M),(M,\delta_M)).$$

Then $\mathrm{End}_A^\bullet((M,\delta_M)), d_{\mathrm{Hom}})$ is a dg-algebra, and $\mathrm{Hom}_A^\bullet((M,\delta_M),(N,\delta_N)), d_{\mathrm{Hom}})$ is a dg-bimodule over $\mathrm{End}_A^\bullet((N,\delta_N)), d_{\mathrm{Hom}})$- $\mathrm{End}_A^\bullet((M,\delta_M)), d_{\mathrm{Hom}})$.

Recall from [17] the definition of a differential graded division algebra.

Definition 1.1. [17] A *dg-division algebra* is a dg-algebra (A,d) such that the only dg-left ideals are 0 and A and the only dg-right ideals are 0 and A.

Differential graded division algebras (A,d) were completely classified in [17] provided the set of left regular homogeneous elements of $\ker(d)$ coincides with the set of right regular homogeneous elements of $\ker(d)$.

The corresponding result is the following.

Theorem 1.2. [17] *Let (A,d) be a dg-algebra. Suppose that the set of left regular homogeneous elements of $\ker(d)$ coincides with the set of right regular homogeneous elements of $\ker(d)$.*

Then

- *(A,d) is a dg-division algebra if and only if $\ker(d)$ is a $\mathbb{Z}$-gr-division algebra (cf* [7]*).*
- *If (A,d) is a dg-division algebra,*
  - *then,*
    - *either $d = 0$ and $\ker(d)$ is a skew field concentrated in degree 0,*
    - *or $H(A,d) = 0$ and there is a skew field R_0 such that $\ker(d) \simeq R_0[X,X^{-1};\phi]$ for an automorphism ϕ of R_0 and $Xr = \phi(r)X$ for any $r \in R_0$.*
  - *If $H(A,d) = 0$, then there is a homogeneous element y with $d(y) = 1$ and $y^2 \in \ker(d)$, and there is a map $D : \ker(d) \longrightarrow \ker(d)$ of degree 1 defined by*

    $$D(a) = -(-1)^{|a|}d(yay) = ya - (-1)^{|a|}ay$$

    for any homogeneous $a \in \ker(d)$, such that A is isomorphic with the quotient of the twisted polynomial ring

    $$A \simeq \ker(d)[T;D]/(T^2 - y^2).$$

Moreover, the algebra structure on the twisted group ring is given by $D(a) = Ta - (-1)^{|a|}aT$ *for any homogeneous* $a \in \ker(d)$*. Furthermore,* $A = \ker(d) \oplus y\ker(d)$*, and the isomorphism is*

$$\begin{array}{rcl} \Phi : \ker(d)[T;D] & \longrightarrow & A \\ b + Ta & \mapsto & b + ya \end{array}$$

for any homogeneous $a, b \in \ker(d)$*. Further, for any homogeneous* $a, b \in \ker(d)$ *we get* $d(b + ya) = a$*.*

Corollary 1.3. *Let* (A, d) *be a dg-algebra. Suppose that the set of left regular homogeneous elements of* $\ker(d)$ *coincides with the set of right regular homogeneous elements of* $\ker(d)$*. Suppose that* (A, d) *is a graded commutative dg-division algebra. Then either* $\ker(d)$ *is a field concentrated in degree* 0*, or else* $\ker(d) = K[X, X^{-1}]$ *for some field* K *and* X *in non zero degree, and, in case* K *is of characteristic different from* 2*, then* X *is in even degree.*

We shall need to recall the definition of a differential graded structure on a tensor product of algebras. Let (A, d_A) be a dg-algebra and let (B, d_B) de a dg-algebra. Consider a dg-homomorphism $(A, d_A) \longrightarrow (B, d_B)$. Then for

$$d_{B\otimes_A B} = d_B \otimes \mathrm{id}_B + \mathrm{id}_B \otimes d_B$$

defines a dg-$B - B$-bimodule structure on $(B \otimes_A B$. If A is a subalgebra of the graded centre of B, then $(B \otimes_A B, d_{B\otimes_A B})$ is a dg-algebra again.

