Tilting Hereditary Orders

Steffen König and Alexander Zimmermann[1]

Mathematisches Institut B
Universität Stuttgart
D–70550 Stuttgart, Germany

Abstract

We give a combinatorial description of the class of tilting complexes over a hereditary order. Moreover, we determine from the combinatorial data the endomorphism rings of these tilting complexes, that is all the rings which are derived equivalent to a hereditary order.

1 Introduction and statement of the main result

Classical tilting theory tries to compare the module categories of two rings by means of functors associated with certain bimodules, the so called *tilting modules*. In the representation theory of finite dimensional algebras, this method has led to a good knowledge of many module categories, in particular by starting with a hereditary algebra and then applying an iterative tilting process. By a result of Happel [3], tilting can be understood as a special case of an equivalence of the derived module categories of the rings involved. Thus, a more general question is to ask for *derived equivalences*. Rickard [6] has given a necessary and sufficient criterion for the existence of a derived equivalence between two rings. The criterion is based on the existence of a so called *tilting complex*, which generalizes the notion of tilting modules.

For non–artinian algebras, in particular for orders, only few examples of derived equivalences are known. Among the most prominent examples are derived equivalences between certain blocks of group rings, for example in the case of cyclic defect. The study of derived equivalences for orders, especially for group rings, is motivated by M.Broué's conjecture [1] which states that a

[1]Both authors acknowledge financial support from the Deutsche Forschungsgemeinschaft.
1991 AMS Mathematics Subject Classification: Primary 16G30, Secondary 18E30, 16E60.

block B of abelian defect and its Brauer correspondent block b in the group ring of the normalizer of the defect group are derived equivalent.

The experience from finite dimensional algebras suggests to study derived equivalences also for hereditary orders. We will give a complete combinatorial description of all tilting complexes over hereditary orders, and also of all rings which are derived equivalent to a hereditary order. It turns out, that for a given hereditary order Λ, up to Morita equivalence and isomorphism, there are only finitely many rings, which are derived equivalent to Λ. We will describe explicitly all these rings. Thus, the situation is completely understood (but of course by far not as rich as in the artinian case).

It turns out, that the derived equivalences involving hereditary orders are rather different from the known examples of derived equivalences involving integral group rings:

Firstly, a ring which is derived equivalent to the hereditary order Λ either is Morita equivalent to Λ or not an order at all.

Secondly, for a discrete valuation domain R, the existence of a derived equivalence between the hereditary R–order Λ and the ring Γ in general does not imply the existence of a derived equivalence of the residue class rings $\Lambda/(\pi\Lambda)$ and $\Gamma/(\pi\Gamma)$ modulo a prime element π of R. This is in contrast to the fact, proved in [8], that each tilting complex over $\Lambda/(\pi\Lambda)$ can be lifted uniquely to a tilting complex over Λ in such a way that the endomorphism ring of the lifted complex reduces modulo π to the endomorphism ring of the original tilting complex. There is no contradiction, however, since there are many rings that reduce to a given $R/(\pi R)$–algebra.

And thirdly, these derived equivalences usually are not induced by a twosided tilting complex T which consists of bimodules which are projective on either side. Derived equivalences induced by twosided tilting complexes of bimodules such that the restriction of the twosided tilting complex to either side is isomorphic to a tilting complex even *in the homotopy category* are studied extensively by J.Rickard [9] and the study is motivated by M.Broué's concept of isotypies [1].

Before we can formulate our main result, the combinatorial description of tilting complexes over hereditary orders, we remind the reader of the rôle tilting complexes are playing in the theory of derived categories.

By Rickard's fundamental theorem [6], the rings which are derived equivalent to a ring Λ are precisely the endomorphism rings of tilting complexes over Λ. A tilting complex T is a complex of finitely generated projective modules, bounded above and bounded below, which does not admit self–extensions and which has the property that the smallest triangulated subcategory of $D^b(\Lambda - mod)$ which contains all direct summands of finite sums of T also contains all the finitely generated projective modules.

Let us also recall the structure theory of hereditary orders.

Let R be a complete, discrete, rank 1 valuation ring with field of fractions K and let Λ be a hereditary R–order in a simple K-algebra A. Then A is Morita equivalent to a skew field D; we denote by Δ the unique maximal order in D and by π_D a prime element in Δ ([5]). The unique maximal ideal in Δ is generated by π_D and is denoted by (π_D). Without loss of generality we may assume that Λ is basic and connected, thus is (up to an isomorphism) of the following form ([2, 4]), say with n rows and columns:

$$\Lambda \simeq \begin{pmatrix} \Delta & \dots & \dots & \dots & \Delta \\ (\pi_D) & \ddots & & & \vdots \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ (\pi_D) & \dots & \dots & (\pi_D) & \Delta \end{pmatrix}$$

Taking left modules, the isomorphism classes of indecomposable projective Λ–lattices are numbered $P(i)$ (where $i \in \mathbb{Z}/n\mathbb{Z}$) in such a way that

$$rad(P(i+1)) \simeq P(i).$$

Thus, $P(i)$ may and will be chosen as the i–th column in the above matrix.

