Auto-equivalences of derived categories acting on cohomology

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To Idun Reiten on the occasion of her 60th birthday

Abstract

Let A be a k-algebra which is projective as a k-module, let M be an A-module whose endomorphisms are given by multiplication by central elements of A, and let $TrPic_k(A)$ be the group of standard self-equivalences of the derived category of bounded complexes of A-modules. Then we define an action of the stabilizer of M in $TrPic_k(A)$ on the Ext-algebra of M. In case M is the trivial module for the group algebra $kG = A$, this defines an action on the cohomology ring of G which extends the well-known action of the automorphism group of G on the cohomology group.

Introduction

Let A and B be R-algebras over the commutative ring R so that A is projective as an R-module. If there is an equivalence between the derived categories of bounded complexes of A-modules $D^b(A)$ and the derived category $D^b(B)$, Rickard and Keller proved that there is a complex X in $D^b(A \otimes_R B^{op})$ so that $F_X := X \otimes^{\mathbb{L}}_B -$ induces such an equivalence. In case B is projective as an R-module as well, there is also an object Y in $D^b(B \otimes_R A^{op})$ so that $F_Y := Y \otimes^{\mathbb{L}}_B -$ is a quasi-inverse to F_X. In case $A = B$, equivalences of this type form a group which, in an earlier work with R. Rouquier [8], is called $TrPic_R(A)$.

Let M be an A-module. Then, it is reasonable to expect that the set $HD_M(A)$ of elements in $TrPic_R(A)$, which fix M up to isomorphism, acts on the Ext-algebra $Ext^*_A(M,M)$ of M as ring automorphisms since $Ext^n_A(M,M) = Hom_{D^b(A)}(M, M[n])$ for any integer n. To get an actual action one has to be a bit more careful. We prove the above statement if any automorphism of M is induced by multiplication by an invertible element of the centre of A. For other modules with more complicated automorphism groups an extension $\widetilde{HD}_M(A)$ of $HD_M(A)$ by some quotient of the automorphism group of M acts on $Ext^*_A(M,M)$.

The above defined action is well behaved with respect to change of base rings. In case A is the group ring RG of a group G, the action of $HD_R(RG)$ extends the well known action of the outer automorphism group of G on the cohomology ring $H^*(G,R)$. This action of $HD_R(RG)$ is functorial with respect to the second variable. In further work [10, 11] we study the functoriality with respect to the first variable. There, the situation is more complicate and we only have partial answers.

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1 Brief review on derived equivalences

We shall present briefly what we will need concerning Rickard's tilting theory. Our notation closely follows [4]. Let A and B be R-algebras over a commutative ring R and assume that A and B are projective as modules over R.

Rickard [6], and under weaker hypotheses Keller [2], show that if there is an equivalence of triangulated categories $D^b(A) \simeq D^b(B)$ then there is a bounded complex X in $D^b(A \otimes_R B^{op})$ so that $X \otimes^{\mathbb{L}}_B -$ is an equivalence. Moreover, the inverse functor is given by the left derived tensor product with a complex Y in $D^b(B \otimes_R A^{op})$. Equivalences given by (left derived) tensor product by a bounded complex of bimodules are called of *standard type*. The complex X is called a *twosided tilting complex*.

We define [8] the group

$$TrPic_R(A) := \{ \text{ isomorphism classes of two-sided tilting complexes in } D^b(A \otimes_R A^{op})\}$$

It is clear by the definition that in case A and B are R-algebras which are projective as R-modules and $D^b(A) \simeq D^b(B)$, then $TrPic_R(A) \simeq TrPic_R(B)$.

2 Operation on Ext-algebras

We are interested in the action of $TrPic_R(A)$ on Ext-algebras and in particular on group cohomology. Let us recall the definitions. For an R-algebra A and two A-modules M and N let $\oplus_{n\in\mathbb{N}} Ext^n_A(M,N) =: Ext^*_A(M,N)$. Then, $Ext^*_A(M,N)$ is a module over the R-algebra $Ext^*(M,M)$. This follows easily by the observation $Ext^n_A(M,N) = Hom_{D^b(A)}(M,N[n])$ for any $n \in \mathbb{N}$. Then, the ring structure of $Ext^*_A(M,M)$ is given by composition of mappings and the $Ext^*_A(M,M)$-module structure of $Ext^*_A(M,N)$ as well is given by composing mappings.