2. dg-Separability

Recall that an algebra A is separable if A is a projective object in the category of $A-A$-bimodules. This is equivalent with the fact that the multiplication map is split as morphism of $A-A$-bimodules. Similarly, a graded algebra A is graded separable if the graded bimodule A is projective in the category of graded bimodules.

Proposition 2.1. [3, Example 2.5] *The extension of graded rings* $R[T^n, T^{-n}] \subseteq S[T, T^{-1}]$ *is separable if and only if the extension* $R \subseteq S$ *is separable and* n *is invertible in* R*.*

Recall (cf e.g. [13]) that an algebra extension $\beta : A \longrightarrow B$ of K-algebras is separable if the multiplication map

$$\mu : B \otimes_A B \longrightarrow B$$

splits as a homomorphism of $B \otimes_K B$-bimodules.

We shall use the analogous concept.

Definition 2.2. Let K be a commutative ring and let (A, d_A) and (B, d_B) be differential graded algebras. A *dg-extension of dg-algebras* is a homomorphism $\beta : (A, d_A) \longrightarrow (B, d_B)$ dg-algebras.

An extension of dg-algebras $\beta : (A, d_A) \longrightarrow (B, d_B)$ is called *dg-separable* if the multiplication map

$$\mu : (B, d_B) \otimes_A (B, d_B) \longrightarrow (B, d_B)$$

is split as morphism of differential graded $B - B$-bimodules.

Note however, for

$$B \otimes_A B \xrightarrow{\mu} B$$

one has $d_{B\otimes_A B} = d_B \otimes 1 + 1 \otimes d_B$ and then, by the Leibniz formula and conventions for the sign rules on graded rings, we always have

$$\mu \circ d_{B\otimes_A B} = d_B \circ \mu.$$

Proposition 2.3. *Let* $(A, d_A) \longrightarrow (B, d_B)$ *be a dg-extension of dg-algebras. This extension is dg-separable if and only if there is* $\omega \in \ker(d_{B\otimes_A B})$ *homogeneous of degree* 0 *with* $b\omega = \omega b$ *for all* $b \in B$ *and* $\mu(\omega) = 1$*.*

Proof. Let $\rho : B \longrightarrow B \otimes_A B$ be a retract with $\mu \circ \rho = 1_B$. Then

$$\rho \circ d_B = d_{B\otimes_A B} \circ \rho$$

is equivalent with

$$d_{B\otimes_A B}(\omega) = 0$$

for $b\omega = \omega b$ and $\mu(\omega) = 1$. Hence, ω has to be a cycle in $B \otimes_A B$. Even better, this is equivalent. Suppose $\omega \in \ker(d_{B\otimes_A B})$ with $b\omega = \omega b$ for all b and $\mu(\omega) = 1$. Then

$$\begin{array}{rcl} d_{B\otimes_A B}(\rho(b)) & = & d_{B\otimes_A B}(b\rho(1)) \\ & = & d_{B\otimes_A B}(b\omega) \\ & = & ((d_B \otimes 1)(b \otimes 1) + (1 \otimes d_B)(b \otimes 1))\omega + (-1)^{|b|} b d_{B\otimes B}(\omega)) \\ & = & d_B(b)\omega \\ & = & d_B(b)\rho(1) \\ & = & \rho(d_B(b)) \end{array}$$

where the last equation holds since ρ is a morphism of bimodules. ■

Lemma 2.4. *Let (A, d_A) and (B, d_B) be dg-algebras and let $\varphi : (A, d_A) \longrightarrow (B, d_B)$ be a dg-extension of dg-algebras. Then $\varphi|_{\ker(d_A)}$ is an extension of graded rings $\ker(d_A) \longrightarrow \ker(d_B)$.*