We fix now the hereditary order Λ with all the data above, in particular n, the 'size' of Λ, is fixed.

We want to study the following set:

Definition 1.1 *The set $\mathfrak{TC}$ contains the equivalence classes of the multiplicity free tilting complexes over Λ modulo isomorphisms and modulo shift in the derived category $D^b(\Lambda - mod)$.*

This set will be shown to be in bijection with the following set which will turn out to be rather easily described.

For the definition we need the following notation:

Fix a circle in the plane and n different vertices on the circle, numbered clockwise by $1, \dots, n$, identified in the natural way with elements of $\mathbb{Z}/n\mathbb{Z}$.

An interval $(\bar{i}, \bar{j})$ with $\bar{i}, \bar{j} \in \mathbb{Z}/n\mathbb{Z}$ is just the smallest non empty image of intervals (i, j) in $\mathbb{Z}/n\mathbb{Z}$ under the residue mapping with $i + n\mathbb{Z} = \bar{i}$ and $j + n\mathbb{Z} = \bar{j}$.[2] Similarly, $(\bar{i}, \bar{j}] = (\bar{i}, \bar{j}) \cup \{\bar{j}\}$ and $[\bar{i}, \bar{j}) = (\bar{i}, \bar{j}) \cup \{\bar{i}\}$.

Definition 1.2 *The elements of the set $\mathfrak{C}$ are quadruples (n_1, n_2, M_1, M_2) satisfying the following conditions:*

[2] see Section 2.3 for a more formal definition

(a) *Both* n_1 *and* n_2 *are integers,* n_1 *is positive,* n_2 *is non–negative and the sum* $n_1 + n_2$ *equals* n *(the size of the hereditary order),*

(b) *the set* M_1 *contains* n_1 *of the fixed vertices (which in the following will be called 'stars'),*

(c) *the set* M_2 *contains* n_2 *ordered pairs* (i, j) *of fixed vertices (which in the following will be seen as 'arrows' going from* i *to* j *and which will be written* $i \longrightarrow j$*),*

(d) **(0)** *there is a star at the vertex 1; if* i *is the starting vertex of an arrow ending at 1 and* j *is the ending vertex of an arrow starting at 1, then the intervals* $(1, j]$ *and* $[i, 1)$ *have empty intersection,*

(I) *two different arrows (drawn as straight lines) do not intersect in interior points of the circle,*

(II) *viewing the union of the arrows as an (non oriented) graph, it is the disjoint union of* n_1 *trees, each of them containing exactly one vertex contained in* M_1*,*

(III) *if there is an arrow leading from* i *to* j*, then there is no star strictly between* i *and* j*.*

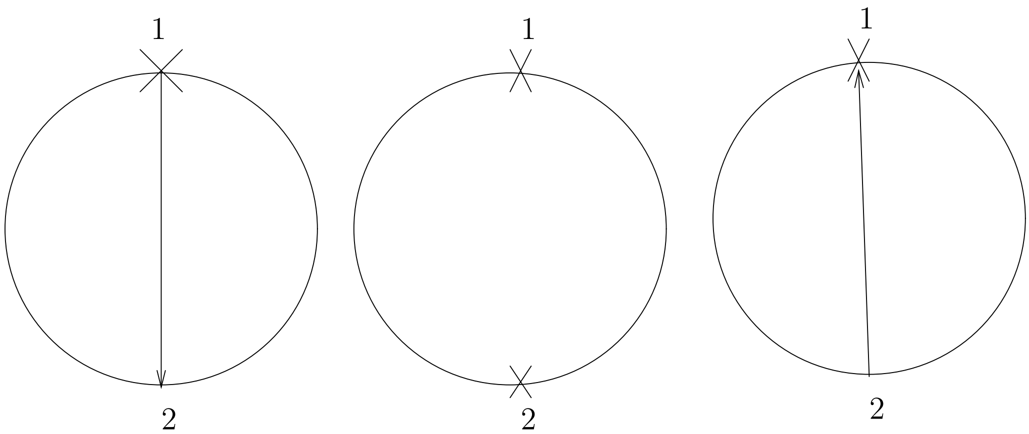
The set $\mathfrak{C}$ is used to describe completely the set $\mathfrak{TC}$:

Theorem *There is an explicit bijection between the sets* $\mathfrak{TC}$ *and* $\mathfrak{C}$*.*

Bijective maps in both directions will be defined explicitly in the course of the proof, but we will illustrate the bijection by an example already now.

The simplest example is a circle with two vertices, $n = 2$. We denote by ι the natural embedding of $P(1)$ into $P(2)$. Then there are three objects in $\mathcal{C}$.

Objects in $\mathcal{C}$



Corresponding tilting complexes

$0 \to P(1)^2 \stackrel{(\iota,0)}{\to} P(2) \to 0$ $\qquad$ $0 \to P(1) \oplus P(2) \to 0$ $\qquad$ $0 \to P(2) \stackrel{\binom{\iota}{0}}{\to} P(1)^2 \to 0,$

$P(2)$ in degree 1, $\qquad$ homology in degree 0 $\qquad$ $P(2)$ in degree -1

Endomorphism rings

$$\begin{pmatrix} \Delta & \Delta/(\pi_D) \\ 0 & \Delta/(\pi_D) \end{pmatrix}, \qquad \Lambda \qquad \begin{pmatrix} \Delta/(\pi_D) & \Delta/(\pi_D) \\ 0 & \Delta \end{pmatrix}.$$

In section 3 we will give a much more illustrative description of the elements of $\mathfrak{C}$, which makes it easy to construct and to imagine examples.

