

STABLE ENDOMORPHISM ALGEBRAS OF MODULES OVER SPECIAL BISERIAL ALGEBRAS

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ABSTRACT. We prove that the stable endomorphism algebra of a module without self-extensions over a special biserial algebra is a gentle algebra. In particular, it is again special biserial. As a consequence, any algebra which is derived equivalent to a gentle algebra is gentle.

Dedicated to Idun Reiten on the occasion of her 60th birthday

1. INTRODUCTION

We consider finite-dimensional (associative) algebras over an algebraically closed ground field k . One of the most important classes of tame algebras is the class of special biserial algebras. These algebras occur naturally in many different contexts, see the introduction of [17] or [18] for more details. We shall prove that for a special biserial algebra A and any A -module M without self-extensions the endomorphism algebra of M modulo the ideal of endomorphisms factoring through a projective A -module is not only special biserial again, but is even a gentle algebra. Gentle algebras are special biserial algebras satisfying certain minimality conditions.

The result is interesting and surprising in its own right, since for arbitrary algebras this quotient of the endomorphism algebra of modules without self-extensions will be very different from the original algebra. Nevertheless, our main motivation came from different considerations, namely the theory of derived equivalences. Let $\text{mod}(A)$ be the category of finite-dimensional right A -modules and let $D^b(A)$ be the derived category of bounded complexes of finite-dimensional A -modules. Two algebras A and B are said to be derived equivalent if the categories $D^b(A)$ and $D^b(B)$ are equivalent as triangulated categories. The stable module category $\underline{\text{mod}}(A)$ is the category with the same objects as $\text{mod}(A)$, and whose morphisms are equivalence classes of A -module homomorphisms modulo those which factor through a projective module.

Rickard shows that two algebras A and B are derived equivalent if and only if there exists a so-called tilting complex T in $D^b(A)$ such that $B \simeq \text{End}_{D^b(A)}(T)$, see [10],[12]. Now, Happel proved in [8] that there is a full

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and faithful embedding $F : D^b(A) \longrightarrow \underline{\text{mod}}(RA)$ of triangulated categories, where RA is the repetitive algebra of A . Since for a tilting complex T one gets $\text{Hom}_{D^b(A)}(T, T[1]) = 0$, the algebra B is isomorphic to the endomorphism algebra in $\underline{\text{mod}}(RA)$ of a module $M = FT$ without self-extensions. Thus, studying stable endomorphism algebras of modules without self-extensions is of central importance when one is interested in derived equivalences.

In our case, since an algebra is gentle if and only if its repetitive algebra is special biserial, (see [11] and also [14] and [15]), we get as a corollary to our main result that the class of gentle algebras is closed under derived equivalences. Examples of classes of algebras closed under derived equivalence are very rare and show that gentle algebras deserve much attention.

To be more precise, let us recall some definitions: Let Q be a (not necessarily finite) quiver. For an arrow $\alpha : a \rightarrow b$, let $s(\alpha) = a$ be its starting point and $e(\alpha) = b$ its end point. A *path* of length $n \geq 1$ in Q is a sequence $\alpha_1 \cdots \alpha_n$ of arrows such that $e(\alpha_i) = s(\alpha_{i+1})$ for all $1 \leq i \leq n-1$. A *relation* for Q is a non-zero k -linear combination of paths of length at least 2 having the same starting point and the same end point. Let ρ be a set of relations for Q . Then (Q, ρ) is called *special biserial* if the following hold:

- (1) Any vertex in Q is the starting point of at most two arrows and also the end point of at most two arrows;
- (2) Given an arrow β , there is at most one arrow α with $e(\alpha) = s(\beta)$ and $\alpha\beta \notin \rho$, and there is at most one arrow γ with $e(\beta) = s(\gamma)$ and $\beta\gamma \notin \rho$;
- (3) Each infinite path in Q contains a subpath which is in ρ .

A special biserial pair (Q, ρ) is *gentle* if additionally the following hold:

- (4) All elements in ρ are paths of length 2;
- (5) Given an arrow β , there is at most one arrow α' with $e(\alpha') = s(\beta)$ and $\alpha'\beta \in \rho$, and there is at most one arrow γ' with $e(\beta) = s(\gamma')$ and $\beta\gamma' \in \rho$.

A k -algebra is called *special biserial*, or *gentle*, if it is Morita equivalent to an algebra $kQ/(\rho)$ for (Q, ρ) special biserial, or gentle, respectively. In these cases, $kQ/(\rho)$ is finite-dimensional if and only if Q contains only finitely many vertices. Here kQ is the path algebra of Q , and (ρ) is the ideal generated by the elements in ρ . Recall that kQ has as a basis the set of all paths in Q including a path e_i of length 0 for each vertex i in Q . By ‘modules’ we always mean finite-dimensional right modules.

Here is our main result.

Theorem 1.1. *Let A be a special biserial algebra, and let M be an A -module. If $\text{Ext}_A^1(M, M) = 0$, then $\underline{\text{End}}_A(M)$ is a gentle algebra.*

Let $T[i]$ be the usual shift of a complex T in $D^b(A)$ by i degrees to the left (we adopt the convention of [10]). Then, this theorem has the following consequence.

Corollary 1.2. *Let A be a finite-dimensional gentle algebra, and let T be a complex in $D^b(A)$. If $\text{Hom}_{D^b(A)}(T, T[1]) = 0$, then $\text{End}_{D^b(A)}(T)$ is a gentle algebra. In particular, any algebra B which is derived equivalent to A is gentle, and (up to Morita equivalence) there are only finitely many such algebras B .*

Classical examples of gentle algebras are hereditary algebras of type \mathbb{A}_n and $\tilde{\mathbb{A}}_n$. It was known before that all algebras which are derived equivalent to these examples are gentle again. For a complete classification of the derived equivalence classes of these examples see [1] and [3]. Recently, Vossieck classified all algebras A such that $D^b(A)$ is discrete. In this case, it turns out that A is either derived hereditary of Dynkin type or a gentle algebra, see [19]. Using Vossieck's result, the derived equivalence classes of algebras with discrete derived category have been classified in [4].

The paper is organised as follows. In Section 2 we recall definitions and give a survey of known results. The main theorem and its corollary are proved in Section 3. Finally, in Section 4 we give some examples.

Although we often write maps on the left hand side, we compose them as if they were on the right. Thus the composition of a map θ followed by a map ϕ is denoted $\theta\phi$.