Proof. Suppose that $\varphi : (A, d_A) \longrightarrow (B, d_B)$ is a dg-extension of dg-algebras. Then φ induces a graded-extension $\varphi|_{\ker(d_A)} : \ker(d_A) \longrightarrow \ker(d_B)$ by restriction. Indeed, if $d_A(x) = 0$, then

$$0 = \varphi(d_A(x)) = d_B(\varphi(x))$$

and hence $\varphi(x) \in \ker(d_B)$ as well. ■

3. Characterisation of dg-separable dg-field extensions

Proposition 3.1. *Suppose that (A, d_A) and (B, d_B) are graded commutative dg-division algebras, suppose that (A, d_A) is acyclic, and suppose that $\varphi : (A, d_A) \longrightarrow (B, d_B)$ is a dg-extension of dg-algebras. If the restriction $\varphi|_{\ker(d_A)}$ is a graded-separable extension, then φ is a dg-separable extension. If the characteristic of A is different from 2, then the converse also holds.*

Proof. Since A and B are graded commutative, the set of left regular elements of $\ker(d_A)$ coincides with the set of right regular homogeneous elements of $\ker(d_A)$, and likewise for B. The algebra (B, d_B) is a left dg-module over (A, d_A) via φ. Therefore, by [1] we get that (B, d_B) is acyclic as well. We may hence suppose that (A, d_A) and (B, d_B) are both acyclic dg-division algebras. Then

$$A = \ker(d_A)[T; D_A]/(T^2 - y_A^2)$$

and

$$B = \ker(d_B)[T, D_B]/(T^2 - y_B^2)$$

for $d_A(y_A) = 1$ and $d_B(y_B) = 1$. Suppose that $\varphi : (A, d_A) \longrightarrow (B, d_B)$ is a dg-homomorphism. Further, $\varphi(1_A) = 1_B$ implies that we may assume that $\varphi(y_A) = y_B$.

By Lemma 2.4 the restriction of φ to $\ker(d_A)$ is an extension of graded rings $\ker(d_A) \longrightarrow \ker(d_B)$.

Suppose now that $\varphi|_{\ker(d_A)}$ is a graded-separable extension. Let $\omega_{\ker}$ be the element from Proposition 2.3 with $\mu(\omega_{\ker}) = 1$ and $b \cdot \omega_{\ker} = \omega_{\ker} \cdot b$ for all homogeneous $b \in \ker(d_B)$. Recall

$$A = \ker(d_A)[T; D_A]/(T^2 - y_A^2).$$

But, $\varphi(T)$ can be used as T in the isomorphism

$$B = \ker(d_B)[T, D_B]/(T^2 - y_B^2)$$

since we can put $\varphi(y_A) = y_B$. But than we only need to show

$$T\omega_{\ker} = \omega_{\ker} T.$$

If (B, d_B) is graded commutative, then for all homogeneous $b_1, b_2 \in \ker(d_B)$ we have that either the characteristic is 2, or else all b_i are of even degree. Hence

$$T(b_1 \otimes_A b_2) = b_1 T \otimes_A b_2 = b_1 \otimes_A T b_2 = b_1 \otimes_A b_2 T = (b_1 \otimes_A b_2)T.$$

Moreover, trivially, T commutes with T, and hence T commutes with any $(b_1 \otimes_A b_2) \in B \otimes_A B$.

As for the converse, suppose that $(A, d_A) \longrightarrow (B, d_B)$ be a dg-separable dg-extension. Therefore, by Proposition 2.3 there is an element $\omega \in B \otimes_A B$ of degree 0. with $b\omega = \omega b$ for all homogeneous $b \in B$, and the image of ω under the multiplication map $B \otimes_A B \to B$ is 1. Further, $\omega \in \ker(d_{B\otimes_A B})$. If B is a graded commutative dg-division ring of characteristic different from 2, then $\ker(d_B)$ has to be concentrated in even degrees, since any homogeneous element is invertible, whence not nilpotent ($x^2 = -x^2$ for elements of odd degree), and by consequence in even degrees.