2 The proof of the theorem

2.1 Indecomposable direct summands of tilting complexes

The arrows and the stars in the combinatorial objects will correspond to the indecomposable direct summands of the tilting complexes. Thus, our first aim is to describe all indecomposable objects in $D^b(\Lambda)$ having no self–extensions.

Since Λ is hereditary, an indecomposable object in $D^b(\Lambda)$ is up to shift an indecomposable module. An indecomposable finitely generated Λ–module X which does not have self–extensions is either a lattice (hence projective) or a uniserial torsion module with all composition multiplicities of simple modules being less or equal to one and not all being one. Thus, its projective cover is of the form $P \to Q$ with P and Q being indecomposable projective (and not isomorphic) and the map being one with maximal possible image. In the following, maps between projectives, which are not given explicitly, always are projective cover maps of torsion modules without self–extensions.

Lemma 2.1 *Let T be a tilting complex of Λ. Then each indecomposable summand S of T is up to a shift isomorphic to an indecomposable projective module, or S is up to a shift isomorphic to a two term complex $P \longrightarrow Q$ with homology concentrated in degree 0 and the projective modules P and Q are indecomposable and the differential is just the mapping with the maximal possible image.* ■

There is an obvious way of representing an indecomposable projective module P by a star at P and a torsion module without self–extensions by an arrow $P \to Q$ describing its projective resolution. Of course, T is up to an isomorphism a direct sum of shifted copies of such modules. We will see later, that these data really determines T.

2.2 The existence of a star at 1

Lemma 2.2 *Let T be a tilting complex of the hereditary R–order Λ. Then, the total homology of T has a projective indecomposable direct summand.*

Proof. Assume to the contrary, all summands would be torsion. Then the endomorphism ring would be artinian and tilting back would imply that Λ is artinian. ■

Thus, there is at least one indecomposable direct summand of T representing a projective module. To assume that the projective indecomposable module is $P(1)$, is no loss of generality, since we always may conjugate by a suitable power of

$$\begin{pmatrix} 0 & 1 & 0 & \dots & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & 0 \\ 0 & \dots & \dots & \dots & 0 & 1 \\ \pi_D & 0 & \dots & \dots & \dots & 0 \end{pmatrix}$$

We shift T in order to bring $P(1)$ into degree 0.

2.3 Extensions between different partial tilting complexes

To produce a tilting complex as a direct sum of partial tilting complexes[3] one has in particular to exclude extensions between them. This is what we study next. We assume that we are given a tilting complex T.

[3]Direct summands of tilting complexes are called partial tilting complexes

Lemma 2.3 *Let $P \longrightarrow Q$ be an indecomposable direct summand of T. The complex has homology concentrated in degree i, say. Then,*

- *if $X := (P \longrightarrow S)$ is another direct summand of T, then the homology of X is concentrated in degree i,*
- *if $X := (S \longrightarrow P)$ is another direct summand of T, then the homology of X is concentrated in degree $i-1$,*
- *if P is another direct summand of T, then $i = 1$.*

Proof. The mappings

$$\begin{array}{ccc} P & \longrightarrow & Q \\ \| & & \downarrow \\ P & \longrightarrow & 0 \end{array}$$

and in case of $Q \subseteq S$

$$\begin{array}{ccc} P & \longrightarrow & Q \\ \| & & \downarrow \\ P & \longrightarrow & S \end{array}$$

as well as

$$\begin{array}{ccc} P & \longrightarrow & Q \\ \| & & \uparrow \\ P & \longrightarrow & S \end{array}$$

in case of $Q \supseteq S$, and

$$\begin{array}{ccccc} 0 & \longrightarrow & P & \longrightarrow & Q \\ \downarrow & & \| & & \downarrow \\ S & \longrightarrow & P & \longrightarrow & 0 \end{array}$$

are all not homotopic to zero. ■

Remark 2.1 Lemma 2.3 tells us, that each projective indecomposable module occurs in at most one degree.

We now determine the precise conditions when two indecomposable complexes without self–extensions do not have extensions in either direction.

Before we formulate the result, we have to remind the reader of our convention, that an interval (Q, R) is defined by starting at the vertex succeeding Q and walking clockwise through the circle until arriving at the vertex preceding R. More formally:

Notation 1 An interval $(i + n\mathbb{Z}, j + n\mathbb{Z})$ with $i, j \in \mathbb{Z}$ is the smallest non empty image of intervals $(i + nz_1, j + nz_2)$ with $z_1, z_2 \in \mathbb{Z}$ under the natural projection $\mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z}$. That means for $i, j \in \mathbb{Z}/n\mathbb{Z}$,

- $(i+n\mathbb{Z}, j+n\mathbb{Z}) := \{k |$ there is a $d \in \mathbb{Z} : i < k < j + dn, \frac{(i-j)}{n} < d \leq 1 + \frac{(i-j)}{n}\}$,
- $(i, j] := (i, j) \cup \{j\}$,
- $[i, j) := (i, j) \cup \{i\}$,
- $[i, j] := [i, j) \cup (i, j]$.