2. KNOWN FACTS ON SPECIAL BISERIAL ALGEBRAS

In this section, we recall some basic facts on special biserial algebras, string modules and homomorphisms between string modules. As a main reference we use [5], see also there for further references.

Let $Q = (Q_0, Q_1)$ be a quiver with set of vertices Q_0 and set of arrows Q_1 . Let ρ be a set of relations for Q such that (Q, ρ) is special biserial, and let $A = kQ/(\rho)$ be the corresponding special biserial algebra. Without loss of generality we can assume that ρ contains only zero relations, i.e. paths, and relations of the form $p - q$ with p and q paths, which are called *commutativity relations*. If $r = p - q$ is a commutativity relation, then we say that p and q are contained in r . By ρ^+ we denote the set ρ together with all paths which are contained in commutativity relations in ρ .

Given an arrow α in Q denote by α^- a formal inverse where $s(\alpha^-) = e(\alpha)$ and $e(\alpha^-) = s(\alpha)$. Also let $(\alpha^-)^- = \alpha$. The set of formal inverses of arrows is denoted by Q_1^- . We use the word 'arrow' only for element in Q_1 , we refer to elements in Q_1^- always as inverse arrows. A *string* (for (Q, ρ)) of length $n \geq 1$ is a sequence $c_1 \cdots c_n$ of arrows or inverse arrows with the following properties:

- (1) $e(c_i) = s(c_{i+1})$ for $1 \leq i \leq n - 1$;
- (2) $c_i \neq c_{i+1}^-$ for $1 \leq i \leq n - 1$;
- (3) For all $1 \leq i < j \leq n$ neither $c_i c_{i+1} \cdots c_j$ nor $c_j^- \cdots c_{i+1}^- c_i^-$ are contained in ρ^+ .

Let $C = c_1 \cdots c_n$ be a string. Define $s(C) = s(c_1)$ and $e(C) = e(c_n)$. Let $C^- = c_n^- \cdots c_1^-$ be the inverse string of C . Additionally, we define for every vertex i in Q two strings $1_{(i,1)}$ and $1_{(i,-1)}$ of length 0 with $s(1_{(i,t)}) = e(1_{(i,t)}) = i$ and $1_{(i,t)}^- = 1_{(i,-t)}$ for $t = -1, 1$. The length of an arbitrary string C is denoted by $|C|$. We call C *direct* if $|C| = 0$ or $C = c_1 \cdots c_n$ with $c_i \in Q_1$ for all i , and C is called *inverse* if C^- is direct.

Similarly as in [5], one can define two maps $\sigma, \epsilon : Q_1 \rightarrow \{-1, 1\}$ which satisfy the following properties:

- (1) If $\alpha_1 \neq \alpha_2$ are arrows with $e(\alpha_1) = e(\alpha_2)$, then $\epsilon(\alpha_1) = -\epsilon(\alpha_2)$;
- (2) If $\beta_1 \neq \beta_2$ are arrows with $s(\beta_1) = s(\beta_2)$, then $\sigma(\beta_1) = -\sigma(\beta_2)$;
- (3) If α and β are arrows with $e(\alpha) = s(\beta)$ and $\alpha\beta \notin \rho^+$, then $\epsilon(\alpha) = -\sigma(\beta)$;
- (4) Let i be a vertex such that there is only one arrow α with $e(\alpha) = i$, and only one arrow β with $s(\beta) = i$. If $\alpha\beta \in \rho^+$, then $\epsilon(\alpha) = \sigma(\beta)$.

Note that σ and ϵ are defined in [5] without condition (4). The set of all strings is denoted by \mathcal{S} . We extend σ and ϵ to maps

$$\sigma, \epsilon : \mathcal{S} \rightarrow \{-1, 1\}$$

as follows: For an arrow α define $\sigma(\alpha^-) = \epsilon(\alpha)$ and $\epsilon(\alpha^-) = \sigma(\alpha)$. If $C = c_1 \cdots c_n$ is a string of length $n \geq 1$, then let $\sigma(C) = \sigma(c_1)$ and $\epsilon(C) = \epsilon(c_n)$. For a string $1_{(i,t)}$ of length 0 define $\sigma(1_{(i,t)}) = t$ and $\epsilon(1_{(i,t)}) = -t$.

Let $C = c_1 \cdots c_n$ and $D = d_1 \cdots d_m$ be strings of length at least 1. If $CD = c_1 \cdots c_n d_1 \cdots d_m$ is a string, then we say that the concatenation of C and D is defined. For an arbitrary string C let $1_{(s(C),t)}C = C$ if $\sigma(C) = t$, and let $C1_{(e(C),t)} = C$ if $\epsilon(C) = -t$. Otherwise, the concatenation with a string of length 0 is not defined.

One uses σ and ϵ to give a certain ‘orientation’ to strings. One of the main properties of σ and ϵ is the following:

Lemma 2.1. *If C and D are strings such that the concatenation CD is defined, i.e. CD is again a string, then $e(C) = s(D)$ and $\epsilon(C) = -\sigma(D)$.*

For each string C , we define a map

$$v_C : \{1, \dots, |C| + 1\} \longrightarrow Q_0$$

as follows. If $C = 1_{(i,t)}$ is a string of length 0, then let $v_C(1) = i$. If $C = c_1 \cdots c_n$ is of length at least 1, then let

$$v_C(i) = \begin{cases} s(c_i) & : 1 \leq i \leq |C|; \\ e(c_n) & : i = |C| + 1. \end{cases}$$

For each string C we construct an A -module $M(C)$ as follows. First, assume that $C = c_1 \cdots c_n$ is a string of length $n \geq 1$. Fix a basis $\{z_1, \dots, z_{n+1}\}$ of $M(C)$. Let e_j be a path of length 0 in Q . Then define

$$z_i \cdot e_j = \begin{cases} z_i & : i \in v_C^{-1}(j); \\ 0 & : \text{otherwise.} \end{cases}$$

Given an arrow α in Q let

$$z_i \cdot \alpha = \begin{cases} z_{i+1} & : \alpha = c_i, 1 \leq i \leq n; \\ z_{i-1} & : \alpha^- = c_{i-1}, 2 \leq i \leq n+1; \\ 0 & : \text{otherwise.} \end{cases}$$

Next, assume $C = 1_{(i,t)}$ is a string of length 0. Then let $\{z_1\}$ be a basis of $M(C)$. For each path e_j of length 0 in Q let

$$z_1 \cdot e_j = \begin{cases} z_1 & : j = i; \\ 0 & : \text{otherwise.} \end{cases}$$

For all α in Q_1 define $z_1 \cdot \alpha = 0$.