Consider the map

$$\Upsilon : \ker(d_B) \otimes_{\ker(d_A)} \ker(d_B) \longrightarrow B \otimes_A B$$

given by the natural inclusion. But then, as $B = \ker(d_B) \oplus T\ker(d_B) = \ker(d_B) \oplus \ker(d_B)T$, and since T is of degree -1, we see that the direct summand $\ker(d_B) \otimes_A T\ker(d_B)$ and $T\ker(d_B) \otimes_A \ker(d_B)$ are in odd degrees. Hence the image of Υ is in the subspace of even degree of $B \otimes_A B$. Further, all $b_1 \otimes b_2$ with $b_1, b_2 \in \ker(d_B)$ are in the image of Υ. Also, for all $b_1, b_2 \in \ker(d_B)$ the elements

$$Tb_1 \otimes Tb_2 = T^2 b_1 \otimes b_2 = y_B^2 b_1 \otimes b_2$$

are in the image of Υ. Further, y_B^2 is homogeneous of degree -2, satisfying $d_B(y_B) = 1$, and since

$$d_B(y_B^2) = d_B(y_B)y_B - y_B d_B(y_B) = y_B - y_B = 0$$

we have in $y_B^2 \in \ker(d_B)$. We have two cases. If $y_B^2 = 0$, then

$$T\ker(d_B) \otimes_A T\ker(d_B) = y_B^2 \ker(d_B) \otimes_A \ker(d_B) = 0.$$

Else, $y_B^2 \in \ker(d_B)^\times$ since (B, d_B) is an acyclic dg-division algebra. Hence, the image of Υ is precisely the subspace of even degree elements of $B \otimes_A B$.

Since ω has to be homogeneous of degree 0, which is even, $\omega \in \mathrm{im}(\Upsilon)$. Let $\omega' \in \ker(d_B) \otimes_A \ker(d_B)$ with $\Upsilon(\omega') = \omega$. Note that Υ is injective. Therefore $\omega' \in \ker(d_B) \otimes_A \ker(d_B)$ can be used as required element to show that $\ker(d_A) \longrightarrow \ker(d_B)$ is graded separable. ∎

Remark 3.2. The case of differential 0 is trivial. A dg-extension $(A, 0) \longrightarrow (B, 0)$ is precisely a graded extension. Note that if (A, d_A) is acyclic, then any extension (B, d_B) of (A, d_A) is acylic as well. The only case left is when $d_A = 0$ and (B, d_B) is acyclic.

Proposition 3.3. *Let $(A, 0)$ be a gr-field and let (B, d) be a graded commutative acyclic dg-division algebra. Suppose that $A \longrightarrow B$ is a dg-extension. If B is of characteristic different from 2, then this extension is not dg-separable.*

Proof. We need to find an element $\omega \in B \otimes_A B$, homogeneous of degree 0 and mapping to 1 under the multiplication map $B \otimes_A B \longrightarrow B$, such that $d_{B\otimes_B A}(\omega) = 0$ and such that $b\omega = \omega b$ for all homogeneous $b \in B$.

If B is a graded commutative dg-division ring over k, then $\ker(d)$ has to be concentrated in even degrees, since any homogeneous element is invertible, whence not nilpotent ($x^2 = -x^2$ for elements of odd degree), and by consequence in even degrees. But then, as $B = \ker(d) \oplus T\ker(d) = \ker(d) \oplus \ker(d)T$, and since T is of degree -1, we see that for the element ω we have

$$\omega \in (\ker(d) \otimes \ker(d)) \oplus (T\ker(d) \otimes T\ker(d)).$$

However, $\omega \in \ker(d_{B\otimes_A B})$. Clearly,

$$(\ker(d) \otimes \ker(d)) \subseteq \ker(d_{B\otimes_A B}).$$

Now, for $x = \sum_{i=1}^n Tb_i \otimes Tb_i' \in (T\ker(d) \otimes T\ker(d))$ we get

$$0 = d_{B\otimes_A B}(x) = \sum_{i=1}^n b_i \otimes Tb_i' + (-1)^{|b_i|+1} Tb_i \otimes b_i' = \sum_{i=1}^n b_i \otimes Tb_i' - Tb_i \otimes b_i'$$