In particular if G is a group and R is a commutative ring, then for any RG-module M we have

$$H^n(G,M) \simeq Ext^n_{RG}(R,M) \simeq Hom_{D^b(RG)}(R,M[n]) \ .$$

Let X be a complex with isomorphism class in $TrPic_R(A)$ and let

$$F_X := X \otimes^{\mathbb{L}}_A - : D^b(A) \longrightarrow D^b(A)$$

be the corresponding self-equivalence. Then, F_X induces a mapping

$$Hom_{D^b(A)}(M,M[n]) \longrightarrow Hom_{D^b(A)}(F_X(M),F_X(M[n])) = Hom_{D^b(A)}(F_X(M),F_X(M)[n]) \ .$$

Assume now that we have an isomorphism $F_X(M) \stackrel{\alpha_X}{\longrightarrow} M$. Then, composing by α_X and its inverse from the left and from the right, the pair (α_X, X) induces automorphisms $F_X^{(n)}$

$$\begin{array}{rccc} F_X^{(n)}: & Hom_{D^b(A)}(M,M[n]) & \stackrel{\alpha_X[n]_*(\alpha_X^{-1})^* F_X}{\longrightarrow} & Hom_{D^b(A)}(M,M[n]) \\ & \lambda & \longrightarrow & \alpha_X[n]\cdot F_X(\lambda)\cdot \alpha_X^{-1} \end{array}$$

for any positive integer n.

We shall discuss what happens if we have two isomorphic functors F_X and F_Y.

Lemma 2.1 *Let F_X and F_Y be two functors $D^b(A) \longrightarrow D^b(A)$ and suppose that $F_Y \stackrel{\eta}{\longrightarrow} F_X$ is an isomorphism of functors. If there exists an isomorphism $\alpha_X : F_X(M) \longrightarrow M$. Then, the action of $(\alpha_X\eta_M, F_Y)$ and the action of (α_X, F_X) on $Hom_{D^b(RG)}(M,M[n])$ coincide.*

Proof. Let $\lambda \in Hom_{D^b(A)}(M, M[n])$. Then, setting $\alpha_Y := \alpha_X \eta_R$ we get a commutative diagram

$$\begin{array}{ccccccc} M[-n] & \stackrel{\alpha_X^{-1}[-n]}{\longrightarrow} & F_X(M)[-n] & \stackrel{F_X(\lambda)}{\longrightarrow} & F_X(M) & \stackrel{\alpha_X}{\longrightarrow} & M \\ \| & & \uparrow \eta_M[-n] & & \uparrow \eta_M & & \| \\ M[-n] & \stackrel{\alpha_Y^{-1}[-n]}{\longrightarrow} & F_Y(M)[-n] & \stackrel{F_Y(\lambda)}{\longrightarrow} & F_Y(M) & \stackrel{\alpha_Y}{\longrightarrow} & M \end{array}$$

This proves the lemma. ∎

Let $F_{X_1} : D^b(A) \longrightarrow D^b(A)$ and $F_{X_2} : D^b(A) \longrightarrow D^b(A)$ be equivalences of triangulated categories and suppose there are isomorphisms $\alpha_{X_i} : F_{X_i}(M) \longrightarrow M$ for $i = 1, 2$. We define the composition of two objects (α_{X_1}, F_{X_1}) and (α_{X_2}, F_{X_2}) by:

$$(\alpha_{X_2}, F_{X_2}) \cdot (\alpha_{X_1}, F_{X_1}) = (\alpha_{X_2} F_{X_2}(\alpha_{X_1}), F_{X_2} F_{X_1})$$

Definition 2.2 Let $\widehat{HD}_M(A)$ be the class of pairs (α_X, X) where $F_X = X \otimes^{\mathbb{L}}_A -$ is a self-equivalence of $D^b(A)$ of standard type so that $X \otimes^{\mathbb{L}}_A M \simeq M$ and $\alpha_X : X \otimes^{\mathbb{L}}_A M \longrightarrow M$ is an isomorphism of complexes of A-modules. Two of these pairs (α_X, X) and (α_Y, Y) are called isomorphic if there is an isomorphism $\eta : Y \longrightarrow X$ so that $\alpha_Y = \alpha_X \circ (\eta \otimes id_M)$. Let $\widetilde{HD}_M(A)$ be the set of isomorphism classes in $\widehat{HD}_M(A)$.

In case $A = RG$ is a group ring and $M = R$ the trivial RG-module, then we denote $\widehat{HD}_R(RG) =: \widehat{HD}_R(G)$ and $\widetilde{HD}_R(RG) =: \widetilde{HD}_R(G)$ for short.

With this definition we get

Lemma 2.3 *For any $\lambda \in Hom_{D^b(A)}(M[-n], M)$ and (α_{X_2}, X_2) and (α_{X_1}, X_1) in $\widehat{HD}_M(A)$ we get*

$$\begin{aligned} (id_R, Id_{D^b(A)})\,(\lambda) &= \lambda \\ ((\alpha_{X_2}, X_2) \cdot (\alpha_{X_1}, X_1))\,(\lambda) &= (\alpha_{X_2}, X_2)\,((\alpha_{X_1}, X_1)\,(\lambda)) \end{aligned}$$