Modules of the form $M(C)$ are called *string modules*, and $\{z_1, \dots, z_{n+1}\}$ is called the *canonical basis* of $M(C)$. The construction of string modules goes back to Gelfand and Ponomarev, see [7] and also [5]. One can easily check that string modules are always indecomposable. For strings C_1 and C_2 define $C_1 \sim C_2$ if $C_1 = C_2$ or $C_1 = C_2^-$. Two string modules $M(C_1)$ and $M(C_2)$ are isomorphic if and only if $C_1 \sim C_2$.

For each vertex i , let $S_i = M(1_{(i,1)})$. Then $\{S_i \mid i \in Q_0\}$ is a complete set of isomorphism classes of simple A -modules. Furthermore, let P_i be the corresponding indecomposable projective A -module with $\text{top}(P_i) = S_i$.

For a string C define $\mathcal{P}(C) = \{(D, E, F) \mid D, E, F \in \mathcal{S} \text{ and } DEF = C\}$. We call (D, E, F) in $\mathcal{P}(C)$ a *factor string* of C if the following hold:

- (1) Either $|D| = 0$ or $D = d_1 \cdots d_n$ where $d_n \in Q_1^-$;
- (2) Either $|F| = 0$ or $F = f_1 \cdots f_m$ where $f_1 \in Q_1$.

Dually, we call (D, E, F) a *substring* of C if the following hold:

- (1) Either $|D| = 0$ or $D = d_1 \cdots d_n$ where $d_n \in Q_1$;
- (2) Either $|F| = 0$ or $F = f_1 \cdots f_m$ where $f_1 \in Q_1^-$.

Let $\text{fac}(C)$ be the set of factor strings of C , and by $\text{sub}(C)$ we denote the set of substrings of C . If C_1 and C_2 are strings, then we call a pair $((D_1, E_1, F_1), (D_2, E_2, F_2))$ in $\text{fac}(C_1) \times \text{sub}(C_2)$ *admissible* if $E_1 \sim E_2$. Denote the set of admissible pairs in $\text{fac}(C_1) \times \text{sub}(C_2)$ by $\mathcal{A}(C_1, C_2)$.

For each $a = ((D_1, E_1, F_1), (D_2, E_2, F_2))$ in $\mathcal{A}(C_1, C_2)$ we define a homomorphism

$$f_a : M(C_1) \longrightarrow M(C_2)$$

as follows: If $E_1 = E_2$, then let

$$f_a(z_{|D_1|+i}) = z_{|D_2|+i}$$

for $1 \leq i \leq |E_1| + 1$, and all other canonical basis vectors of $M(C_1)$ are mapped to 0. If $E_1 = E_2^-$, then let

$$f_a(z_{|D_1|+i}) = z_{|D_2 E_2|+2-i}$$

for $1 \leq i \leq |E_1| + 1$, and all other canonical basis vectors are mapped to 0. Such homomorphisms are called *graph maps*. It is proved in [6] that the graph maps $\{f_a \mid a \in \mathcal{A}(C_1, C_2)\}$ form a basis of the homomorphism space $\text{Hom}_A(M(C_1), M(C_2))$.

There is the following multiplicative behaviour of graph maps: The composition $f_a f_b$ of graph maps is either 0 or a graph map. This fact is very important and has numerous applications.

If $a = ((D_1, E_1, F_1), (D_2, E_2, F_2))$ is admissible, then we call a and f_a *oriented* if $E_1 = E_2$. We say that a and f_a are *left-sided*, or *right-sided* if they are oriented and $|D_1| = |D_2| = 0$, or $|F_1| = |F_2| = 0$, respectively. In case a is left-sided or right-sided, one calls a and f_a *one-sided*. Finally, a and f_a are *weakly one-sided* if a or $((D_1, E_1, F_1), (F_2^-, E_2^-, D_2^-))$ are one-sided, and they are *two-sided* if they are not weakly one-sided.

Let $C_1 = D_1 E_1 F_1$ and $C_2 = D_2 E_2 F_2$. A graph map f_a can always be transformed to an oriented graph map. This is done by composing f_a with one of the canonical isomorphisms $M(C_1^-) \rightarrow M(C_1)$ or $M(C_2) \rightarrow M(C_2^-)$. Note that these are both graph maps. Since there are two possibilities, we define $a(l) = a(r) = a$ if a is oriented, and $a(l) = ((F_1^-, E_1^-, D_1^-), (D_2, E_2, F_2))$ and $a(r) = ((D_1, E_1, F_1), (F_2^-, E_2^-, D_2^-))$, otherwise. It follows that $a(l)$ is one-sided if and only if $a(r)$ is one-sided if and only if a is weakly one-sided. The next lemma follows directly from the definitions.

Lemma 2.2. *Let f_a and f_b be weakly one-sided graph maps. If $f_{a(l)}$ and $f_{b(r)}$ are both left-sided, or both right-sided, then $f_{a(l)} f_{b(r)}$ is again left-sided, or right-sided, respectively. In particular, $f_a f_b$ is again weakly one-sided. Otherwise, $f_a f_b$ is 0 or two-sided.*

In [16] we use the term ‘perfect’ instead of ‘oriented’. Note also that we use left modules in [16] instead of right modules. Thus strings are written in the opposite way. One of the important steps in our work is to prove a stable version of the following proposition.

Proposition 2.3. *Let $A = kQ/(\rho)$ be a special biserial algebra, and let C_i , $1 \leq i \leq t$, be strings for (Q, ρ) . If each $a \in \mathcal{A}(C_i, C_j)$, $1 \leq i, j \leq t$, is weakly one-sided, then $\text{End}_A(\bigoplus_{i=1}^t M(C_i))$ is a gentle algebra.*

Proof. For A a string algebra, this statement is proved in [16, Proposition 4.8]. Now assume A is special biserial, and let $p - q$ be a commutativity relation in ρ . If M is a direct sum of string modules, then $M \cdot p$ and $M \cdot q$ are both 0. Thus we can define $B = A/(p_i, i \in I)$ where the p_i are all paths which are contained in commutativity relations. Clearly, B is a string algebra, and $\text{End}_A(M)$ is isomorphic to $\text{End}_B(M)$. \square

3. PROOF OF THE MAIN RESULT

Throughout this section, we assume that $A = kQ/(\rho)$ is special biserial, with ρ containing only zero relations or commutativity relations.