since all b_i are of even degree. Since,

$$B \otimes_A B = (\ker(d) \otimes \ker(d)) \oplus (T\ker(d) \otimes \ker(d)) \oplus (\ker(d) \otimes T\ker(d)) \oplus (T\ker(d) \otimes T\ker(d))$$

we get

$$(T\ker(d) \otimes \ker(d)) \cap (\ker(d) \otimes T\ker(d)) = 0$$

and hence
$$\sum_{i=1}^{n} b_i \otimes Tb_i' = 0 = \sum_{i=1}^{n} Tb_i \otimes b_i' = T \cdot (\sum_{i=1}^{n} b_i \otimes b_i')$$
which shows that
$$\sum_{i=1}^{n} b_i \otimes b_i' = 0.$$
Therefore $x = 0$. But for $\omega \in \ker(d) \otimes_A \ker(d)$ we get that $T\omega \neq \omega T$ since the left hand side is in $T\ker(d) \otimes_A \ker(d)$ and the right hand side lies in $\ker(d) \otimes_A \ker(d)T$, whose intersection is 0. ∎

We illustrate the argument by a simple

Example 3.4. We illustrate the proof of Proposition 3.3 with an example. Let $A = K[X]/X^2$ for $d(X) = 1$, and X in degree -1. Here, $\ker(d) = K \cdot 1$. Then we need to see if the multiplication map
$$\begin{array}{ccc} A \otimes_K A & \longrightarrow & A \\ (a + bX) \otimes (c + dX) & \mapsto & ac + (ad + bc)X \end{array}$$
is split as $A-A$-dg-bimodules. As we have a K-basis $\{1, X\}$ of A, we also have a K-basis $\{1 \otimes 1, 1 \otimes X, X \otimes 1, X \otimes X\}$ of $A \otimes_K A$. The multiplication map splits as dg-map if and only if there is an element ω of degree 0 with $v\omega = \omega v$ for all $v \in A$ and mapping to 1 under the multiplication map. As $\ker(d)$ is commutative and central, we only need to verify this property for $v = X$. The degree 0 component of $A \otimes_K A$ is of dimension 1, generated by $1 \otimes 1$. An element $\lambda \cdot (1 \otimes 1)$ maps to 1 under the multiplication if and only if $\lambda = 1$. However,
$$X \cdot (1 \otimes 1) = (X \otimes 1) \neq (1 \otimes X) = (1 \otimes 1) \cdot X$$
Hence the extension is not separable.

We summarise the results in the following

Theorem 3.5. *Let K be a field, let (A, d_A) and let (B, d_B) be graded commutative dg-division rings over K. Let $\varphi : (B, d_B) \to (A, d_A)$ be a dg-extension.*

- *Then $\ker(d_A)$ and $\ker(d_B)$ are graded-commutative graded-division rings.*
- *If (B, d_B) is acyclic, then also (A, d_A) is acyclic and*
  - *the dg-extension is dg-separable if the induced graded-extension $\ker(d_B) \to \ker(d_A)$ is graded separable.*
  - *If the characteristic of K is different from 2, then the dg-extension is dg-separable if and only if the induced graded-extension $\ker(d_B) \to \ker(d_A)$ is graded-separable.*
- *If K is of characteristic different from 2, and if $d_B = 0$ and (A, d_A) is acyclic, then the dg-extension is not dg-separable.*
- *If K is of characteristic different from 2, and C is a graded-commutative gr-division ring, then there is a field D such that $C \simeq D[T, T^{-1}]$ for some T in non zero even degree. An extension $D_1[T^n, T^{-n}] \to D_2[T, T^{-1}]$ is graded-separable if and only if the field extension $D_1 \longrightarrow D_2$ is separable and n is invertible in D_1.*

4. General consequences of dg-separability

Remark 4.1. Recall that we have two concepts of semisimplicity. An abelian category $\mathcal{A}$ is semisimple if every short exact sequence of objects in $\mathcal{A}$ is split. An (graded) algebra A is J-semisimple (Jacobson-semisimple) if every graded A-module is a direct sum of simple (graded) A-modules. It is well-known that if A is artinian, then the two concepts coincide for $\mathcal{A}$ being the category of finitely generated (graded) A-modules. Similar concepts hold for dg-modules instead of graded modules.