Proof. The first assertion is immediate. For the second assertion we compute

$$\begin{aligned} (\alpha_{X_2}, X_2)\,((\alpha_{X_1}, X_1)\,(\lambda)) &= (\alpha_{X_2}, X_2)\,(\alpha_{X_1} F_{X_1}(\lambda) \alpha_{X_1}^{-1}[-n]) \\ &= \alpha_{X_2}(F_{X_2}(\alpha_{X_1}) F_{X_2} F_{X_1}(\lambda) F_{X_2}(\alpha_{X_1})^{-1}[-n]) \alpha_{X_2}^{-1}[-n] \\ &= ((\alpha_{X_2}, X_2) \cdot (\alpha_{X_1}, X_1))\,(\lambda) \end{aligned}$$

This proves the second assertion. ∎

Lemma 2.4 *The set $\widetilde{HD}_M(A)$ is a group with the above defined multiplication. The identity element is (nat, A) and the inverse of (α_X, X) is $(nat \circ (\eta_X \otimes id_M) F_X^{-1}(\alpha_X^{-1}), F_X^{-1})$ where $\eta_X : X^{-1} \otimes_A X \longrightarrow A$ is an isomorphism in the derived category of $A \otimes_R A^{op}$-bimodules, and $nat : A \otimes_A M \longrightarrow M$ is the natural isomorphism.*

Moreover, projection onto the second component gives a group homomorphism

$$\widetilde{HD}_M(A) \longrightarrow TrPic_R(A)$$

whose image is the fix point stabilizer of the trivial module

$$HD_M(A) := \{[X] \in TrPic_R(A) \mid X \otimes^{\mathbb{L}}_A M \simeq M\}$$

Proof. Let (α_X, X), (α_Y, Y) and (α_Z, Z) be elements of $\widehat{HD}_M(A)$. It is clear that $(nat, A) \cdot (\alpha_X, X) = (\alpha_X, X)$ in $\widetilde{HD}_M(A)$. Now,

$$\begin{aligned}(nat \circ (\eta_X \otimes id_M) \circ F_X^{-1}(\alpha_X^{-1}), F_X^{-1}) \cdot (\alpha_X, F_X) \\ &= (nat \circ (\eta_X \otimes id_M) \circ F_X^{-1}(\alpha_X^{-1}) \circ F_X^{-1}(\alpha_X), F_X^{-1} F_X) \\ &= (nat \circ (\eta_X \otimes id_M) \circ F_X^{-1}(id), F_X^{-1} F_X) \\ &= (nat, A)\end{aligned}$$

Associativity is the following computation:

$$\begin{aligned}((\alpha_X, X)(\alpha_Y, Y))(\alpha_Z, Z) &= (\alpha_X F_X(\alpha_Y), X \otimes_A Y)(\alpha_Z, Z) \\ &= (\alpha_X F_X(\alpha_Y) F_{X \otimes_A Y}(\alpha_Z), X \otimes_A Y \otimes_A Z) \\ &= (\alpha_X F_X(\alpha_Y F_Y(\alpha_Z)), X \otimes_A Y \otimes_A Z) \\ &= (\alpha_X, X)(\alpha_Y F_Y(\alpha_Z), Y \otimes_A Z) \\ &= (\alpha_X, X)((\alpha_Y, Y)(\alpha_Z, Z))\end{aligned}$$

The fact that the projection onto the second component is a group homomorphism with image $HD_R(G)$ described above is immediate. ■

Observe that $\widehat{HD}_M(A)$ is not a group in general. Moreover, we use the fact that we deal with standard equivalences and we use the fact that we work in algebras, which are projective as a module over the base ring. So, composition of functors is associative since we may replace a complex by its projective resolution and there we may use the ordinary tensor product which is associative.

Lemma 2.5 *Let A be an R-algebra and let M be an A-module so that any A-linear automorphism of M is induced by multiplication by an invertible element of the centre $Z(A)$ of A. Then, $\widetilde{HD}_M(A) \simeq HD_M(A)$. In particular, for any group G and any commutative ring R we have $\widetilde{HD}_R(G) \simeq HD_R(G)$.*

Proof. The kernel of the canonical surjection $\widetilde{HD}_M(A) \longrightarrow HD_M(A)$ is formed by the set of A-linear automorphisms of M modulo the group of automorphisms $nat \circ (\eta \otimes id_M) \circ nat^{-1}$ for automorphisms η of A as bimodule. Automorphisms of A as bimodules are precisely the multiplications by central invertible elements of A. Now, $Aut(M)$ is generated by multiplication by invertible elements of $Z(A)$. Therefore, the kernel of the surjection $\widetilde{HD}_M(A) \longrightarrow HD_M(A)$ is trivial and this surjection induces the isomorphism as stated.