An interval of vertices in the circle is always identified with the interval in $\mathbb{Z}/n\mathbb{Z}$ under the natural identification of the vertices of the circle with $\mathbb{Z}/n\mathbb{Z}$.

An interval of projective modules is defined to be the interval in $\mathbb{Z}/n\mathbb{Z}$ via the natural identification of the vertices with $\mathbb{Z}/n\mathbb{Z}$. Thus, *an interval never is empty.*

Lemma 2.4 *Let $Q \to R$ and $S \to T$ be two indecomposable complexes, not homotopic to the zero complex, and assume that both do not have self–extensions. Assume that Q is in degree 0 and S in degree $i \geq 0$. Then all homomorphisms of non zero degree between these complexes are homotopic to zero if and only if the following conditions are satisfied:*

(1) if $i = 0$: $Q \neq T$ and $R \neq S$ and the intervals (Q, R) and (S, T) have empty intersection.

(2) if $i = 1$: $Q \neq S, T$ and $R \neq T$ and the intervals (Q, R) and (S, T) have empty intersection.

(3) if $i > 1$: Q, R, S, T are pairwise different and the intervals (Q, R) and (S, T) have empty intersection.

Proof. We start with case (1). Here we have to consider when all homomorphisms

$$\begin{array}{ccccc} 0 & \to & Q & \to & R \\ \downarrow & & \downarrow & & \downarrow \\ S & \to & T & \to & 0 \end{array}$$

and also all homomorphisms

$$\begin{array}{ccccc} 0 & \to & S & \to & T \\ \downarrow & & \downarrow & & \downarrow \\ Q & \to & R & \to & 0 \end{array}$$

are homotopic to zero. All the homomorphisms in the first diagram are homotopic to zero if and only if $Q \neq T$ and moreover $S \in [Q, T)$ or $R \in (Q, T]$. All the homomorphisms in the second diagram are homotopic to zero if and only if $S \neq R$ and moreover $Q \in [S, R)$ or $T \in (S, R]$. Altogether, this is equivalent to the condition in (1).

In case (2) we have to consider the first of the situations above and also the situations

$$\begin{array}{ccc} Q & \rightarrow & R \\ \uparrow & & \uparrow \\ S & \rightarrow & T \end{array}$$

and

$$\begin{array}{ccc} S & \rightarrow & T \\ \uparrow & & \uparrow \\ Q & \rightarrow & R \end{array}$$

which altogether give the condition in (2). (Note that both complexes represent torsion modules and there are no homomorphisms in either direction if and only if either there are no common composition factors or one module is a proper quotient of a proper submodule of the other module.)

In case (3) we have to consider all the situations of (1) and (2). ■

In a very similar way one proves the next result:

Lemma 2.5 *Let P be an indecomposable projective module viewed as a complex concentrated in degree 0 and $Q \rightarrow R$ an indecomposable complex not homotopic to the zero complex and without self-extensions such that Q is in degree i. Then all homomorphisms of non–zero degree between P and $Q \rightarrow R$ are homotopic to zero if and only if:*

(1) in case $i = 0$: $P \in (R, Q]$.

(2) in case $i = -1$: $P \in [R, Q)$.

(3) in all other cases: $P \in (R, Q)$. ■

2.4 Starting with a tilting complex

Now we define the map from $\mathfrak{TC}$ to $\mathfrak{C}$.

We attach n_1 stars P and n_2 arrows $Q \rightarrow R$ to T in the following way: an indecomposable projective direct summand P of T is attached to a star (also named P) and an indecomposable direct summand $Q \rightarrow R$ is attached to an arrow $Q \rightarrow R$. Observe that Q and R are necessarily indecomposable projective.

Now we have to check that this object satisfies all the combinatorial rules.

Lemma 2.4 and Lemma 2.5 give rule (I) and (III). The first part of rule (0) is shown in Lemma 2.2. The second part of rule (0) is shown in Lemma 2.4 (1).

The remaining part is rule (II). This will be done in the following lemmata.

Lemma 2.6 *Let T be a tilting complex and associate to its direct summands stars and arrows as above. Then,*

1. *for each projective indecomposable module P there is a projective indecomposable module Q such that one of the following indecomposable complexes is a direct summand of T:*

   (a) $P \longrightarrow Q$

   (b) $Q \longrightarrow P$

   (c) P

2. *each arrow lies on a path and each path contains a star.*

Here, as before, by path we mean a connected component of the (non–oriented) graph underlying the union of the arrows.

Proof.

1. Each projective module must occur, since for every simple module S there is a $j(S) \in \mathbb{Z}$ such that $Hom_{D^b(\Lambda)}(T, S[j(S)]) \neq 0$.