Remark 4.2. Let $\mathcal{C}$ be an abelian category in which every object is projective. Then $\mathcal{C}$ is semisimple in the sense that every short exact sequence of objects in $\mathcal{C}$ splits.

Theorem 4.3. *Let (A, d) be a dg-algebra.*

- [1, Proposition 3.3] *If a dg-module (M, δ) over (A, d) is a projective object in the category of dg-modules, then (M, δ) is acyclic.*

- [1, Theorem 4.7] *If* (A,d) *is acyclic, then every dg-module over* (A,d) *is acyclic and the functor*
$$A\otimes_{\ker(d)} - : gr-\ker(d)-mod \longrightarrow dg-(A,d)-mod$$
is an equivalence with quasi-inverse being the functor taking cycles.
- [1, Definition 5.1 and Theorem 5.3] *The category of dg-modules over* (A,d) *is J-semisimple if and only if* (A,d) *is acyclic and* $\ker(d)$ *is graded-J-semisimple.*

We consider consequences which can be derived for dg-separable dg-extensions of dg-algebras.

Theorem 4.4. *Let* (A,d_A) *is a dg-algebras over some graded commutative acyclic dg-division ring* (K,d_K) *and suppose that* $\varphi:(K,d)\to(A,d_A)$ *is a dg-separable dg-extension. Let* (L,d_L) *be a graded commutative dg-division ring being a dg-extension of* (K,d_K).

Then, any dg-module (M,δ_M) *over* $(A\otimes_K L, d_{A\otimes_K L})$ *is a direct summand of* $(A\otimes_K L, d_{A\otimes_K L})^I$ *for some index set* I. *More precisely,* I *is a* $\ker(d_L)$*-basis of* $\ker(\delta_M)$.

Proof.
$$\mu: A\otimes_K A\longrightarrow A$$
is split as dg-morphism by $\rho: A\longrightarrow A\otimes_K A$, satisfying $\mu\circ\rho=\mathrm{id}_A$. Then $\rho\otimes\mathrm{id}_L$ is a split of
$$\mu_L:(A\otimes_K L)\otimes_L(A\otimes_K L)\longrightarrow(A\otimes_K L).$$
Indeed,
$$(A\otimes_K L)\otimes_L(A\otimes_K L)=(A\otimes_K A)\otimes_K L$$
and with this identification we get
$$(\rho\otimes\mathrm{id}_L)\circ(\mu\otimes\mathrm{id}_L)=(\rho\circ\mu)\otimes\mathrm{id}_L=\mathrm{id}_A\otimes\mathrm{id}_L=\mathrm{id}_{A\otimes_K L}.$$
We therefore may assume that $K=L$ from the beginning.

Doing so
$$\mu\otimes_A\mathrm{id}: A\otimes_K M=A\otimes_K A\otimes_A M\longrightarrow A\otimes_A M=M$$
is split by $\rho\otimes\mathrm{id}$. Hence (M,δ_M) is a direct factor of $(A\otimes_K M,\delta_{A\otimes_K M})$.

We need to analyze $(A\otimes_K M,\delta_{A\otimes_K M})$.