By $\underline{\text{Hom}}_A(M, N)$ we denote the A -module homomorphisms from M to N modulo the homomorphisms which factor through projective A -modules. Let $\underline{\text{mod}}(A)$ be the stable category of finite-dimensional A -modules. Here the objects are the same as in $\text{mod}(A)$, but the morphism space from M

to N is $\underline{\text{Hom}}_A(M, N)$. For a homomorphism f let \underline{f} be the corresponding morphism in $\underline{\text{mod}}(A)$.

It is known that any finite-dimensional indecomposable A -module is a string module, a band module, or a non-serial projective-injective module, see [5]. If P is an indecomposable non-serial projective-injective A -module, then the radical $\text{rad}(P)$ is isomorphic to a string module $M(CD)$ with C a direct and D an inverse string. Similarly, the socle factor $P/\text{soc}(P)$ is isomorphic to a string module $M(C'D')$ with C' an inverse and D' a direct string.

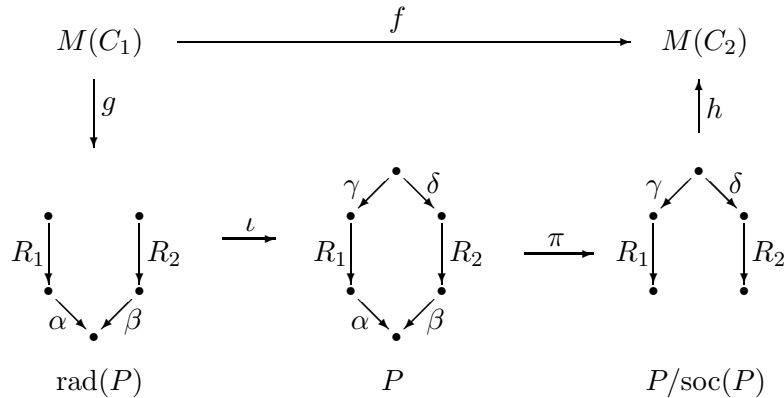
If N is a band module, then $\text{Ext}_A^1(N, N) \neq 0$ since band modules lie in homogeneous tubes of the Auslander-Reiten quiver of A , i.e. $\tau_A N = N$, where τ_A is the Auslander-Reiten translation. For basic facts on Auslander-Reiten theory we refer to [2] or [13].

So, if M is an A -module with $\text{Ext}_A^1(M, M) = 0$, then M is a direct sum of string modules and of non-serial projective-injectives. Since we are interested in stable homomorphisms, it is enough to consider only string modules.

For the rest of this section, let C_1 and C_2 be (not necessarily different) strings for (Q, ρ) .

Lemma 3.1. *Let $f : M(C_1) \rightarrow M(C_2)$ be a homomorphism which factors through a direct sum of non-serial projective-injective A -modules. Then f is a linear combination of two-sided graph maps.*

Proof. Clearly, it is enough to prove the lemma for the case that f factorizes through an indecomposable non-serial projective-injective A -module. Thus, let P be an indecomposable non-serial projective-injective A -module, and let $f_1 : M(C_1) \rightarrow P$ and $f_2 : P \rightarrow M(C_2)$ homomorphisms such that $f = f_1 f_2$. By $\iota : \text{rad}(P) \rightarrow P$ we denote the canonical radical inclusion, and by $\pi : P \rightarrow P/\text{soc}(P)$ the projection of P onto its socle factor. Clearly, f_1 must factor through $\text{rad}(P)$, and f_2 must factor through $P/\text{soc}(P)$. Thus there are homomorphisms g and h such that $f_1 = g\iota$ and $f_2 = \pi h$. The following picture describes the situation:



Now $g = \sum_i \lambda_i f_{a_i}$ and $h = \sum_j \mu_j f_{b_j}$ where a_i is of the form

$$((D_{1i}, E_{1i} E'_{1i}^-, F_{1i}), (D_{2i}, E_{2i}, F_{2i}))$$

with E_{1i} and E'_{1i} being direct strings, and b_j is of the form

$$((D_{3j}, E_{3j}^- E'_{3j}, F_{3j}), (D_{4j}, E_{4j}, F_{4j}))$$

with E_{3j} and E'_{3j} being direct strings. Thus we get

$$f = \sum_{i,j} \lambda_i \mu_j f_{a_i} \iota \pi f_{b_j}.$$

It is straightforward to check that for all i, j the composition $f_{a_i} \iota \pi f_{b_j}$ is either 0 or a sum of either one or two graph maps which are two-sided. \square

Lemma 3.2. *Let $f_{a_i} : M(C_1) \rightarrow M(C_2)$, $1 \leq i \leq s$, be pairwise different graph maps, which are weakly one-sided, with $\underline{f_{a_i}} \neq 0$. Then the $\underline{f_{a_i}}$ are linearly independent in $\underline{\text{Hom}}_A(M(C_1), M(C_2))$.*

Proof. Assume that $\sum_{i=1}^s \lambda_i \underline{f_{a_i}} = 0$ with $\lambda_i \neq 0$ for some i , and define $f = \sum_{i=1}^s \lambda_i f_{a_i}$. Thus there is a projective A -module $P = \bigoplus_{i=1}^l P(i)$, $P(i)$ indecomposable for all i , and homomorphisms $f_1 = [f_{11}, \dots, f_{1l}] : M(C_1) \rightarrow P$ and $f_2 = [f_{21}, \dots, f_{2l}]^t : P \rightarrow M(C_2)$ such that $f = f_1 f_2$. Thus $f_1 f_2 = \sum_{i=1}^l f_{1i} f_{2i}$. If $P(i)$ is not a non-serial projective-injective for some i , then f_{1i} and f_{2i} are both linear combinations of graph maps. By the multiplicativity of graph maps, we get that $f_{1i} f_{2i}$ is a linear combination of graph maps as well, and each of these factors through $P(i)$. Otherwise, if $P(j)$ is a non-serial projective-injective for some j , Lemma 3.1 yields that $f_{1j} f_{2j}$ is a linear combination of two-sided graph maps. But different graph maps are linearly independent, and the f_{a_i} are by assumption weakly one-sided. So all f_{a_i} with $\lambda_i \neq 0$ must factor through a projective, a contradiction. \square

Lemma 3.3. *Let f_a and f_b be weakly one-sided graph maps. If $f_{a(l)}$ and $f_{b(r)}$ are both left-sided, or both right-sided, and if $\underline{f_a} \neq 0 \neq \underline{f_b}$, then $\underline{f_a f_b} \neq 0$.*

Proof. Without loss of generality assume that f_a and f_b are both right-sided. Define $f_c = f_a f_b$. For the sake of brevity we just write ‘1’ instead of $1_{(e(E), -e(E))}$. Thus c is of the form

$$((D_1, E, 1), (D_2, E, 1)).$$

Assume that $\underline{f_a} \neq 0 \neq \underline{f_b}$. Note that this implies that $M(D_1 E)$ and $M(D_2 E)$ are both not projective. We have to show that $\underline{f_a f_b} \neq 0$.