2. Let $Q \to R$ be an arrow. Since Λ is connected, there is a projective P and a sequence of non–trivial (i.e. not homotopic to zero) maps between indecomposable summands of T, which connects P and $Q \to R$.

In order to show that this defines a path, we have to study the non–zero maps between the various direct summands. We will show that non–zero maps occur between arrows only if they have a common point and between an arrow and a star only if the star lies on either the ending or the starting point of the arrow.

There is a non–zero map

$$\begin{array}{ccc} Q & \to & R \\ \downarrow & & \downarrow \\ P & \to & 0 \end{array}$$

from an arrow to a star if and only if $P \in [Q, R)$. But the vanishing of $Ext^1(T, T)$ tells us, that also $P \in (R, Q]$, hence $P = Q$. Similarly, for an arrow $S \to T$ with homology concentrated in degree 0 we obtain $P = S$.

For maps between arrows representing indecomposable complexes with homology in the same degree, there is the following argument: an arrow represents a torsion module, and between torsion modules there can be a non–zero map only if they share a composition factor, that is if and only if the arrows have a common point (note that arrows must not intersect by Lemma 2.4).

The last case is that of an arrow $Q \to R$ and another arrow $S \to T$, where the homology is concentrated in, is one. Assume $Q \neq T$. What we need is a homomorphism

$$\begin{array}{ccccc} 0 & \to & Q & \to & R \\ \downarrow & & \downarrow & & \downarrow \\ S & \to & T & \to & 0 \end{array}$$

which is not homotopic to zero (note that in the other direction, 0 is the only homomorphism). This is possible if and only if both R and S lie in (T, Q). Moreover, we know that $S \in (T, R)$ since $Ext^1(T, T) = 0$. There exists a star P and Lemma 2.5 implies that $P \in [R, Q]$ and also that $P \in [T, S]$. This implies $S = R$, which gives a contradiction. So we have $Q = T$ and hence the existence of a non–trivial homomorphism is obvious. ■

Lemma 2.7 *The number of (non–isomorphic) direct summands of T is exactly n.*

Proof. The Grothendieck group of the endomorphism ring A of T is isomorphic to the Grothendieck group of Λ, but is also free of rank l, where l is the number of indecomposable direct summands of T. ■

From the arguments in the proof of the preceding lemma, one may get the structure of A. The precise structure of this endomorphism ring will be given in section 4.

Lemma 2.8 *A path contains exactly one star, and each path is a tree.*

Proof. Consider a path and a star P on it (which we already know to exist by Lemma 2.6). We count the number a of indecomposable projectives occurring in the polygon and also the number b of stars and arrows in it. We prove the claim by induction on the number b. The start is obvious. If we add an arrow to an existing path, this increases both a and b by one, except if the arrow closes a cycle. In the latter case, a remains unchanged whereas b is increased by one. If we add another star to a path, a again remains unchanged, and b again is increased by one. Thus, a equals b if and only if the path contains exactly one star and there is no cycle, and otherwise, a is strictly smaller than b. Now we sum up both numbers over all paths, and by the previous lemma the result for both numbers must be n. ■

This finishes the first part of the proof: The combinatorial object which has been associated with the tilting complex T satisfies all the rules from (0) to (III).
In the following subsection we show the converse.

2.5 Starting with a combinatorial object

Assume we are given an element of $\mathfrak{C}$.

From Subsection 2.4 together with Lemma 2.3 it follows how to construct a complex T which is a direct sum of indecomposable projective lattices P in degree 0 and of complexes $Q \to R$ with homology in various other degrees. We retain the notation of the first part of the proof and note that the Lemmata 2.4 and 2.5 show already that T does not have self-extensions.

What we still have to show is that the smallest triangulated subcategory $\mathfrak{X}$ of $D^b(\Lambda)$ which contains the direct summands of T also contains all direct summands of Λ (up to isomorphism).

Let P be an indecomposable projective lattice and $Q \to P$ and $P \to R$ be arrows. Then there are maps

$$\begin{array}{ccc} Q & \to & P \\ \uparrow & & \uparrow id \\ 0 & \to & P \end{array}$$

and

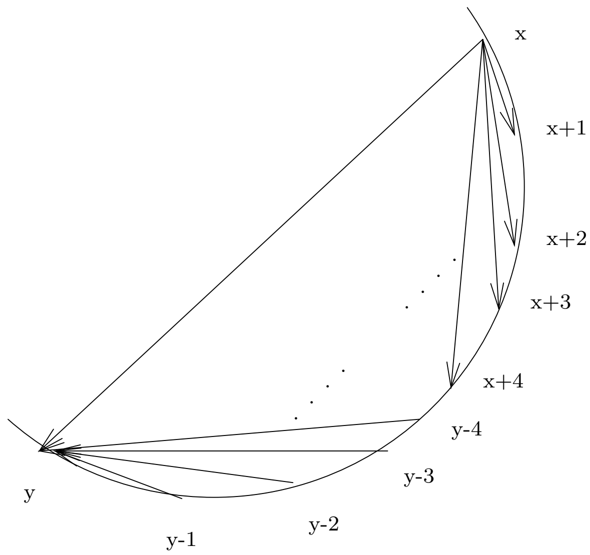
$$\begin{array}{ccc} P & \to & R \\ \downarrow id & & \downarrow \\ P & \to & 0 \end{array}$$

which are not homotopic to zero. If P and one of these arrows lie in the triangulated category $\mathfrak{X}$, then $\mathfrak{X}$ must contain the mapping cone of the respective map, which is Q or R, respectively. Thus, starting with a star and walking through the paths we finish the proof of the Theorem. ■

3 More on the combinatorics: cascades of fans

In this section we give a more illustrative and more practical method to construct the elements of $\mathfrak{C}$.