For a group ring RG, any automorphism of the trivial module is multiplication by a unit in R. It is clear that the group of units R^* of R is in $Z(RG)$. ■

Remark 2.6 Bass observed that there is a monomorphism of the outer automorphism group $Out_R(A)$ of the R-algebra A to $Pic_R(A)$. This monomorphism maps an automorphism f of the R-algebra A to the bimodule ${}_fA_1$ on which $a \in A$ acts by multiplication by a on the right and by multiplication by $f(a)$ on the left. $Pic_R(A)$ in turn is a subgroup of $TrPic_R(A)$ and we get a group homomorphism

$$\{\phi \in Aut_R(A) \mid {}^{\phi}M \simeq M\} \longrightarrow HD_M(A).$$

In particular, define $HA_R(G)$ as the image of the automorphism group of G in $HD_R(G)$ by mapping an automorphism f of G to the bimodule ${}_fRG_1$ which is RG from the right and on which $g \in G$ operates by multiplication by $f(g)$ from the left.

Note that the question when an automorphism becomes inner in the group ring (and hence induces the identity in $HD_R(G)$) is far from trivial. The reader may consult for example M. Mazur [5] or Roggenkamp-Zimmermann [7].

We have obtained the following result.

Theorem 1 *Let R be a commutative ring and let A be an R-algebra, projective as an R-module and let M be an A-module. Then, for any integer n the group $Ext^n_A(M,M)$ is an $R\,\widetilde{HD}_M(A)$-module.*

If in addition any automorphism of M is induced by multiplication by an invertible element in $Z(A)$, then for any integer n the group $Ext^n_A(M,M)$ is an $R\,HD_M(A)$-module.

In particular, for any commutative ring R and any group G the cohomology group $H^n(G,R)$ is an $R\,HD_R(G)$-module.

Proof. The results in Lemma 2.3, Lemma 2.4 and Lemma 2.5 imply the statement immediately. ∎

Remark 2.7 The action of $HD_R(G)$ extends the well known and well studied action of $Out(G)$ on $H^n(G,R)$. Our interest into the above defined action partly comes from this fact.

3 Properties of the action

We recall the trivial cases.

Proposition 3.1 *[8] Let A be a commutative indecomposable R-algebra, or let A be a local algebra. Then $TrPic_R(A) = Pic_R(A) \times < [1] >$.*

We shall study change of rings properties and functoriality with respect to the coefficient ring R.

Remark 3.2 What happens in the case of a group ring RG for a finite group G ? Since $H^0(G,R) \simeq R$ is the trivial $HD_R(G)$-module, we may restrict our attention to $H^n(G,R)$ for $n \geq 1$. Moreover, assume that R is finitely generated over $\mathbb{Z}$. Then, the universal coefficient theorem gives an exact sequence

$$0 \longrightarrow H^n(G,\mathbb{Z}) \otimes_{\mathbb{Z}} R \longrightarrow H^n(G,R) \longrightarrow Tor_1^{\mathbb{Z}}(H^{n+1}(G,\mathbb{Z}),R) \longrightarrow 0$$

So, for certain coefficient rings R and for certain questions we may restrict attention to the coefficient ring $\mathbb{Z}$. Moreover, for $n \geq 1$, we get $H^n(G,\mathbb{Z}) \simeq \prod_{p \in Spec\,\mathbb{Z}} H^n(G,\hat{\mathbb{Z}}_p)$. If p does not divide the order of the finite group G, one gets $H^n(G,\hat{\mathbb{Z}}_p) = 0$.

So, part of the problem is reduced to the case $H^n(G,\hat{\mathbb{Z}}_p)$ for all prime numbers dividing the order of G.

Let S be a commutative R-algebra. Since we assumed that A is projective as an R-module, [6] proves that

$$S \otimes_R - :\ TrPic_R(A) \longrightarrow TrPic_S(S \otimes_R A)$$

is a homomorphism of groups. It is not true in general that this homomorphism is surjective or injective as the following example shows.

Example: Since $\mathbb{Z}C_3$ is commutative indecomposable, Proposition 3.1 shows that $TrPic_{\mathbb{Z}}(\mathbb{Z}C_3) =< [1] >$ while $TrPic_{\hat{\mathbb{Z}}_7}(\hat{\mathbb{Z}}_7 C_3) \simeq TrPic_{\hat{\mathbb{Z}}_7}(\hat{\mathbb{Z}}_7 \times \hat{\mathbb{Z}}_7 \times \hat{\mathbb{Z}}_7) \simeq C_\infty \wr \mathfrak{S}_3$ is a wreath product since $\hat{\mathbb{Z}}_7$ contains a primitive third root of unity. So, the above mapping $S \otimes_R - : TrPic_R(A) \longrightarrow TrPic_S(S \otimes_R A)$ is not surjective in general. We should remark that $HD_{\hat{\mathbb{Z}}_7}(C_3) = C_\infty \wr C_2$ while $HD_{\mathbb{Z}}(C_3) = Gal(\mathbb{Z}[\zeta_3] : \mathbb{Z}) = C_2$ is generated by the Galois automorphism of $\mathbb{Z}[\zeta_3]$ over $\mathbb{Z}$.