Assume that $f_c = f_1 f_2$ is a factorization of f through a projective module P . Since f_c is weakly one-sided, and since different graph maps are linearly independent, it follows from Lemma 3.1 that we can assume that P does not contain non-serial projective-injective direct summands. Using the multiplicativity of graph maps, we can assume without loss of generality that $P = M(L^- R E)$ with L and R direct strings, $|R| \geq 1$, and f_1 and f_2

right-sided graph maps. In particular, E must be a direct string. Note that $D_2 = D'_2 R$ for some string D'_2 . We know that a is of the form

$$((D_{11}, D_{12}E, 1), (D_3, D_{12}E, 1)),$$

and b is of the form

$$((D_4, D_{22}E, 1), (D_{21}, D_{22}E, 1)).$$

Now let D_{122} be a direct string of maximal length such that $D_{12} = D_{121}D_{122}$ for some string D_{121} , and let D_{222} be a maximal direct string such that $D_{22} = D_{221}D_{222}$ for some string D_{221} . Since $f_c = f_a f_b$, we get that $|D_{12}| = 0$ or $|D_{22}| = 0$.

If $|D_{22}| = 0$, then f_b clearly factors through $P = M(L^-RE)$, a contradiction. Thus, assume $|D_{22}| \geq 1$. This implies $|D_{12}| = 0$. If now $|R| \leq |D_{222}|$, then f_a factors through P . Otherwise, if $|R| > |D_{222}|$, then $D_{22} = D_{222}$, since $D_2 = D'_2 R$ and since b is admissible. This implies that f_b factors through P . This finishes the proof. \square

Proposition 3.4. *Let $A = kQ/(\rho)$ be a special biserial algebra, and let C_i , $1 \leq i \leq t$, be strings for (Q, ρ) . If each f_a , $a \in \mathcal{A}(C_i, C_j)$, $1 \leq i, j \leq t$, is weakly one-sided or factors through a projective A -module, then $\underline{\text{End}}_A(\bigoplus_{i=1}^t M(C_i))$ is a gentle algebra.*

Proof. Since we are interested in algebras only up to Morita equivalence, we can assume that the $M(C_i)$ are pairwise not isomorphic. We can also assume that the $M(C_i)$ are not projective. For the sake of brevity let $M = \bigoplus_{i=1}^t M(C_i)$. By definition $\underline{\text{End}}_A(M) = \text{End}_A(M)/\mathcal{P}$ where \mathcal{P} is the ideal of all endomorphisms which factor through projectives.

The graph maps in $\text{Hom}_A(M(C_i), M(C_j))$, $1 \leq i, j \leq t$, form a basis of $\text{End}_A(M)$. By our assumption, we know that all two-sided graph maps are contained in \mathcal{P} . This implies that the weakly one-sided graph maps generate $\underline{\text{End}}_A(M)$ as a vectorspace. Let \mathcal{B} be the set of weakly one-sided graph maps f_a in $\text{End}_A(M)$ which satisfy $\underline{f_a} \neq 0$. It follows from Lemma 3.2 that $\underline{\mathcal{B}} = \{\underline{f_a} \mid f_a \in \mathcal{B}\}$ is a basis of $\underline{\text{End}}_A(M)$. This basis behaves again multiplicatively. Namely, let f_a and f_b be in \mathcal{B} . Then it follows from Lemma 2.2 that $\underline{f_a f_b} = 0$, or $\underline{f_a f_b} \in \underline{\mathcal{B}}$.

Now we can compute the quiver with relations of the algebra $\underline{\text{End}}_A(M)$. The vertices of this quiver are the identity maps $M(C_i) \rightarrow M(C_i)$. The arrows are the maps $\underline{f_a}$ in $\underline{\mathcal{B}}$ which are not of the form $\underline{f_b f_c}$ with $\underline{f_b}$ and $\underline{f_c}$ non-invertible elements in $\underline{\mathcal{B}}$. Now we proceed as in the proof of [16, Proposition 4.8]. As in [16, Lemmas 4.1, 4.2 and Corollary 4.3] we get that there are at most two arrows ending and at most two arrows starting at each vertex of the quiver of $\underline{\text{End}}_A(M)$. Thus we get property (1) in the definition of a gentle algebra. Then property (2) and (5) are proved as in [16, Lemma 4.4, 4.5]. However, to get (5) we have to use also Lemma 3.3. Since M is a finite-dimensional module, also $\underline{\text{End}}_A(M)$ must be finite-dimensional, thus we get (3). Then we copy the proof of [16, Lemma 4.6] and use Lemma

3.3 to show that each path $\underline{f_{a_1}} \cdots \underline{f_{a_l}}$, which is 0 in $\underline{\text{End}}_A(M)$, has to be of length 2. As in [16, Lemma 4.7] we get that there are no commutativity relations. Thus property (4) holds. This finishes the proof. \square

The following lemma is based on an idea of Ringel.

