Tilting selfinjective algebras and Gorenstein orders

by

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Abstract

Endomorphism rings of a special class of tilting complexes for selfinjective algebras and for Gorenstein orders are computed. As applications we get short proofs of generalizations of results of Rickard and Linckelmann on (artinian or integral) blocks with cyclic defect groups. Our construction also covers examples of derived equivalences for certain blocks with non–cyclic defect groups, and for algebras which are not group algebras.

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1 Introduction

By Rickard's fundamental theorem [7], the rings which are derived equivalent to a ring Λ are precisely the endomorphism rings of tilting complexes over Λ. A tilting complex T is a finitely generated complex of finitely generated projective modules, which does not admit self–extensions and which has the property that the smallest triangulated subcategory of $D^b(\Lambda)$ which contains T also contains all the finitely generated projective modules. Rickard constructed tilting complexes between blocks of cyclic defect group over algebraically closed fields of characteristic p and their Brauer correspondent blocks in the group ring of the normalizer of their defect group. [8]. Linckelmann [4] proved an analogous result for an extension of the p-adic integers having large enough residue field. Other known examples are derived equivalences between certain blocks with dihedral defect groups (Linckelmann [5]) including the 2-adic principal blocks of the alternating groups A_4 and A_5 over an algebraically closed field of characteristic 2 (Rickard [9], [7, Section 8]).

The aim of this note is to show that tilting complexes giving all these, and many more, derived equivalences can be constructed and studied in a unified way. The innovation in our approach is that we describe endomorphism rings of tilting complexes via endomorphism rings of modules. To compute homomorphisms in the module category turns out to be easier than the usual computation in the homotopy category.

In case Λ is an order, the endomorphism ring $\Gamma := End_{D^b(\Lambda)}(T)$ of the tilting complex T with homology concentrated in the two degrees 0 and 1 (to be defined in section 2) is the pullback in the following diagram (see proposition 2.1):

$$\begin{array}{ccccc} \Gamma & \rightarrow & End_\Lambda(H^0(T)) & \rightarrow & 0 \\ \downarrow & & \downarrow & & \\ End_\Lambda(H^1(T)) & \rightarrow & \underline{End_\Lambda}(H^0(T)) & \rightarrow & 0 \\ \downarrow & & \downarrow & & \\ 0 & & 0 & & \end{array}$$

Here $\underline{End_\Lambda}(H^0(T))$ denotes the quotient of $End_\Lambda(H^0(T))$ modulo those morphisms which factor through a projective module. The proof of this statement is only based on a certain distinguished triangle in the homotopy category and two long exact sequences coming from this triangle.

In section 2 we define certain tilting complexes, and compute their endomorphism rings as pullbacks of endomorphism rings of modules. The construction, and the computation, works almost all the way both for artinian algebras and for orders.

The following sections give applications: In section 3 we discuss blocks with cylic defect groups, both in the artinian and in the integral case. Thus in the artinian case we have to study Brauer tree algebras, whereas in the integral case we have to work with Green orders in the sense of Roggenkamp [11, 12]. Our result here generalizes results of Rickard (in case of a field) and of Linckelmann (in case of a certain complete discrete valuation domain).

Using our construction it is possible to give explicitly a twosided tilting complex for Green orders and blocks of cyclic defect groups [14].

In section 4 we collect other examples, in order to underline that our construction works for some blocks with non–cyclic defect groups, too, and to show that this kind of tilting works for some algebras which are not group algebras.

2 Endomorphism rings of certain tilting complexes

We will cover simultaneously the case of finite dimensional algebras and of orders. Thus we fix the following notations for this section.

- By R we denote a commutative ring which is either artinian or a local Dedekind domain.
- If R is artinian, then Λ denotes an Artin R–algebra.
  If R is a Dedekind domain, then Λ is assumed to be a finitely generated R–free R–algebra with $frac(R) \otimes_R \Lambda$ being a semisimple $frac(R)$–algebra.
- If X, Y, and Z are Λ–modules, then by $Hom_\Lambda(X, Y, Z)$ we denote the Λ–homomorphisms from X to Z which factor over Y.

