Contents

# STRUCTURE OF BLOCKS WITH CYCLIC DEFECT GROUP AND GREEN CORRESPONDENCE

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## 1. Introduction

This paper is a based on a course I gave at the Ovidius university of Constanta on "Structure of blocks with cyclic defect group and Green correspondence". However, I added many details to the script of the course I have given there. We shall divide the material into three parts.

- In the first part we will present a new way due to Auslander and Kleiner [1] to get a form of Green correspondence. The classical Green correspondence follows easily from that and this more general point of view might have some impact to other fields of interest. The classical Green correspondence will be used in the following sections with good success.
- In the second part we shall present the classical and very beautifully written paper of Green [5]. Here, the Green correspondence in its classical form is used intensively. We just cite from the classical paper of Dade the fundamental properties of blocks with cyclic defect group. We also give results of Michler [13, 14] on the structure of blocks with cyclic defect groups.
- The third part deals with K. W. Roggenkamp's paper [16] on Green orders in which he firstly defined Green orders, secondly used the results of Green to prove that a block with cyclic defect group is a Green order, and thirdly determines the structure of a Green order in great detail. As far as is known to the author, this is the most far reaching result on the structure of blocks with cyclic defect group. For the proof we follow [16].

The reader is assumed to know very basic facts on categories, not much more than the definition of a category, functors and natural transformations. A good reference is [15, 12]. Also the basic notions in noetherian ring theory are assumed such as the notion of a radical of a ring and a module, a socle and a top. As a good reference we give here [7]. Furthermore, some very basic algebraic number theory is assumed such as the basic definitions of a Dedekind domain and the ramification index for local fields. Certainly [6] covers more than what is enough as reference. Besides the deep theory of Dade, all proofs are included. In this sense the paper is self–contained.

The course, I gave in Constanta is contained, for the part dealing with the Green correspondence in the Sections 2.1, 2.2, 2.3, 2.4.1, 2.4.2 and the beginning parts of 2.4.4. The reader who is only interested in the classical Green correspondence is adviced to read only Sections 2.1, 2.3 and 2.4.1. Of course, the proof is given, as is done in [1], in the abstract terms. The nicest part of the abstract Green

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correspondence is collected in the Section 2.4.2 and 2.4.3. For the classical theory on blocks with cyclic defect groups, the material presented in Constanta is located in the Sections 3.1, 3.2 and 3.3. Section 3.4 and 3.5 are devoted to prove Green's walk around the Brauer tree, in the way as Green did. For Roggenkamp's description of blocks with cyclic defect groups the parts which were presented in Constanta are located in Section 4.1 and 4.2. The rest of the sections contain mainly proofs, which especially in Section 2 are uncomfortly technical, and which are not needed for the understanding of the other parts.

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## 2. Green correspondence

2.1. **Motivation.** Let $G$ be a finite group and let $R$ be a complete discrete valuation ring of characteristic 0 with residue field $k$ of characteristic $p$. A large part of modular representation theory of finite groups deals with trying to relate the representation theory of $G$ to the representation theory of the "$p$–local structure" of $G$. The first and perhaps most elementary attempt to do so is the Green correspondence.

By $RF-\overline{mod}^{\circ}$ we denote for any finite group $F$ the *stable module category*. The objects of ${}_{RF}\overline{mod}^{\circ}$ are the same as those of $RG-mod^{\circ}$, namely finitely generated $R$–projective $RF-$modules, called in the sequel $RF-$lattices. To define the morphism set we have to put an equivalence relation onto the morphism sets of $RF-mod^{\circ}$. Let $M$ and $N$ be two $RF$–lattices. Two morphisms $f, g \in Hom_{RF}(M,N)$ are called equivalent, $f \cong g$, if and only if $f-g$ factors through a projective $RF$–module. Then,

$$Hom_{RF-\overline{mod}^{\circ}}(M,N) =: \overline{Hom}_{RF}(M,N) := Hom_{RF}(M,N)/_{\cong}.$$