Lemma 3.5. *Let $f_a : M(C_1) \rightarrow M(C_2)$ be a two-sided graph map with $a = ((D_1, E_1, F_1), (D_2, E_2, F_2))$. If $D_1E_1F_2$ and $D_2E_1F_1$ are strings, or if $D_1E_1D_2^-$ and $F_2^-E_1F_1$ are strings, then $\text{Ext}_A^1(M(C_2), M(C_1)) \neq 0$.*

Proof. Without loss of generality assume $E_1 = E_2$, and set $E = E_1$. Thus $D_1ED_2^-$ and $F_2^-EF_1$ cannot be strings. If D_1EF_2 and D_2EF_1 are strings, we get a short exact sequence

$$0 \longrightarrow M(D_1EF_1) \longrightarrow M(D_1EF_2) \oplus M(D_2EF_1) \longrightarrow M(D_2EF_2) \longrightarrow 0$$

of A -modules, see [16] for a precise construction. Since a is two-sided, it follows that the middle term of this sequence is not isomorphic to the direct sum of its end terms. Thus the sequence does not split, which implies $\text{Ext}_A^1(M(C_2), M(C_1)) \neq 0$. \square

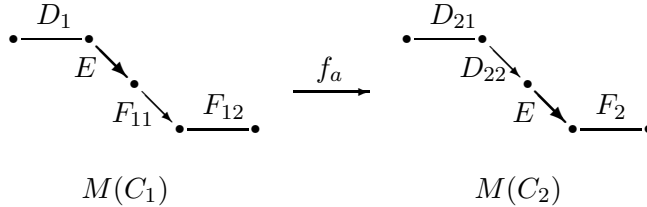
Corollary 3.6. *Let $f_a : M(C_1) \rightarrow M(C_2)$ be a two-sided graph map with $a = ((D_1, E_1, F_1), (D_2, E_2, F_2))$. If E_1 is not a direct or an inverse string, then $\text{Ext}_A^1(M(C_2), M(C_1)) \neq 0$.*

Proposition 3.7. *Let $f_a : M(C_1) \rightarrow M(C_2)$ be a two-sided graph map, and assume $\text{Ext}_A^1(M(C_2), M(C_1)) = 0$. Then $\underline{f_a} = 0$.*

Proof. Let $a = ((D_1, E_1, F_1), (D_2, E_2, F_2))$. Without loss of generality we can assume that $E_1 = E_2$. Set $E = E_1$. If D_1EF_2 and D_2EF_1 are both strings, then by Lemma 3.5 we get $\text{Ext}_A^1(M(C_2), M(C_1)) \neq 0$, a contradiction.

Thus, assume that D_2EF_1 is not a string. This implies that E must be a direct string.

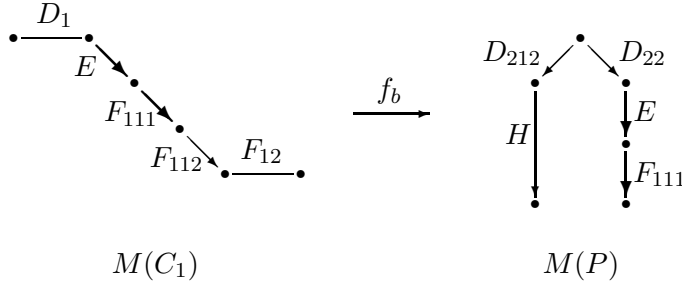
Let F_{11} be a direct string of maximal length such that $F_1 = F_{11}F_{12}$ for some string F_{12} . Similarly, let D_{22} be a maximal direct string such that $D_2 = D_{21}D_{22}$ for some D_{21} . Observe that $D_{22}EF_{11}$ is a string if and only if D_2EF_1 is a string. Thus $D_{22}EF_{11}$ is not a string. Since $D_{22}E$ and EF_{11} are both strings, we get that $|D_{22}| \geq 1$ and $|F_{11}| \geq 1$. Thus $f_a : M(C_1) \rightarrow M(C_2)$ can be visualised as follows:



Now let F_{111} be a maximal direct string such that $D_{22}EF_{111}$ is still a string. Thus there exists a (direct) string F_{112} with $|F_{112}| \geq 1$ such that $F_{11} = F_{111}F_{112}$. Next, let D_{212} be a maximal direct string such that $D_{21} = D_{211}D_{212}^-$ for some string D_{211} . Let H be a maximal direct string such that $D_{212}H$ is still a string. Define $P = H^-D_{212}^-D_{22}EF_{111}$. Since $|D_{22}| \geq 1$ and $|F_{112}| \geq 1$ we get a graph map $f_b : M(C_1) \rightarrow M(P)$ with

$$b = ((D_1, EF_{111}, F_{112}F_{12}), (H^-D_{212}^-D_{22}, EF_{111}, 1_{(e(F_{111}), -e(F_{111}))})).$$

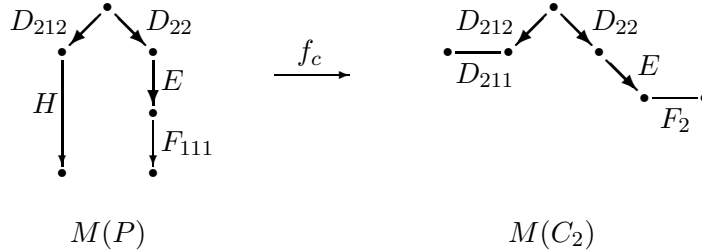
The corresponding picture looks as follows:



Observe that D_{211} is either of length 0 or ends with an arrow. Thus we get another graph map $f_c : M(P) \rightarrow M(C_2)$ with

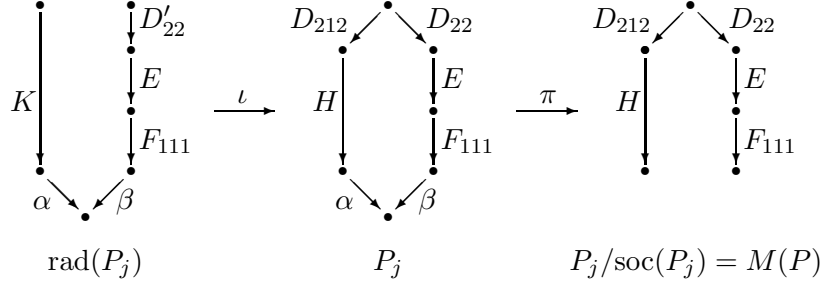
$$c = ((H^-, D_{212}^-D_{22}E, F_{111}), (D_{211}, D_{212}^-D_{22}E, F_2)),$$

and the corresponding picture is the following:



Next, one checks easily that $f_a = f_b f_c$. Thus f_a factors through $M(P)$. In case $M(P)$ is projective, we get $\underline{f_a} = 0$.