Proposition 2.1 *Let Λ be as above and let L be either a Λ–module (in case Λ is artinian) or a Λ–lattice (otherwise).*

- *Let*
  $$\ldots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P \xrightarrow{\varphi} Q \longrightarrow L \longrightarrow 0$$
  be the first terms of a minimal projective resolution of L.
- *Let $P' \oplus Q$ be a progenerator for Λ such that P is a direct summand of P' but Q and P' do not have a common direct summand.*
- *Let T be the complex*
  $$0 \longrightarrow P \oplus P' \stackrel{(\varphi,0)}{\rightarrow} Q \longrightarrow 0$$
  with homology concentrated in degree -1 and 0.
- *Let $K' := ker\varphi$, let $K = K' \oplus P'$.*
- *Assume that*
  $$Hom_\Lambda(P \oplus P', L) = 0 = Hom_\Lambda(L, P \oplus P').$$
- *Assume moreover that the projective Λ–modules are injective (in the category of Λ–modules with Λ–module–homomorphisms in case of Λ being artinian, or in the category of Λ–lattices with R–pure Λ–homomorphisms otherwise). That is, Λ is a self–injective algebra or a Gorenstein order.*

Then, T is a tilting complex and its endomorphism ring (inside the derived category) occurs in the following two short exact sequences:

$$0 \to (K, P \oplus P', K) \to (T,T) \to (L,L) \to 0$$

$$0 \to (Q, L, Q) \to (T,T) \to (K,K) \to 0$$

In case Λ is an order, the endomorphism ring (T,T) is the pullback in the following diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & (Q,L,Q) & = & (L,Q,L) & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & (K,P\oplus P',K) & \to & (T,T) & \to & (L,L) & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & (K,P\oplus P',K) & \to & (K,K) & \to & (L,L)/(L,Q,L) & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Here, the mapping $(K,K) \to (L,L)/(L,Q,L)$ is defined as lifting any endomorphism of K along the exact sequence $0 \longrightarrow K \longrightarrow P \oplus P' \longrightarrow P \longrightarrow Q \longrightarrow L \longrightarrow 0$ using that Λ is Gorenstein and the resulting endomorphism of L is well defined modulo (L,Q,L).

We note that the first assumption implies the vanishing of $Ext^1_\Lambda(L,L)$.

We will frequently use in case of an order that for an exact sequence of lattices

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

every morphism $X \longrightarrow I$ for an injective lattice I can be lifted to a morphism $Y \longrightarrow I$. This is well known and seen most easily by taking the pushout Ω along $X \longrightarrow Y$ and $X \longrightarrow I$ and observing that by the injectivity of I in the category of lattices the sequence

$$0 \longrightarrow I \longrightarrow \Omega \longrightarrow Z \longrightarrow 0$$

splits.

We split the **Proof** into several claims.

We abbreviate for any two complexes X and Y of Λ modules $(X,Y) := Hom_{D^b(\Lambda)}(X,Y)$ and ${}^i(X,Y) := Hom_{D^b(\Lambda)}(X, Y[i])$.

In the following we will often have to identify various spaces of homomorphisms, which are either endomorphism rings or contained in such. The identifications (which will be written as equalities) will be done by applying Hom–functors. Thus the ring structure by composition of maps always will be preserved.

Claim 2.2 *[9] T is a tilting complex.*

Proof of the claim. We decompose T into $T_1 \oplus T_2$, where T_1 is the complex

$$0 \to P \to Q \to 0$$

and T_2 equals $P'[1]$. The assumption implies that every homomorphism from $P \oplus P'$ to Q has to factor via φ. Also by assumption, a homomorphism $\psi : Q \to P \oplus P'$ with $\varphi \cdot \psi = 0$ must be zero. This proves that ${}^i(T_1,T_1) = 0$ for all $i \neq 0$, and that ${}^i(T_1,T_2) = {}^i(T_2,T_1) = 0$ for all $i \neq 0$. Thus, T does not have self–extensions.