**Theorem 2.1.** *(Green) Let $D$ be a Sylow $p-$subgroup of $G$ and let $H := N_G(D)$ be the normalizer of $D$ in $G$. Assume that for all $g \in G \setminus H$ we get $gDg^{-1} \cap D = \{1\}$. Then, induction*

$$ind_H^G := RG \otimes_{RH} - : RH - \overline{mod}^{\circ} \longrightarrow RG - \overline{mod}^{\circ}$$

*and restriction*

$$res_H^G : RG - \overline{mod}^{\circ} \longrightarrow RH - \overline{mod}^{\circ}$$

*are mutually inverse equivalences of categories. These equivalences of categories preserve the indecomposability of modules. More precisely, for every indecomposable non projective $RG$–module $M$ there is an indecomposable non projective $RH$–module $f(M)$ such that $f(M)|res_H^G(M)$ and $M/f(M)$ is a projective $RH$–module. For every indecomposable non projective $RH$–module $N$ there is an indecomposable non projective $RG$–module $g(N)$ such that $g(N)|ind_H^G(N)$ and $(ind_H^G N)/g(N)$ is a projective $RG$–module.*

**Remark 2.2.**
- The situation described by the hypotheses of Theorem 2.1 is commonly known and will be referred to as the TI–situation. Here TI stands for 'trivial intersection'.
- One should note that even the classical Green correspondence is much more general than expressed here. The theorem above is just a very special case where the correspondence appears in the most illustrative way.

This theorem is a very special case of a much more general statement which was proven by M. Auslander and M. Kleiner in [1]. They give a categorical and much more general approach to the theory of J. A. Green [4] which establishes an equivalence between certain quotient categories of finitely generated $RG$–modules and finitely generated $RH$–modules where $H$ is a subgroup of $G$ containing the normalizer in $G$ of a certain $p$–subgroup $D$ of $G$. The theorem above is the case for which the categorical equivalence of Green is most easily formulated.

2.2. **Adjoint functors.** The method of Auslander and Kleiner intensively use adjoint functors. We shall give a very brief account on what this is about.

We assume the reader to be familiar with the notion of a category, functors and of natural transformations between functors. As basic reference one might see [15] or [12].

**Definition 2.3.** Let $\mathcal{A}$ and $\mathcal{B}$ be two categories and let $F : \mathcal{A} \longrightarrow \mathcal{B}$ and $G : \mathcal{B} \longrightarrow \mathcal{A}$ be two functors. The category of sets is called $\mathcal{E}\backslash\int$.

If there is an equivalence

$$\mathcal{B}(F-, -) \simeq \mathcal{A}(-, G-)$$

of bifunctors

$$\mathcal{A} \times \mathcal{B} \longrightarrow \mathcal{E}\backslash\int$$

then the functor $F$ is said to be *left adjoint* to the functor $G$ and the functor $G$ is said to be *right adjoint* to $F$. The pair $(F, G)$ is said to be an adjoint pair.

Let $(F, G)$ be an adjoint pair. By the defining relation we get an isomorphism of bifunctors

$$\eta' : \mathcal{B}(F-, F-) \simeq \mathcal{A}(-, GF-)$$

and hence we get a natural transformation

$$\eta : 1_{\mathcal{A}} \longrightarrow GF$$

by just putting $\eta(A) := \eta'(id_{FA})$. The natural transformation $\eta$ is called the *unit of the adjointness*. Of course, it depends not only on the two functors $F$ and $G$ but also on the choice of the isomorphisms in the defining relation.

*We give an example.* Let $R$ be a commutative ring and let $G$ be a finite group with subgroup $H$. We denote by $\iota : RH \longrightarrow RG$ the canonical embedding. Since an $RG$–module $M$ is nothing else than a $R$–module $M$ together with a ring homomorphism $RG \longrightarrow End_R(M)$, one defines the restriction $res_H^G(M)$ just as $RH \longrightarrow RG \longrightarrow End_R(M)$. The corresponding mapping is denoted by $\iota_*$. One should observe that this amounts to saying that the $RH$–module structure of $res_H^G(M)$ is just $M$ as $R$–module and now only $H$ operates on $M$ in the same way as $G$ does.