As is shown in [9] the kernel $TrI_{\hat{\mathbb{Z}}_3}(\hat{\mathbb{Z}}_3\mathfrak{S}_3)$ of the homomorphism $TrPic_{\hat{\mathbb{Z}}_3}(\hat{\mathbb{Z}}_3\mathfrak{S}_3) \longrightarrow TrPic_{\hat{\mathbb{Q}}_3}(\hat{\mathbb{Q}}_3\mathfrak{S}_3)$ is a non abelian free group. So, the above mapping is not injective in general neither.

Theorem 2 *Let $\phi : R \longrightarrow S$ be a homomorphism of commutative rings, let A be an R-algebra which is projective as an R-module and let M be an A-module. Then, $S \otimes_R - : TrPic_R(A) \longrightarrow TrPic_S(S \otimes_R A)$ lifts to a homomorphism*

$$H\phi : \widetilde{HD}_M(A) \longrightarrow \widetilde{HD}_{S\otimes_R M}(S \otimes_R A) .$$

Moreover,

$$S \otimes_R - \ : \ Ext^*_A(M, M) \longrightarrow Ext^*_{S\otimes_R A}(S \otimes_R M, S \otimes_R M)$$

is $R\,\widetilde{HD}_M(A)$-linear.

Proof. Let X be a complex with isomorphism class in $HD_M(A)$. Since A is assumed to be projective as an R-module, we may and will assume that the homogeneous components of X are projective as A-modules and projective as A^{op}-modules. Then, we can replace the left derived tensor product by the ordinary tensor product.

First we observe that there is an isomorphism

$$\nu_X : \ (S \otimes_R X) \otimes_{S\otimes_R A} (S \otimes_R M) \xrightarrow{\sim} S \otimes_R (X \otimes_A M)$$

given by

$$(s \otimes x) \otimes (t \otimes m) = (st \otimes x) \otimes (1 \otimes m) \mapsto st \otimes (x \otimes m)$$

in each homogeneous component of X. We define

$$\alpha_{S\otimes_R X} : (S \otimes_R X) \otimes_{S\otimes_R A} (S \otimes_R M) \xrightarrow{\sim} S \otimes_R M$$

by $\alpha_{S\otimes_R X} := (id_S \otimes \alpha_X) \circ \nu_X$.

In order to prove the first statement we define

$$\begin{array}{ccc} \widetilde{HD}_M(A) & \xrightarrow{H\phi} & \widetilde{HD}_{S\otimes_R M}(S \otimes_R A) \\ (\alpha_X, X) & \mapsto & (\alpha_{S\otimes_R X}, (S \otimes_R X)) \end{array}$$

If $\eta : X \longrightarrow Y$ is an isomorphism, then $id_S \otimes_R \eta : S \otimes_R X \longrightarrow S \otimes_R Y$ is an isomorphism as well. It follows that $H\phi$ does not depend on the chosen representative in the isomorphism class in $\widetilde{HD}_M(A)$. Moreover, the above shows that

$$(\alpha_{S\otimes_R X}, (S \otimes_R X)) \in \widehat{HD}_{S\otimes_R M}(S \otimes_R A) .$$

This proves the first statement.

We have to show that

$$S \otimes_R - : Hom_{D^b(A)}(M, M[n]) \longrightarrow Hom_{D^b(S\otimes_R A)}(S \otimes_R M, S \otimes_R M[n])$$

is linear under the action of $\widetilde{HD}_M(A)$. This is equivalent to proving that the diagram

$$\begin{array}{ccc}
Hom_{D^b(A)}(M,M[n]) & \longrightarrow & Hom_{D^b(S\otimes_R A)}(S\otimes_R M, S\otimes_R M[n]) \\
\downarrow & & \downarrow \\
Hom_{D^b(A)}(X\otimes_A M, X\otimes_A M[n]) & & Hom_{D^b(S\otimes_R A)}((SX)\otimes_{SA}(SM), ((SX)\otimes_{SA}(SM[n])) \\
\downarrow \alpha_X[n]_* \cdot (\alpha_X^{-1})^* & & \downarrow \alpha_{S\otimes_R X}[n]_* \cdot (\alpha_{S\otimes_R X}^{-1})^* \\
Hom_{D^b(A)}(M,M[n]) & \longrightarrow & Hom_{D^b(S\otimes_R A)}(S\otimes_R M, S\otimes_R M[n])
\end{array}$$

is commutative. (We wrote SM for $S \otimes_R M$ and likewise SX for $S \otimes_R X$.) The diagram is commutative if and only if

$$\alpha_{S\otimes_R X}[n] \circ ((S\otimes_R X)\otimes f)\circ \alpha_{S\otimes_R X}^{-1} = id_S \otimes (\alpha_X[n]\circ (X\otimes f)\circ \alpha_X^{-1})\ .$$