It remains to verify that the indecomposable direct summands of T generate the triangulated category $K^b(\Lambda)$. In fact, by definition $P'[1]$ is a direct summand of T. By construction of T, P is a direct summand of a direct sum of copies of P'. Sending T to $P \oplus P'$ via the identity on $P \oplus P'$ defines a complex homomorphism with mapping cone Q. However, $Q \oplus P'$ is a progenerator for Λ.

Claim 2.3 *There is an exact sequence* $0 \to (K, P' \oplus P, K) \to (T,T) \to (L,L) \to 0.$

Proof of the claim: There is a triangle $T \to L \to K[2] \rightsquigarrow T[1]$ which gives rise to a long exact sequence part of which looks as follows:

$$\ldots \to (T, L[-1]) \to (T, K[1]) \to (T,T) \to (T,L) \to (T, K[2]) \to \ldots$$

Now $(T, L[-1]) = 0$ by the following argument:

$$\begin{array}{ccccccccc} & & 0 & \longrightarrow & P \oplus P' & \longrightarrow & Q & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \ldots & \longrightarrow & P_2 & \longrightarrow & P_1 & \longrightarrow & P & \longrightarrow & Q & \longrightarrow & 0 \end{array}$$

gives a commutative diagram only if the vertical morphism $Q \longrightarrow P$ factors through the kernel of the horizontal morphism $P \longrightarrow Q$, hence through its projective cover P_1. The morphism is henceforth homotopic to 0. A totally analogous argument gives that also the vertical homomorphism $P \longrightarrow P_1$ is homotopic to 0. Also, $(T, K[2]) = 0$, since T is concentrated in degrees where each projective resolution of $K[2]$ is zero.

Now, (T,L) coincides with (L,L) by the following argument. Take an endomorphism of L. Represent L by its projective resolution. We map a morphism

$$\begin{array}{ccccccccccc} \ldots & \longrightarrow & P_2 & \longrightarrow & P_1 \oplus P' & \stackrel{(d,1_{P'})}{\longrightarrow} & P \oplus P' & \longrightarrow & Q & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow f_P & & \downarrow f_Q & & \\ \ldots & \longrightarrow & P_2 & \longrightarrow & P_1 & \longrightarrow & P & \longrightarrow & Q & \longrightarrow & 0 \end{array}$$

in (L,L) to the morphism

$$\begin{array}{ccccccccccc} & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & P \oplus P' & \longrightarrow & Q & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \| & & \\ \ldots & \longrightarrow & P_2 & \longrightarrow & P_1 \oplus P' & \stackrel{(d,1_{P'})}{\longrightarrow} & P \oplus P' & \longrightarrow & Q & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow f_P & & \downarrow f_Q & & \\ \ldots & \longrightarrow & P_2 & \longrightarrow & P_1 & \longrightarrow & P & \longrightarrow & Q & \longrightarrow & 0 \end{array}$$

in (T,L). This gives a monomorphism: In fact, two of these liftings differ by a mapping $f = (f_Q, f_P, 0, \ldots)$ of complexes which induces 0 in (L,L) and henceforth the mapping f_Q from Q to Q in the first degree of the projective resolution has image in $ker(Q \longrightarrow L)$. Since P is a projective cover of this kernel, f_Q can be factorized via P_1 and is homotopic to 0. For the mapping f_P in degree 1 we argue analogously. On the other hand, a mapping (T,L) clearly induces a mapping (L,L) and since any endomorphism of L can be lifted uniquely up to homotopy to a mapping between any two projective resolutions, this assignment is inverse to the previous one.