Since (K,d_K) is an acyclic dg-division algebra, since (A,d_A) is a dg-module over (K,d_K), as well as (M,δ), we get that (A,d_A) and (M,δ_M) are acyclic (cf Theorem 4.3). Then
$$\Phi_K: K\otimes_{\ker(d_K)} - : \ker(d_K)-gr-mod\longrightarrow(K,d_K)-dg-mod$$
is an equivalence of categories with inverse the functor given by taking cycles. Hence, the unit $\mathrm{id}\longrightarrow\Phi_K\circ\Phi_K^{-1}$ is an isomorphism of functors. Moreover,
$$\Phi_A: A\otimes_{\ker(d_A)} - : \ker(d_A)-gr-mod\longrightarrow(A,d_A)-dg-mod$$
is an equivalence of categories. Then
$$\begin{aligned} A\otimes_K M &\simeq A\otimes_K(\Phi_K\circ\Phi_K^{-1}M)\\ &= A\otimes_K(K\otimes_{\ker(d_K)}\ker(\delta_M))\\ &= A\otimes_{\ker(d_K)}\ker(\delta_M)\end{aligned}$$
Since $\ker(d_K)$ is a gr-field, by [10, Lemma 1.7] $\ker(\delta_M)$ has a $\ker(d_K)$-basis I of homogeneous elements. Hence, $A\otimes_K M=A^I$. ■

Corollary 4.5. *Let* (A,d_A) *be an acyclic dg-algebras over some graded commutative acyclic dg-division ring* (K,d_K) *and suppose that* $\varphi:(K,d)\to(A,d_A)$ *is a dg-separable dg-extension. Then any dg-module over* (A,d_A) *is a projective object in the category of dg-modules over* (A,d_A). *Moreover, if* (A,d_A) *is dg-artinian and dg-Noetherian, then the category of dg-modules over* (A,d) *is semisimple and* $\ker(d_A)$ *is graded-semisimple.*

Proof. Indeed, by Theorem 4.4 every dg-module over (A,d_A) is a direct factor of $(A,d_A)^I$ for some index set I. By Theorem 4.3, since (A,d_A) is assumed to be acyclic, (A,d_A) is a projective object in the category of dg-modules over (A,d_A). Remark 4.2 shows that this implies that the category of dg-modules over (A,d_A) is semisimple. For dg-artinian and dg-Noetherian algebras the concepts of semisimplicity and of J-semisimplicity coincides for finitely generated dg-modules. But by Theorem 4.3 we get that this implies that $\ker(d_A)$ is graded-semisimple. ∎

We can prove an analogue to [6, Proposition 1.3]. Recall that for $B\otimes_A B$-bimodules M_1 and M_2 we denote by M_1^B the subset of elements x in M_1 with $bx = xb$ for all $b\in B$, and likewise for M_2. Then for a homomorphism $\alpha : M_1 \longrightarrow M_2$ of $B\otimes_A B$-bimodules we get that $\alpha(M_1^B)\subseteq M_2^B$. Indeed,

$$b\alpha(x) = \alpha(bx) = \alpha(xb) = \alpha(x)b$$

for all $x\in M_1^B$ and $b\in B$.

Theorem 4.6. *Let (A,d_A) and (B,d_B) be dg-algebras over some graded-commutative dg-division ring (K,d_K) and suppose that $\varphi : (A,d_A)\longrightarrow(B,d_B)$ is a dg-separable dg-extension. Then any short exact sequence of dg-modules*

$$0\longrightarrow (L,\delta_L)\xrightarrow{f}(M,\delta_M)\xrightarrow{g}(N,\delta_N)\longrightarrow 0$$

over (B,d_B) is split if and only if it is split considered as a sequence of dg-modules over (A,d_A).

Proof. The space $\mathrm{Hom}^\bullet_{(K,d_K)}((N,\delta_N),(M,\delta_M))$ is a dg-bimodule over $(B,d_B)-(B,d_B)^{op}$ given by $b_1\otimes b_2$ acts on $\Phi\in\mathrm{Hom}^\bullet_{(K,d_K)}((N,\delta_N),(M,\delta_M))$ by $((b_1\otimes b_2)\cdot\Phi)(n) = b_1\Phi(b_2n)$.