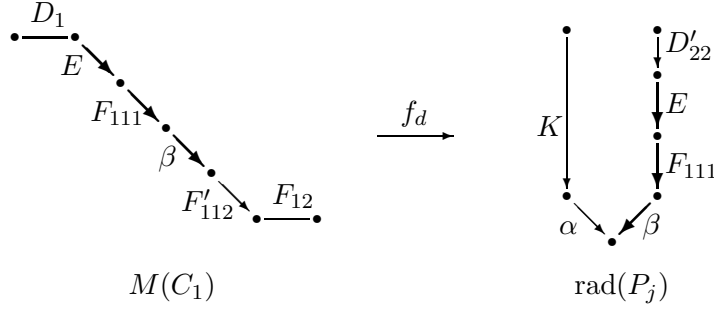
Otherwise, the indecomposable projective module P_j with top S_j , $j = s(D_{22})$, must be a non-serial projective-injective module. Assume we are in this case. This implies that there exists a commutativity relation $D_{212}H\alpha - D_{22}EF_{111}\beta$ in ρ where α and β are arrows. We know that $D_{212}H = \gamma K$ for some arrow γ and some (direct) string K , and $D_{22} = \delta D'_{22}$ for some arrow δ and some (direct) string D'_{22} . Thus the radical $\text{rad}(P_j)$ of P_j is isomorphic to $M(D'_{22}EF_{111}\beta\alpha^-K^-)$, and the socle factor $P_j/\text{soc}(P_j)$ is isomorphic to $M(P)$. The next picture describes P_j , its radical and socle factor together with the canonical inclusion and projection, respectively.



If $|F_{112}| > 1$, then it is easy to check that f_a factors through P_j . Namely, we have $F_{112} = \beta F'_{112}$ for some (direct) string F'_{112} of length at least 1. Then define $f_d : M(C_1) \rightarrow \text{rad}(P_j)$ where

$$d = ((D_1, EF_{111}\beta, F'_{112}F_{12}), (D'_{22}, EF_{111}\beta, \alpha^- K^-)).$$

Thus f_d looks as follows:



Now one checks easily that

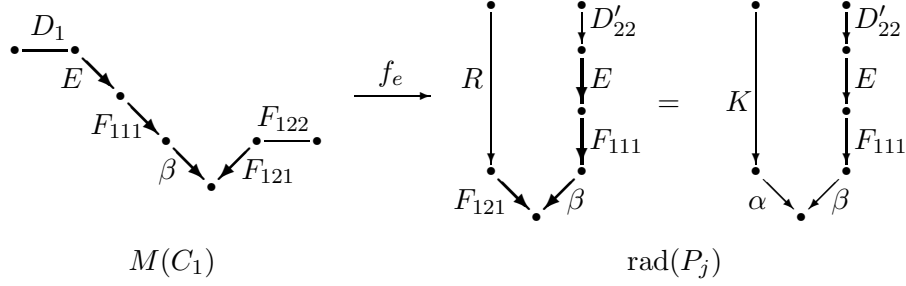
$$f_a = f_d \iota \pi f_c.$$

Thus f_a factors through a projective module.

Next, assume $|F_{112}| = 1$. This implies $F_{112} = \beta$. Let F_{121} be a maximal direct string such that $F_{12} = F_{121}^- F_{122}$ for some string F_{122} . Then there is a graph map $f_e : M(C_1) \rightarrow \text{rad}(P_j)$ with

$$e = ((D_1, EF_{111}\beta F_{121}^-, F_{122}), (D'_{22}, EF_{111}\beta F_{121}^-, R^-))$$

where R is a (direct) string such that $RF_{121} = K\alpha$. Thus f_e looks as in the following picture:



If $|F_{121}| < |H\alpha|$, then one easily checks that

$$f_a = f_e \iota \pi f_c.$$

Thus, again f_a factors through a projective.

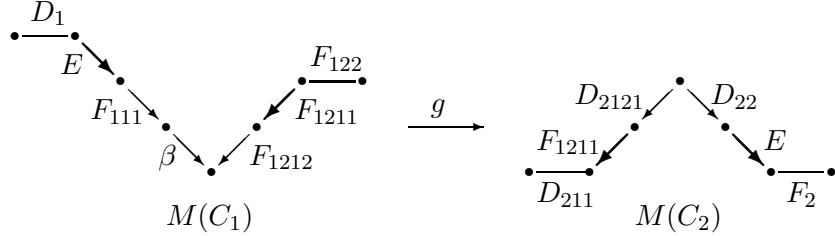
It remains to consider the case $|F_{121}| \geq |H\alpha|$. Note that this implies $|D_{212}| \geq 1$. It follows that there is some (direct) string F_{1211} such that $F_{121} = F_{1211}F_{1212}$ and $D_{212} = D_{2121}F_{1211}$ for some (direct) strings F_{1212} and D_{2121} . Note that $|F_{1212}| \geq 1$. Thus we get a graph map $f_l : M(C_1) \rightarrow M(C_2)$ with

$$l = ((D_1 E F_{111} \beta F_{1212}^-, F_{1211}^-, F_{122}), (D_{211}, F_{1211}^-, D_{2121}^- D_{22} E F_2)).$$

We get

$$f_a + f_l = f_e \iota \pi f_c.$$

Define $g = f_a + f_l$. Clearly, we have $\underline{g} = 0$. The following picture illustrates the situation:



Now we need some Auslander-Reiten theory: If $|D_1| \geq 1$, then f_a factors through $\tau_A^{-1}M(C_1)$. This can be checked by using the construction of Auslander-Reiten sequences for string algebras as explained in [5]. Here we use also that $|F_{111}| \geq 1$. Then we use the Auslander-Reiten formula

$$\text{Ext}_A^1(M(C_2), M(C_1)) \simeq D \underline{\text{Hom}}_A(\tau_A^{-1}M(C_1), M(C_2)).$$

Thus, since $\text{Ext}_A^1(M(C_2), M(C_1)) = 0$, we get $\underline{f_a} = 0$.

Similarly, if $|F_{122}| \geq 1$, then f_l factors through $\tau_A^{-1}M(C_1)$. Here we use that $|F_{1212}| \geq 1$. Again, since $\text{Ext}_A^1(M(C_2), M(C_1)) = 0$, we get $\underline{f_l} = 0$. But if $\underline{g} = \underline{f_l} = 0$, then we get also $\underline{f_a} = 0$.

Thus, it remains to consider the case $|D_1| = |F_{122}| = 0$. Then we can construct a short exact sequence

$$0 \longrightarrow M(C_1) \longrightarrow P_j \oplus M(D_{211}F_{1211}^-) \oplus M(EF_2) \xrightarrow{h} M(C_2) \longrightarrow 0.$$

Here

$$h = [\pi f_c, f_m, f_n]^t$$

where f_m and f_n are graph maps with

$$m = ((1, D_{211}F_{1211}^-, 1), (1, D_{211}F_{1211}^-, D_{212}^- D_{22}EF_2))$$

and

$$n = ((1, EF_2, 1), (D_{211}F_{1211}^- D_{212}^- D_{22}, EF_2, 1)).$$

For the sake of brevity, in the definition of m and n we just wrote ‘1’ for all strings of length 0. Next, one checks easily that h is an epimorphism with kernel isomorphic to $M(C_1)$. This sequence does obviously not split, thus $\text{Ext}_A^1(M(C_2), M(C_1)) \neq 0$, a contradiction. Altogether, we proved the following: If D_2EF_1 is not a string, then the Proposition holds.