We have to show that $(T, K[1])$ equals $(K, P \oplus P', K)$. Again, representing K by its projective resolution, an element in $(T, K[1])$ is fixed by a homomorphism α from $P \oplus P'$ to P_1.

$$\begin{array}{ccccccccc} & \longrightarrow & 0 & \longrightarrow & P \oplus P' & \longrightarrow & Q & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \alpha & & \downarrow & & \\ \ldots & \longrightarrow & P_2 & \longrightarrow & P_1 & \longrightarrow & 0 & \longrightarrow & 0 \end{array}$$

Such an element is zero if α can be factored via $P \oplus P' \to Q$ and $P_2 \to P_1$. Composing α with the inclusion $K \to P$ gives a map $(T, K[1]) \to (K, P \oplus P', P_1)$ where on the right hand side elements are homomorphisms modulo factorization of $P \oplus P' \to P_1$ via $P_2 \to P_1$:

$$\begin{array}{ccccccccc} \ldots & \longrightarrow & 0 & \longrightarrow & K & \longrightarrow & 0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & \longrightarrow & 0 & \longrightarrow & P \oplus P' & \longrightarrow & Q & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \alpha & & \downarrow & & \\ \ldots & \longrightarrow & P_2 & \longrightarrow & P_1 & \longrightarrow & 0 & \longrightarrow & 0 \end{array}$$

This assignment clearly is surjective, and it is injective since P/K' is a pure sublattice of Q, and P_1 is injective.

As K is a pure sublattice of $P \oplus P'$ and P_2 is injective, the factorization condition can be replaced by saying that the whole map $K \to P \oplus P' \to P_1$ is to be taken modulo factorization via $P_2 \to P_1$.

Composing such a map with the epimorphism $P_1 \to K$ produces a map β in $(K, P \oplus P', K)$. If β is zero, then α factors via the kernel of the epimorphism $P_1 \to K$, hence over the projective cover of this kernel, which is P_2. Since $P \oplus P'$ is projective, any β is in the image. Thus we have an identification $(T, K[1]) = (K, P \oplus P', K)$.

Claim 2.4 *There is an exact sequence* $0 \to (Q, L, Q) \to (T, T) \to (K, K) \to 0$.

Proof of the claim: From the above triangle $T \to L \to K[2] \rightsquigarrow T[1]$ we get another long exact sequence:

$$\ldots \to (K[2], T) \to (L, T) \to (T, T) \to (K[1], T) \to (L[-1], T) \to \ldots.$$

Now $(K[2], T)$ is zero, since T is concentrated in degrees 0 and -1, where each projective resolution for $K[2]$ is zero.

The image of $(K[1], T)$ in $(L[-1], T)$ is zero for the following reason: An element of $(K[1], T)$ is given by a map $\alpha : P_1 \oplus P' \to K \to P \oplus P'$. The image of α in $(L[-1], T)$ sends P_1 to P via α, whereas P and Q are being sent to zero as is shown by the following diagram, representing L by its minimal projective resolution $\ldots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P \longrightarrow Q \longrightarrow 0 \longrightarrow \ldots$

$$\begin{array}{cccccccccccc}
L[-1] & : & \ldots & \longrightarrow & P_2 & \longrightarrow & P_1 & \longrightarrow & P & \longrightarrow & Q & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
K[1] & : & \ldots & \longrightarrow & P_2 & \longrightarrow & P_1 \oplus P' & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
T & : & \ldots & \longrightarrow & 0 & \longrightarrow & P \oplus P' & \longrightarrow & Q & \longrightarrow & 0 & \longrightarrow & 0
\end{array}$$

Since P_2 is being sent to zero, α factors via K'.

But P_1 is an injective lattice and the sequence

$$0 \to K' \to P \to Q \to L \to 0$$

is pure. Thus the mapping α factors via P, and the lifting is a homotopy equivalence to zero.

Moreover, $(K[1], T)$ equals (K, K) by construction of T.

Finally, $(L, T) = (Q, L, Q)$. In fact, represent L by a minimal projective resolution

$$\ldots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P \longrightarrow Q \longrightarrow 0 \longrightarrow \ldots$$

with homology concentrated in degree 0. An element $\alpha \in (L, T)$ is given by a sequence of mappings $(\alpha_i)_{i \in \mathbb{N}_0}$, where $\alpha_0 \in (Q, Q)$; $\alpha_1 \in (P, P \oplus P')$ and $\alpha_i = 0$ for $i \geq 2$. Therefore, α_1 induces a mapping in $Hom(coker(P_1 \longrightarrow P), (P \oplus P'))$. But the sequence

$$0 \longrightarrow coker(P_1 \longrightarrow P) \longrightarrow Q \longrightarrow L \longrightarrow 0$$

is pure exact, $P \oplus P'$ is an injective lattice, and hence we get a homotopy such that $\alpha_1 = 0$. Therefore, we have to deal with a mapping in (L, Q), or equivalently with its composition with the natural epimorphism $Q \to L$, thus with $\alpha \in (Q, L, Q)$. On the other hand, each element in (Q, L, Q) defines an element in (L, T).