But,

$$\begin{aligned}
\alpha_{S\otimes_R X}[n] \circ ((S\otimes_R X)\otimes f)\circ&\alpha_{S\otimes_R X}^{-1} \\
&= (id_S\otimes_R \alpha_X[n])\circ \nu\circ ((S\otimes_R X)\otimes f)\circ \nu^{-1}\circ (\alpha_X^{-1}\otimes id_S) \\
&= (id_S\otimes_R \alpha_X[n])\circ ((S\otimes_R X)\otimes f)\circ (\alpha_X^{-1}\otimes id_S)
\end{aligned}$$

since conjugation by ν acts trivially on the morphism $(S \otimes_R X) \otimes_A f$. Now, since $(S \otimes_R X) \otimes_A f \simeq S \otimes_R (X \otimes_A f)$, we see that under this isomorphism we obtain

$$(id_S\otimes_R \alpha_X[n])\circ ((S\otimes_R X)\otimes f)\circ (\alpha_X^{-1}\otimes id_S) = id_S\otimes (\alpha_X[n]\circ (X\otimes f)\circ \alpha_X^{-1})\ .$$

This proves the second statement. ∎

As a consequence, one might get a rather different group acting on $H^*(G, S)$ than on $H^*(G, R)$ even though it might happen that $H^{\geq 1}(G, S) \simeq H^{\geq 1}(G, R)$. For example $\hat{\mathbb{Z}}_p$ contains a primitive $p-1$-th root of unity but $\mathbb{Z}_p$ does not. Nevertheless, $H^{\geq 1}(G, \hat{\mathbb{Z}}_p) \simeq H^{\geq 1}(G, \mathbb{Z}_p)$ for a finite group G.

Lemma 3.3 *Let $\phi : R \longrightarrow S$ and $\psi : S \longrightarrow T$ be homomorphisms of commutative rings and let A be an R-algebra which is projective as an R-module. Let M be an A-module. Then, $H\psi \circ H\phi = H(\psi \circ \phi)$.*

Proof. It is clear that there is an isomorphism $T \otimes_S (S \otimes_R X) \simeq T \otimes_R X$ and so, the only thing to prove is that under this isomorphism $(\alpha_{T\otimes_S(S\otimes_R X)}, T \otimes_S (S \otimes_R X))$ becomes equivalent to $(\alpha_{T\otimes_R X}, T \otimes_R X)$. But, this is immediate. ∎

Remark 3.4 Let R be any commutative ring, let A and B be R-algebras which are projective as R-modules and let M be an A-module. Assume now that we have an equivalence of triangulated categories $D^b(A) \simeq D^b(B)$. Then, by [6], there is a complex Y in $D^b(B \otimes_R A^{op})$ so that the left derived tensor product $Y \otimes^{\mathbb{L}}_A -$ is an equivalence and a (quasi-)inverse equivalence is given by left derived tensor product with a complex X in $D^b(A \otimes_R B^{op})$. Therefore, the group $Y \otimes_A HD_M(A) \otimes_B X \subseteq TrPic_R(B)$ fixes $Y \otimes_A M$. It is now immediate that $\widetilde{HD}_M(A) \simeq \widetilde{HD}_{Y\otimes_A M}(B)$ We should note however that one has to enlarge the definition of $\widetilde{HD}_M(A)$ in the obvious way in case $Y \otimes_A M$ is not necessarily isomorphic to a module.

Proposition 3.5 *Let R be a commutative ring, let A and B be R-algebras which are projective as R-modules and let M be an A-module. Let $F_U := U \otimes^{\mathbb{L}}_A - : D^b(A) \longrightarrow D^b(B)$ be an equivalence of standard type of triangulated categories.*

Then, F_U induces an isomorphism

$$HU : \widetilde{HD}_M(A) \stackrel{\simeq}{\longrightarrow} \widetilde{HD}_{U\otimes_A M}(B)$$

and an isomorphism

$$Ext^*_A(M,M) \stackrel{\simeq}{\longrightarrow} {}^{HU}Ext^*_B(U\otimes_A M, U\otimes_A M)$$

as $\widetilde{HD}_M(A)$*-modules.*

Proof. We note by U^{-1} a complex of $A\otimes_R B^{op}$-modules so that $U\otimes^{\mathbb{L}}_A U^{-1}\simeq B$ and $U^{-1}\otimes^{\mathbb{L}}_B U\simeq A$. Since A and B are projective as R-modules, we may and will assume that the homogeneous components of U are projective as A^{op}-modules and as B-modules and that the homogeneous components of U^{-1} are projective as B^{op}-modules and as A-modules. Let $\rho_T : T\otimes_A M \longrightarrow M$ be an isomorphism fixed for each T which has isomorphism class in $HD_M(A)$. Set

$$\rho_{UTU^{-1}} : U\otimes_A T\otimes_A U^{-1}\otimes_B U\otimes_A M \simeq U\otimes_A T\otimes_A M \stackrel{id_U\otimes\rho_T}{\longrightarrow} U\otimes_A M$$