A fan is attached to an arrow $a = (v_\alpha \longrightarrow v_\omega)$, called the basic arrow, with either $v_\omega = 1$ or else $v_\alpha < v_\omega$, and consists of a set S_e of arrows all ending in v_ω and a *disjoint* set S_b of arrows all starting in v_α. Furthermore, the arrows in S_e have beginning vertex larger than v_α and the arrows in S_b have ending vertex smaller than v_ω. The smallest vertex that occurs as a beginning vertex of an arrow in S_e is strictly larger than the largest vertex that occurs as an ending vertex of an arrow in S_b, i.e. there is no crossing. So, a fan looks like the following:



where $x = v_\alpha$ and $y = v_\omega$. Since the elements of a fan are arrows, each of these arrows itself can serve as a basic arrow a of another fan. The resulting oriented graph is called cascade of fans if for each basic arrow a there is no directed path from the beginning vertex of a to the ending vertex of a besides the basic arrow itself, i.e. if the underlying graph is a tree. The largest interval $[i_l, i_r]$ such that there is a path from i_l to i_r only using arrows belonging to the cascade of fans is called the interval the cascade of fans is based on. A cascade of fans is called complete if for all basic arrows a of a fan in the cascade between the beginning vertex b and the ending vertex e all the vertices in the interval (b, e) are ending or beginning vertices of an arrow.

Lemma 3.1 *A connected component of a combinatorial object in $\mathcal{C}$ is a complete cascade of fans. Conversely, given a complete cascade of fans, one can choose a star at a vertex belonging to the interval (according to rule (III)), the cascade of fans is based on, and this then gives rise to a connected component of an object of $\mathcal{C}$. A set of cascades of fans in pairwise disjoint intervals gives rise to an element of $\mathcal{C}$.*

Proof. Let X be a connected component of an object in $\mathcal{C}$. We prove the statement by induction on the longest undirected path in the component beginning from the star and induction on the number of the paths of longest length. Take a longest path. If the length of a longest path is 0, we are done. Let p be the vertex that is reached last walking along the longest path and let p' be the vertex that is reached before. Since this is the longest path, the arrows ending at p' are a set of arrows S_e such that if i_e is the smallest number of the vertex with ending index p', and since all the vertices have to occur, for all the vertices with numbers j in (i_e, p') there is an arrow $j \longrightarrow p'$. Analogously, if we choose i_b to be the largest number of the vertices k such that there is an arrow $p' \longrightarrow k$, all the vertices k in (p', i_b) are linked to p' by an arrow $p' \longrightarrow k$. If we remove all the vertices in $[i_e, i_b]$ and the arrows ending or starting there, by induction we arrive at a complete cascade of fans. Together with the arrows and vertices in $[i_e, i_b]$ we obtain a complete cascade of fans. Since we removed two complete fans, i.e. that attached to $i_e \longrightarrow p'$ and that attached to $p' \longrightarrow i_b$, the whole object is a complete cascade of fans.

Assume we are given a complete cascade of fans based on the interval $[l, r]$. If $l > r$ we attach a star to 1 and stars to each vertex in (r, l). If $l \leq r$ we attach a star to l. The object fulfills rule (0) by the definition of a 'basic arrow'. It fulfills rule (I) also by definition of a fan. Rule (II) is fulfilled by the definition of a cascade of fans and the fact that S_e and S_b are disjoint. Rule (III) can be fulfilled by the definition of a fan and the completeness.

Completing a set $\mathcal{S}$ of complete cascades of fans to an object of $\mathcal{C}$ is done just by adding stars to certain vertices. If a vertex does not belong to any of the intervals, one of the cascades of fans in $\mathcal{S}$ is based on, we attach a star to it. Moreover, we attach a star to 1. For a cascade of fans $\mathcal{X}$ in $\mathcal{S}$ which is not based on an interval containing 1, we attach a star to the smallest vertex that belongs to the interval $\mathcal{X}$ is based on. The rest of the proof is analogous to the previous paragraph. ■

4 Rings which are derived equivalent to hereditary orders

From the Theorem it follows that tilting complexes over Λ are purely combinatorial objects, and now we show the same for their endomorphism rings. We define a class $\mathfrak{A}$ of algebras which turn out to be these endomorphism rings.

We will work with a complex which is defined by a combinatorial object in $C \in \mathfrak{C}$. We number the stars and arrows in such a way that first the stars come in their natural order, then the arrows in any order. Then we have to find all the non–zero homomorphisms between the indecomposable direct summands of T.