Now, if Λ is an order, (Q, L, Q) equals (L, Q, L). (This is false in general if Λ is an artinian algebra over a field!) In fact, let $\pi : Q \longrightarrow L$ be the natural epimorphism. $(Q, L, Q) = \pi \cdot (L, Q)$ and we map $(Q, L, Q) \ni \pi\alpha \longrightarrow \alpha\pi \in (L, Q, L)$. To show that this mapping is well defined we pass to the field of fractions and see that $\pi\alpha = \pi\beta$ implies $\alpha = \beta$. Analogously, we give an inverse, proving the statement. ∎

As a consequence, we get the following diagram, in which (T,T) appears. Using the above observation in case Λ is an order we can extend this to a pullback diagram, which proves the proposition:

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & & \downarrow & & & & \\ & & & & (Q,L,Q) & & & & \\ & & & & \downarrow & & & & \\ 0 & \rightarrow & (K,P\oplus P',K) & \rightarrow & (T,T) & \rightarrow & (L,L) & \rightarrow & 0 \\ & & & & \downarrow & & & & \\ & & & & (K,K) & & & & \\ & & & & \downarrow & & & & \\ & & & & 0 & & & & \end{array}$$

The homomorphism $(K,K) \rightarrow (L,L)/(L,Q,L)$ is given by lifting maps along the exact sequence $0 \rightarrow K \rightarrow P \oplus P' \rightarrow Q \rightarrow L \rightarrow 0$. (Note that in the artinian case, (Q,L,Q) need not be contained in (L,L), hence an analogous pullback need not exist in that case.) ∎

We observe that in the case of Λ being a Gorenstein order, $(L,L)/(L,Q,L)$ is just the endomorphism ring of L viewed in the R–stable category. The syzygy functor is an autoequivalence of the R–stable category and hence

$$\begin{array}{rcl} (L,Q,L) & \simeq & (K,K)/\text{endomorphisms factoring through a projective} \\ & \simeq & (K,P\oplus P',K) \end{array}$$

3 Blocks with cyclic defect groups

In this section we are going to apply the proposition to blocks with cyclic defect groups. We deal only with the case of orders, here, since the proof for finite dimensional algebras is exactly the same, except of the notation (which is rather different in the two cases).

We recall from [11, 12] that a Green–order Λ is determined by the following data:

- A finite non–oriented tree A with n vertices embedded into the plane (this is equivalent to saying that we impose to each vertex a cyclic ordering of the vertices adjacent to this vertex),
- a local finitely generated R–torsion R–algebra $\Omega(\Lambda)$,
- for each vertex i of A a local R–order $\Omega_i(\Lambda)$ and a surjective ringhomomorphism $\Omega_i(\Lambda) \longrightarrow \Omega(\Lambda)$ with kernel being a principal ideal.

More generally we will prove:

Theorem 1 *Let Λ and Γ be two Green orders which are Gorenstein such that the associated trees $\mathfrak{T}(\Lambda)$ and $\mathfrak{T}(\Gamma)$ both are connected and have the same number n of vertices. Assume moreover that $\Omega(\Lambda)$ equals $\Omega(\Gamma)$ and that the n–element set of ring homomorphisms $\Omega_i(\Lambda) \rightarrow \Omega(\Lambda)$ associated with Λ coincides (as an unordered set) with the n–element set $\Omega_i(\Gamma) \rightarrow \Omega(\Gamma)$ of ring homomorphisms associated with Γ.*

Then Λ and Γ are derived equivalent.

A different, more computational, proof of this result has been given by K.W.Roggenkamp and the second author in [13].