Suppose that the sequence is split as dg-modules over (A,d_A). Let ρ be an (A,d_A)-splitting of the epimorphism g on the right. Then

$$\begin{array}{ccc} B\otimes_A B & \xrightarrow{\sigma} & \mathrm{Hom}^\bullet_{(K,d_K)}((N,\delta_N),(M,\delta_M)) \\ b_1\otimes b_2 & \mapsto & b_1\rho b_2 \end{array}$$

is a dg-$B\otimes_A B$-module homomorphism since ρ is A-linear. Let $\omega\in\ker(d_{B\otimes_A B})$ mapping to 1 under the multiplication μ, such that $b\omega=\omega b$ for all $b\in B$ from Proposition 2.3. But then we claim that

$$\tau := \sigma(\omega)\in\mathrm{Hom}^\bullet_{(B,d_B)}((N,\delta_N),(M,\delta_M)).$$

Indeed, σ is a homomorphism of $B-B$-bimodules, and hence

$$\tau\in\left(\mathrm{Hom}^\bullet_{(K,d_K)}((N,\delta_N),(M,\delta_M))\right)^B$$

by the remarks preceding the statement of Proposition 4.6. But

$$\left(\mathrm{Hom}^\bullet_{(K,d_K)}((N,\delta_N),(M,\delta_M))\right)^B = \mathrm{Hom}^\bullet_{(B,d_B)}((N,\delta_N),(M,\delta_M)).$$

Further, for all $n\in N$ we get

$$(g\circ\tau)(n) = (g\circ\sigma(\omega))(n) = ((g\circ\sigma)(\omega))(n) = \mu(\omega)\cdot n = 1\cdot n = n$$

If the short exact sequence is split as sequence of dg-(B,d_B)- modules, then trivially it is split as sequence of (A,d_A)-modules as well. This proves the statement. ∎

Corollary 4.7. *Let K be a graded commutative $\mathbb{Z}$-graded-division ring, let (A,d) be a dg-algebra, and let $(K,0)\longrightarrow(A,d)$ be a dg-separable dg-extension. Then any short exact sequence of dg-modules over (A,d) is split. Hence the category of dg-modules over (A,d) is semisimple and therefore* $\ker(d)$ *is graded-semisimple and (A,d) is acyclic.*

Proof. Theorem 4.6 shows that any sequence of dg-modules is split if and only if it is split as sequence of graded K-modules. Since the sequence is a sequence of graded modules, and since K is a graded-division ring, the squence is indeed split as a sequence of K-modules. Hence it is split as a sequence of dg-modules over (A,d). But this is equivalent with the property that the category of dg-modules over (A,d) is semisimple. Theorem 4.3 then implies the statement. ∎

Remark 4.8. Note that by Proposition 3.3 if K is a graded commutative $\mathbb{Z}$-graded-division ring, and (A,d) is a dg-division algebra, such that $(K,0)\longrightarrow(A,d)$ is a dg-separable dg-extension, then.the characteristic of K is 2 or A cannot be graded commutative.

Recall the concept of a separable functor introduced by Nastasescu, van den Bergh, and van Oystaen [8].

Definition 4.9. [8] A *covariant functor* $F : \mathcal{C} \longrightarrow \mathcal{D}$ between categories $\mathcal{C}$ and $\mathcal{D}$ is called *separable* if the canonical map $\Phi^F_{A,B} : \mathcal{C}(A,B) \longrightarrow \mathcal{D}(FA,FB)$ is a naturally split monomorphism.

Proposition 4.10. *Let (A,d_A) and (B,d_B) be dg-algebras and let $\varphi : (A,d_A) \to (B,d_B)$ be a dg-extension. Then φ is dg-separable if and only if the restriction $dg-(B,d_B)-mod \to dg-(A,d_A)-mod$ is a separable functor.*

Proof. Suppose that the multiplication $B \otimes_A B \to B$ is split. Then the restriction is right adjoint to the induction $B \otimes_A -$ as is well-known (cf Yekutieli [12, 12.6.5]). The counit of the adjoint pair is the multiplication map $B \otimes_A B \to B$. By [11, 2.2.(ii)] this is equivalent with the fact that the restriction functor is separable. ■

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