The precise rules are described below (we use the notation of the previous section). All the arguments needed here are already contained in the proof of the Theorem (see in particular Subsection 2.3); we do not repeat them here.

If P and Q are stars, the homomorphisms are $Hom_\Lambda(P, Q)$, which is either Δ or (π_D). If P is a star and $Q \to P$ has homology in degree 0, then the homomorphism set is $\overline{\Delta}$ respectively 0; if $P \to Q$ has homology in degree 1, then we get again $\overline{\Delta}$ respectively 0, there are no more maps between stars and arrows. If two arrows $Q \to R$ and $Q \to S$ have homology in the same degree, then the set of homomorphisms from the smaller to the larger module is $\overline{\Delta}$ and is 0 in the other direction. If two arrows $R \to Q$ and $S \to Q$ have homology in the same degree, then the homomorphism set is 0 from the smaller to the larger module and $\overline{\Delta}$ in the other direction. If $Q \to R$ has homology in degree $i+1$ and $R \to S$ has homology in degree $i+2$, then the set of homomorphisms is $\overline{\Delta}$ (and 0 in the other direction). There are no further non–trivial maps between arrows. The composition of these maps follows the same rules, thus, two maps have a non–trivial composition if there is a composition at all.

Now, from the combinatorial structure of $\mathfrak{C}$, it follows that we have to deal with two different kinds of rings, which have to be pasted together in a way which will be explained after having defined these rings.

For each vertex m we define

1. $m_e :=$ number of arrows ending at that vertex
2. $m_b :=$ number of arrows beginning at that vertex
3. $m_s :=$ number of stars at that vertex (i.e. 0 or 1)

We form a matrix ring of size $m_e + m_b + m_s$ for each vertex. The columns are numbered in the following way: The first m_b columns correspond to the arrows beginning at the vertex and they are numbered according to increasing ending vertex, that is the first column corresponds to the smallest ending vertex and the column number m_b corresponds to the largest ending vertex. If there is a star at the vertex then column $m_b + 1$ corresponds to the star. (Of course, if there is no star at the vertex, then there is no column corresponding to it.) The next m_e columns correspond to the arrows ending at the vertex. Here they are again numbered according to increasing beginning vertex, that is the lowest beginning index corresponds to column $m_s + m_b + 1$, the highest corresponds to the last column. We form then the upper triangular matrix with entries $\overline{\Delta}$ everywhere except if there is a star at the vertex then at the entry (m_b+1, m_b+1) we insert a Δ. Multiplication is just defined via the natural mapping $\Delta \longrightarrow \overline{\Delta}$.

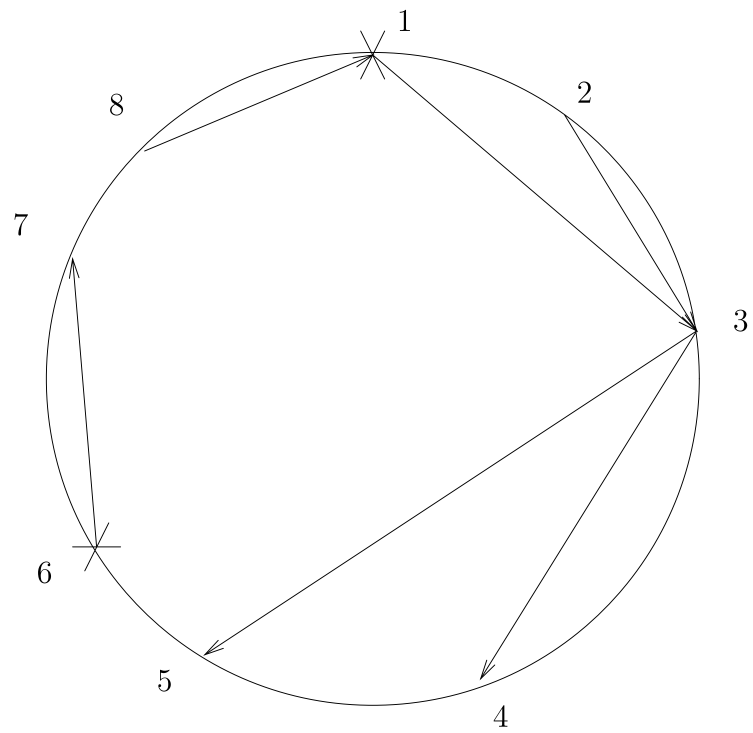
Furthermore, we form a hereditary order with the number of non isomorphic indecomposable projectives being equal to the 'number of stars in the cascade of fans'. They are numbered according to increasing vertex where they are attached, i.e. the first column corresponds to the star at 1, the second column corresponds to the star with the lowest number of the vertex where it is attached to, and so on.

Then, we form the subring of the product of these matrix rings consisting of those elements which have equal entries at those diagonal entries which correspond to the same arrows or stars[4].