For the definition and notation of Green orders we refer to Roggenkamp [11, 12]. There it is also explained why integral blocks of cyclic defect over a discrete valuation ring with splitting quotient field are Green orders, and all information is provided for deducing from the above theorem the results of Rickard and Linckelmann mentioned in the introduction.

We note that Green orders need not be Gorenstein. However, the Gorenstein property of a Green order is determined by the Gorenstein property of the local orders Ω_i. Hence in the theorem, Λ is Gorenstein if and only if Γ is. For any finite group G the R–order RG is a Gorenstein order.

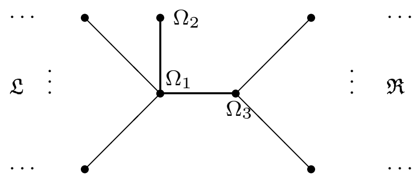
Proposition 3.1 *Let Λ be a Green order with connected tree $\mathfrak{T}$ having n vertices, and with ring homomorphisms $\Omega_i \to \Omega$ associated with the vertices of $\mathfrak{T}$.*

Let Γ be a Green order having as tree a star $\mathfrak{S}$ with n vertices such that the homomorphisms $\Omega_i \to \Omega$ are associated with the vertices of the star (in any order), where the homomorphisms $\Omega_i \to \Omega$ are the same as for Λ.

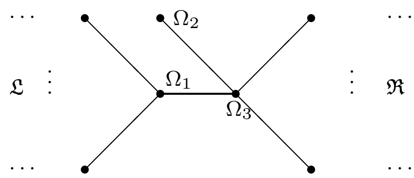
Then Λ and Γ are derived equivalent.

Proof. We proceed in two steps: First we show how to go step by step from an arbitrary tree $\mathfrak{T}$ to a star $\mathfrak{S}$. After that we show how to change the assignment of the rings Ω_i to the vertices.

For the first step, we proceed by induction on the sum $\sum_{i,j} d(i,j)$ where $d(i,j)$ denotes the distance of vertices in the connected graph $\mathfrak{T}$. Obviously, the smallest value of the sum is for $\mathfrak{T}$ a star, so the induction start is trivial. In the induction step we have a tree $\mathfrak{T}$ of the following form (where $\mathfrak{L}$ and $\mathfrak{R}$ represent the rest of the tree $\mathfrak{T}$):



We will move the edge between Ω_2 and Ω_1 to an edge between Ω_3 and Ω_2 as indicated in the following picture.



Denote the Green order associated with this new tree $\mathfrak{T}'$ by Γ (where the Ω_i are arranged as for Λ except for the obvious change with respect to Ω_2). We have to construct a derived equivalence between Λ and Γ, which we will do by using proposition 2.1. We define a tilting complex $\mathfrak{T}$ over Λ in the following way: Let P be the projective Λ–module associated with the edge $\Omega_1 - \Omega_3$, and let Q be associated with $\Omega_1 - \Omega_2$. Then Q has an irreducible quotient L, and the minimal projective resolution of L starts

$$0 \leftarrow L \leftarrow Q \leftarrow P \leftarrow P_1 \ldots$$

where P_1 is associated with an edge between Ω_3 and a vertex in the part $\mathfrak{R}$ of $\mathfrak{T}$. The structure of Λ guarantees the validity of all assumptions in 2.1. Thus we get a tilting complex $\mathfrak{T}$ which has endomorphism ring given by the pullback diagram in 2.1.

The ingredients of the pullback are: (L, L) which is Ω_2, its quotient is Ω, and (K, K). We may decompose K into $K_1 \oplus P'$ where P' is the sum of all indecomposable projective Λ–modules except Q, and K_1 is an irreducible lattice contained in P, and attached to the vertex Ω_3. The endomorphism ring of P' is again a Green order, which is described by the tree obtained from $\mathfrak{T}$ by deleting the edge attached to Q. Since K_1 is irreducible, the homomorphisms between K_1 and P' are multiples of the inclusion $K_1 \to P$ and the projection $P_1 \to K_1$. Altogether, we can now check that the endomorphism ring of T is the Green order defined by $\mathfrak{T}'$.