Thus, assume now that D_2EF_1 is a string. If D_1EF_2 is also a string, then we apply Lemma 3.5 and get that $\text{Ext}_A^1(M(C_2), M(C_1)) \neq 0$, again a contradiction. Otherwise, if D_1EF_2 is not a string, then we proceed as before and get that the Proposition holds. This finishes the proof. \square

Proof of Theorem 1.1. Let A be a special biserial algebra, and let M be an A -module with $\text{Ext}_A^1(M, M) = 0$. Thus M does not contain direct summands which are isomorphic to band modules. Furthermore, since we are interested in stable endomorphism algebras, we can assume that M does not contain projective direct summands. Also, since we consider algebras only up to Morita equivalence, we assume that M is a direct sum of pairwise different indecomposable modules. Thus M is isomorphic to a direct sum $\bigoplus_{i=1}^t M(C_i)$ of pairwise different non-projective string modules $M(C_i)$.

The endomorphism algebra $\text{End}_A(M)$ has a basis consisting of the graph maps in $\text{Hom}_A(M(C_i), M(C_j))$, $1 \leq i, j \leq t$. By Proposition 3.7 we know that any two-sided graph map $M(C_i) \rightarrow M(C_j)$ factorizes through a projective module. Hence we can apply Proposition 3.4 and get that $\underline{\text{End}}_A(M)$ is a gentle algebra. \square

Proof of Corollary 1.2. If B is a finite-dimensional k -algebra, then $\underline{\text{mod}}(B)$ is a cosuspended category with loop functor Ω , where Ω is the syzygy functor. For B -modules M and N and $n \geq 1$ we have

$$\underline{\text{Hom}}_B(\Omega^n(M), N) = \text{Ext}_B^n(M, N)$$

where the $\text{Ext}_B^n(M, N)$ are the usual groups of module extensions. Now let A be a finite-dimensional k -algebra. Then there is a full and faithful embedding

$$F : D^b(A) \longrightarrow \underline{\text{mod}}(RA)$$

of triangulated categories, where RA is the repetitive algebra of A . For details we refer to [8, Chapter 2] and [9, Section 3]. For any two bounded complexes M and N of A -modules, we get

$$\begin{aligned} \mathrm{Hom}_{D^b(A)}(M[-1], N) &= \underline{\mathrm{Hom}}_{RA}(F(M[-1]), F(N)) \\ &= \underline{\mathrm{Hom}}_{RA}(\Omega(F(M)), F(N)) \\ &= \mathrm{Ext}_{RA}^1(F(M), F(N)), \end{aligned}$$

since F is full, faithful and respects the triangulated structure.

Now assume that A is a finite-dimensional gentle algebra, and let T be a complex in $D^b(A)$ such that $\mathrm{Hom}_{D^b(A)}(T, T[1]) = 0$. Using the above considerations, we get $\mathrm{Ext}_{RA}^1(F(T), F(T)) = 0$. Since A is gentle, we know that RA is special biserial. Furthermore, $\mathrm{End}_{D^b(A)}(T)$ and $\underline{\mathrm{End}}_{RA}(F(T))$ are isomorphic. Thus we can apply Theorem 1.1 and get that $\mathrm{End}_{D^b(A)}(T)$ is a gentle algebra.

If T is a tilting complex, then by definition $\mathrm{Hom}_{D^b(A)}(T, T[i]) = 0$ for every $i \neq 0$. But two algebras A and B are derived equivalent if and only if there is a tilting complex T in $D^b(A)$ such that B is isomorphic to $\mathrm{End}_{D^b(A)}(T)$, see [10] or [12]. Thus any algebra which is derived equivalent to a finite-dimensional gentle algebra is gentle again.

It is known that two derived equivalent finite-dimensional algebras A and B have the same number of isomorphism classes simple modules [10, Lemma 6.3.3]. Since there are (up to Morita equivalence) only finitely many gentle algebras with a given number of isomorphism classes of simple modules, also the last statement in Corollary 1.2 holds. \square

4. EXAMPLES AND REMARKS

Example 1. Let A be the algebra $k[x, y]/(x^2, y^2, xy)$ where $k[x, y]$ is the polynomial algebra in two commuting variables. Observe that A is special biserial. Now let M be the string module $M(x^-y^-y)$. One easily checks that $\mathrm{Ext}_A^1(M, M) = 0$ and $\mathrm{End}_A(M) = k[x_1, \dots, x_6]/(x_i x_j, 1 \leq i, j \leq 6)$, which is obviously not gentle and also far from being special biserial. But we get $\underline{\mathrm{End}}_A(M) = k$. Thus Theorem 1.1 cannot be extended to a result on endomorphism algebras, rather than stable endomorphism algebras.

Example 2. Now let $A = k[x, y]/(x^3, y^3, xy)$ which is again special biserial. Let M be the band module $M(x^-x^-yx^-yy, 1, 1)$, see [5] for the construction of M . Since M is a band module we know that $\mathrm{Ext}_A^1(M, M) \neq 0$. The endomorphism algebra $\mathrm{End}_A(M)$ is a factor of the free algebra in 5 non-commuting variables, and we get $\underline{\mathrm{End}}_A(M) = k$. On the other hand, if we interpret M as a module over the algebra $B = k[x, y]/(x^5, y^5, xy)$, then one easily checks that $\mathrm{End}_A(M) \simeq \mathrm{End}_B(M) \simeq \underline{\mathrm{End}}_B(M)$.

Example 3. The algebra $A = k[x]/(x^2)$ is gentle, the simple module S satisfies $\mathrm{Ext}_A^1(S, S) \neq 0$, and $\mathrm{End}_A(S) = \underline{\mathrm{End}}_A(S) = k$ is gentle.

Remark. There are still several important open problems: One needs criteria to decide when two given gentle algebras are derived equivalent.

Also it is still not clear whether tilting-cotilting equivalence is the same as derived equivalence. This holds in the case of gentle one-cycle algebras, see [3] and [4].

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