Then, define a mapping

$$\begin{array}{rccl} HU: & \widetilde{HD}_M(A) & \longrightarrow & \widetilde{HD}_{U\otimes_A M}(B) \\ & (\rho_T, T) & \longrightarrow & (\rho_{UTU^{-1}}, UTU^{-1}) \end{array}$$

The above is well defined since if there is an isomorphism $\eta : T\longrightarrow T'$, then η induces $\rho_{T'} = \rho_T\circ(\eta\otimes id_M)$ and so $\rho_{UT'U^{-1}} = \rho_{UTU^{-1}}\circ(id_U\otimes\eta\otimes id_M)$ which gives

$$HU(\rho_T,T) = (\rho_{UTU^{-1}}, UTU^{-1}) = (\rho_{UTU^{-1}}\circ(id_U\otimes\eta\otimes id_M), UT'U^{-1}) = HU(\rho_{T'},T')\ .$$

It is clear that the mapping HU is bijective since the inverse is given in the very same way, replacing U by U^{-1}. Moreover, for any (ρ_T,T) and (ρ_S,S) in $\widetilde{HD}_M(A)$, we get

$$\begin{array}{rcl} HU(\rho_T\circ F_T(\rho_S), TS) & = & (F_U(\rho_T\circ F_T(\rho_S)), UTSU^{-1}) \\ & = & (F_U(\rho_T), UTU^{-1})(F_U(\rho_S), USU^{-1}) \\ & = & HU(\rho_T,T)\cdot HU(\rho_S,S) \end{array}$$

and this proves that HU is multiplicative. We obtained the first statement.

Recall that the action of (ρ_T,T) with equivalence class in $HD_M(A)$ on $Ext^n_A(M,M)$ is given by the following construction. An $\alpha\in Hom_{D^b(A)}(M[-n],M)$ is mapped by (ρ_T,T) to $\rho_T[-n]T(\alpha)\rho_T^{-1}$.

Take $\beta\in Hom_{D^b(B)}(U\otimes_A M[-n], U\otimes_A M)$.

$$\begin{array}{rcl} (HU(\rho_T,T))\cdot\beta & = & (F_U(\rho_T), UTU^{-1})\cdot\beta \\ & = & F_U(\rho_T)\cdot F_{UTU^{-1}}(\beta)\cdot F_U(\rho_T)[-n] \\ & = & F_U(\rho_T\cdot F_T F_{U^{-1}}(\beta)\cdot\rho_T[-n]) \\ & = & F_U((\rho_T,T)\cdot F_{U^{-1}}(\beta)) \end{array}$$

This proves the second statement. ∎

Now, we shall study the case of a group ring in more detail. In modular representation theory of finite groups it has proved useful to look at the stable module category. Most of what follows works equally well for a selfinjective R-algebra, which is projective as an R-module. We will not go into these details.

Let R be a Dedekind domain. Then, the R-stable module category $\overline{mod}(RG)$ is the quotient category of $RG-mod$ by the subcategory of R-projective modules.

Let $StPic(RG)$ be the group of isomorphism classes of $RG \otimes_R RG^{op}$-modules X up to isomorphism in $\overline{mod}(RG \otimes_R RG^{op})$ so that $X \otimes_{RG} -$ induces an equivalence $\overline{mod}(RG) \longrightarrow \overline{mod}(RG)$. Set

$$HSt_R(G) := \{[X] \in StPic(RG) \mid X \otimes_{RG} R \simeq R \text{ in } \overline{mod}(RG)\}$$

Analogously to $\widetilde{HD}_R(G)$ and $\widehat{HD}_R(G)$ we define $\widetilde{HSt}_R(G)$ and $\widehat{HSt}_R(G)$.

Following [8] we have a group homomorphism

$$TrPic_R(RG) \xrightarrow{\sigma} StPic_R(RG) .$$

This is defined as follows: Take a two-sided tilting complex X with isomorphism class in $TrPic_R(RG)$. Then, choose any projective resolution (X', d') of X as complex of $RG \otimes_R RG^{op}$ bimodules. Let n be the highest degree (differentials are of degree -1) so that X has non zero homology in degree n. Then, $X_\infty := ker\ d'_{n+1}$. Since RG is a Gorenstein order (that is $Ext^1_{RG}(M, RG) = 0$ for any RG-lattice M), the functor $\Omega : \overline{mod}(RG) \longrightarrow \overline{mod}(RG)$ is a self-equivalence.

Define $\sigma(X) := \Omega^{-n}(X_\infty)$. It can be shown that this does not depend on the chosen projective resolution of X.

Lemma 3.6 $\sigma(HD_R(G)) \subseteq HSt_R(G)$.