For the second part we use again proposition 2.1.

We can assume that the two trees of Λ and Γ is a common stem. We have to verify that we may interchange any two of the vertices by applying proposition 2.1. We have the tree

$$\bullet_1 - \bullet_2 - \bullet_3 - \ldots - \bullet_n$$

and choosing for L the lattice corresponding to the vertex 1, we get as ring of endomorphisms the Green order associated to the following tree:

$$\begin{array}{ccccccc} \bullet_1 & & & & & & \\ | & & & & & & \\ \bullet_3 & - & \bullet_4 & - & \ldots & - & \bullet_n \\ | & & & & & & \\ \bullet_2 & & & & & & \end{array}$$

If we choose the lattice corresponding to 2 in this Green order we get a tree with the indices 1 and 2 interchanged. We reached the permutation (1 2) for the vertices of the stem. If we keep on to choose for L the lattice associated to 1, and this $n-2$ times, until we reach again a stem, we get the permutation $(1\ 2\ \ldots\ n)$ of the tree. These two permutations generate, as is well known, the whole symmetric group on n letters. ■

Remark We want to mention that this last combinatorial manipulation on the tree is also valid without having the assumption on Λ to be Gorenstein.

4 Further examples

In this section we apply proposition 2.1 to certain blocks with non–cyclic defect groups. We do not give new results, but show how proposition 2.1 simplifies the proofs of known results.

Linckelmann constructed in [5] derived equivalences between certain blocks of dihedral defect, or more general, between certain algebras appearing in the lists in Erdmann's classification [2] of tame blocks. There are three series of algebras, A_n, B_n, and C_n (for $n \in \mathbb{N}$). All algebras considered here have three simple modules, and the indecomposable projective modules are uniquely defined by their composition series which can be found in [5].

The tilting complexes used in [5] satisfy the assumptions of 2.1. Over A_n we choose $L = L(1)$, which has projective resolution starting

$$0 \leftarrow L \leftarrow P(1) \leftarrow P(2) \oplus P(3) \leftarrow \ldots$$

Thus, K is a direct sum of $P(2)$, $P(3)$, and a module M having composition length three, simple top $L(1)$ and semisimple socle containing $L(2)$ and $L(3)$ each with multiplicity one. In this situation, (Q, L, Q) has dimension one, and is contained in the socle of the algebra (T, T). Thus it remains to compute the quotient algebra (K, K). The endomorphisms of $P(2)$ and of $P(3)$ are known already, and the endomorphisms of M as well as all homomorphisms between M and $P(2)$ or $P(3)$ are one–dimensional. Thus one can write down a basis for (T, T) and sees that it is isomorphic to B_n. A very similar computation shows that B_n is the endomorphism ring over C_n of the tilting complex associated with the simple module $L(2)$. In the same way, one may also check that associated with the simple module $L(2)$ is an autoequivalence of $D^b(C_n - mod)$.

We obtained

Corollary 4.1 *[5] Let G be a finite group and let R be an algebraically closed field of characteristic $p \geq 0$. Let B be a block of RG with dihedral defect group D. If there are exactly three non isomorphic simple modules in B, then B is derived equivalent to its Brauer correspondent in $RN_G(D)$, the group ring of the normalizer of D in G over R.*

This includes the case of the principal block of the alternating group on 5 letters at characteristic 2 and the group ring of the alternating group on 4 letters at characteristic 2 which are here proven to be derived equivalent. That result has already been announced by J. Rickard in [7] and a proof can be found in [9, 10].

Finally we mention a situation where the method used in proposition 2.1 can be applied, too, although the algebras are not group algebras. In [3], the derived equivalence class of hereditary orders has been determined. The tilting complexes used there satisfy the assumptions of 2.1 with K' being zero, but with L not being a lattice. A careful examination of the proof of proposition 2.1 shows that the exact sequences in 2.1 still are valid, but the pullback diagram is not correct any more. In fact, in this situation, the endomorphism ring of T need not be an order any more.

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