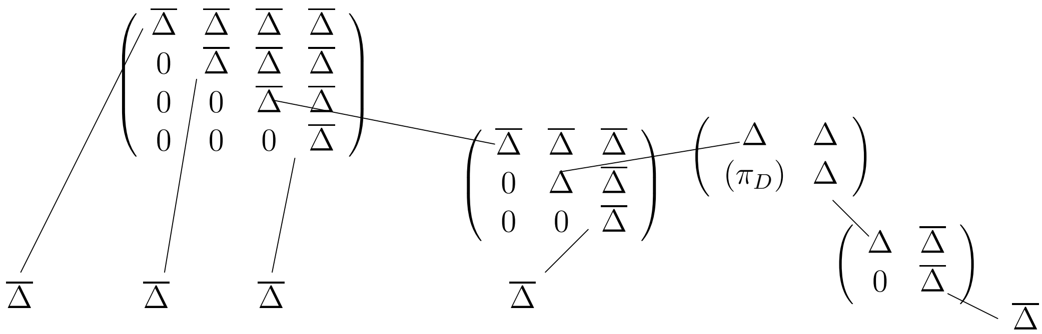
[4]Observe that each arrow is counted twice, first as arrow beginning at the beginning vertex and second as arrow ending at the ending vertex. Similarly each star is counted firstly as part of the hereditary order and secondly as part of the matrix ring corresponding to a vertex.

The summary above of the morphisms between the various components shows that this precisely describes the endomorphism ring of the tilting complex attached to the cascade of fans.

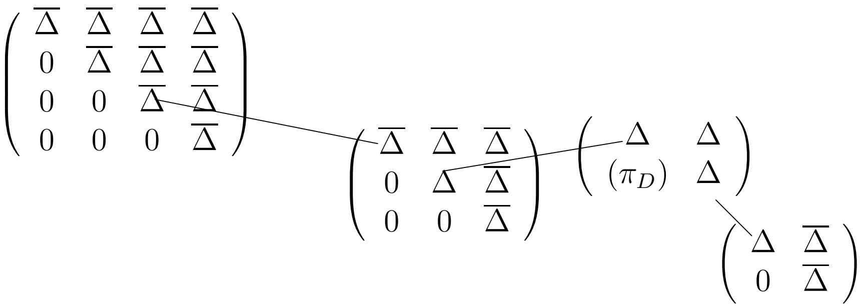
Example: We have for example the following cascade of fans for a hereditary order with 8 indecomposable projectives.



The endomorphism ring here, obtained by the above algorithm, is as follows. The line between two rings indicates the image of the *diagonal embedding* into the two copies of the ring.



Obviously, this ring is isomorphic to the following. Again, a line indicates the diagonal embedding into the direct product of two copies.



Corollary 4.1 *A ring* Γ *is derived equivalent to* Λ *if and only if it is of the form just described.* ■

Now we give some consequences of this combinatorial description of $End(T)$.

Corollary 4.2 *Up to isomorphisms and Morita equivalence there is only a finite number of rings which are derived equivalent to the fixed hereditary order* Λ.

Actually, the endomorphism ring $End(T)$ *of the tilting complex* T *determines the graph* $C \in \mathfrak{C}$ *and thus, the tilting complex* T. *Thus, there is a bijection between the set of tilting complexes and the set of their endomorphism rings.* ■

The last assertion also follows immediately from the observation that there are no auto–equivalences of the derived category of a hereditary order except the auto–equivalences of the module category and a shift in case of a hereditary order. The fact that the group of derived auto–equivalences of hereditary orders is modulo a shift just the usual Picard group of the hereditary order is observed by Raphaël Rouquier and the second author [10] and the proof in [10] uses quite different methods.

If the ring Γ is not Morita equivalent, but derived equivalent to the hereditary order Λ, then it has (as R–module) torsion elements. Thus, a twosided complex inducing the derived equivalence cannot consist of bimodules whose restriction to either side is projective. For blocks of group rings, if there is a complex of bimodules whose restriction to either side is isomorphic to a tilting in the homotopy category and the bimodules are moreover p-permutation modules, then the equivalence induced by the left derived tensor product with this complex is called a splendid equivalence and the complex is called a splendid tilting complex [9]. Its relevance becomes apparent in the fact that a splendid equivalence induces an isotypie in the sense of M. Broué.

If Γ is derived equivalent to Λ, then, there need not exist a derived equivalence between $\Lambda/(\pi\Lambda)$ and $\Gamma/(\pi\Gamma)$. In fact, if there are at least two indecomposable projective modules, $\Lambda/(\pi\Lambda)$ has infinite global dimension, whereas $\Gamma/(\pi\Gamma)$ has finite global dimension if and only if the combinatorial object associated with the derived equivalence has exactly one star. However, a derived equivalence preserves the property of having finite global dimension. The precise formula for the global dimension of Γ itself is as follows:

Corollary 4.3 *Let T be a tilting complex with associated graph $\mathfrak{C}$ and endomorphism ring A. Then the global dimension of A is $1+m$, where m is the length of the longest sequence $(P_1 \to P_2) \oplus (P_2[1] \to P_3[1]) \oplus (P_3[2] \to \ldots \oplus \ldots \to P_{m+1}[m])$, which equals the length of the longest directed path in $\mathfrak{C}$.* ■

ACKNOWLEDGEMENT

The authors thank Klaus Roggenkamp for many fruitful discussions and generous help.

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