Proof. Let X be a two-sided tilting complex with isomorphism class in $HD_R(G)$. Replace X by an isomorphic copy which has projective homogeneous components, also denoted by X. Since now all components are projective as right-RG-modules, one can replace the left derived tensor product by the ordinary tensor product. Now, X is isomorphic to the complex Y whose homogeneous components of degree higher than $n+2$ are 0, the component in degree $n+1$ is $X_\infty = ker\ d_{n+1}$ and all the other homogeneous components of Y are identical to those of X. The differentials of Y are the obvious ones. The image of X in $StPic_R(RG)$ is $\Omega^{-n}(X_\infty)$ where Ω is the syzygy operator. We have to prove that $X_\infty \otimes_{RG} R \simeq \Omega^n(R)$ in $\overline{mod}(RG)$.

If M is a projective $RG \otimes_R RG^{op}$-module, then $M \otimes_{RG} R$ is a projective RG-module. In fact, this is true for free $RG \otimes_R RG^{op}$-modules, and hence it holds for projective modules as well. Since, $X \otimes_{RG} R \simeq R$, the complex $X \otimes_{RG} R$ gives the first n terms of a projective resolution of the trivial module R. Since $X \otimes_{RG} R \simeq Y \otimes_{RG} R \simeq R$ we get a commutative diagram with exact rows

$$\begin{array}{ccccccccccc}
0 & \to & X_\infty \otimes_{RG} R & \to & X_n \otimes_{RG} R & \to & \dots & \to X_m \otimes_{RG} R & \to & R & \to & 0 \\
\uparrow & & \uparrow & & \| & & & \| & & \| & & \\
X_{n+2} \otimes_{RG} R & \to & X_{n+1} \otimes_{RG} R & \to & X_n \otimes_{RG} R & \to & \dots & \to X_m \otimes_{RG} R & \to & R & \to & 0
\end{array}$$

This means $X_\infty \otimes_{RG} R \simeq \Omega^n(R)$. ■

Lemma 3.7 *If RG has the Krull-Schmidt property on lattices, then $\widetilde{HSt}(RG)$ acts on $H^*(G, R)$ and the action of $HD_R(G)$ on $H^*(G, R)$ factors through the action of $\widetilde{HSt}_R(G)$.*

Proof. If RG has the Krull-Schmidt property on lattices one gets

$$H^n(G, R) = Ext^n_{RG}(R, R) = Hom_{\overline{mod}(RG)}(\Omega^n(R), R)$$

It is now clear that $HSt_R(G)$ acts on $H^*(G, R)$ and that the action of $HD_R(G)$ factors through the action of $HSt_R(G)$. ■

It should be noted that the question when RG has the Krull Schmidt property on lattices is a delicate one if R is only local and not complete. We refer to [1, § 36].

Let R be a complete discrete valuation ring of characteristic 0 with residue field of characteristic p and let G be a finite group with cyclic p-Sylow subgroup so that the principal block has no exceptional vertex. Lemma 3.7 enables us to prove that the action of $HD_R(G)$ on $H^*(G,R)$ is trivial.

In fact, Linckelmann computed $StPic_R(B_0(RG))$ for groups with cyclic p-Sylow subgroup P. He obtained in [4, Theorem 11.4.9] and the preceding remarks that in this case $StPic_R(B_0(RG)) \simeq C_{2e} \times Aut(P)/E$ where E is the inertia quotient and $e = |E|$ is the number of simple modules. In our case, in the absence of an exceptional vertex one gets $StPic_R(B_0(RG)) \simeq C_{2(p-1)}$ and this cyclic group correspond to the $2(p-1)$ syzygies of the trivial module. So, $HSt_R(G) = \{1\}$. We note that in [9], using a deep result of Khovanov and Seidel [3], we prove that $HD_R(G)$ contains a braid group on $e = p-1$ strings.

Moreover, Linckelmann obtained in [4, Theorem 11.4.10] for R being a complete discrete valuation domain with residue field of characteristic 2 and $\mathfrak{A}_4$ be the alternating group of order 12 that $StPic(R\mathfrak{A}_4) \simeq C_\infty \times Pic_R(R\mathfrak{A}_4)$. Here C_∞ is the group consisting of taking syzygies, which is the image of the subgroup of $TrPic_R(R\mathfrak{A}_4)$ generated by shift in degrees. Hence, the action of $HD_R(\mathfrak{A}_4)$ on $H^*(\mathfrak{A}_4, R)$ is the action of $Aut_R(R\mathfrak{A}_4)$ on $H^*(\mathfrak{A}_4, R)$. Again, $HD_R(\mathfrak{A}_4)$ contains a braid group on 3 strings as is shown in [12].

The above proof uses the functor $TrPic_R(RG) \longrightarrow StPic_R(RG)$. The existence of this functor needs some hypotheses on the coefficient domain R. It should be noted that it is actually possible to compute the action of a self-equivalence on the cohomology explicitly without passing through the stable category. The result is hence valid in a more general context.

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