One defines functors

$$ind_H^G := RG \otimes_{RH} - : RH - mod \longrightarrow RG - mod$$

and

$$res_H^G := \iota_* : RG - mod \longrightarrow RH - mod$$

We claim that $(ind_H^G, res_H^G)$ is an adjoint pair. This fact is commonly known as *Frobenius reciprocity.*

We have to give for all $RG$–modules $M$ and for all $RH$–modules $N$ natural isomorphisms

$$Hom_{RG}(ind_H^G(N), M) \simeq Hom_{RH}(N, res_H^G(M)).$$

We define

$$\begin{array}{rcll} Hom_{RG}(ind_H^G(N), M) & \simeq & Hom_{RH}(N, res_H^G(M)) & \\ \phi & \stackrel{\Phi}{\longrightarrow} & (n \longrightarrow \phi(1 \otimes n)) & \forall n \in N \\ (g \otimes n \longrightarrow g \cdot \psi(n)) & \stackrel{\Psi}{\longleftarrow} & \psi & \forall n \in N, g \in G \end{array}$$

and observe that the second mapping is well defined since $\psi$ is $RH$–linear. Now, one immediately checks that $\Phi\Psi(\psi) = \psi$ and $\Psi\Phi(\phi) = \phi$ for all $\phi \in Hom_{RG}(ind_H^G(N), M)$ and $\psi \in Hom_{RH}(N, res_H^G(M))$. For the functoriality of $\Phi$

and $\Psi$ we observe that for a homomorphism $\alpha : N \longrightarrow N'$ and for a homomorphism $\beta : M \longrightarrow M'$ we get

$$\begin{aligned}\beta([\Phi(\phi)](\alpha(n))) &= [\beta\circ\phi](1\otimes\alpha(n))\\ &= [\Phi(\beta\circ\phi\circ(id_{RG}\otimes\alpha))](n)\end{aligned}$$

and

$$\begin{aligned}\beta[\Psi(\psi)](g\otimes\alpha(n)) &= \beta(g\cdot(\psi\circ\alpha)(n))\\ &= (g\cdot[\beta\circ\psi\circ\alpha](n))\\ &= [\Psi(\beta\circ\psi\circ\alpha))](n)\ .\end{aligned}$$

This proves the functoriality.

2.3. **Some more background from modular representation theory.** Though we do not need to know of modular representation theory for formulating and proving the Green correspondence for adjoint functors we shall give some background to see what the Green correspondence is about and to be able to give examples.

In this subsection we shall use the following notation.

- $R$ is a commutative Noetherian ring.
- $G$ is a finite group.
- For any subgroup $S$ of $G$ we set $mod(G,S) := mod(RG,S)$ the full[1], additive subcategory of $RG-mod$ whose objects are finitely generated $RG-$modules $M$ for which there is an $RS$–module $L$ such that $M$ is a direct summand of $RG\otimes_{RS} L$.

With these notations we state the following results which also provide a brief introduction into some of the elementary techniques in modular representation theory.

1. We claim that choosing $R$ a local complete discrete valuation ring of characteristic 0 with $pR \neq R$ or a field of characteristic $p$, for a prime number $p$, and choosing $D$ a Sylow $p-$subgroup of $G$, $mod(G,D) = RG-mod$.

   Proof. Given $M \in mod(G,D)$. Then, the mapping

$$\begin{aligned}RG\otimes_{RD} M &\longrightarrow M\\ g\otimes m &\longrightarrow gm\end{aligned}$$

   is split by

$$\begin{aligned}M &\longrightarrow RG\otimes_{RD} M\\ m &\longrightarrow \frac{1}{|G:D|}\sum_{Dh\in D\backslash G} h^{-1}\otimes hm\end{aligned}$$

   The last is clearly well defined and is a $G$–linear map since

$$g\cdot\sum_{Dh\in D\backslash G} h^{-1}\otimes hm = \sum_{Dh\in D\backslash G}(hg^{-1})^{-1}\otimes(hg^{-1})gm$$

   and to run over a coset $\{h\}$ is the same as to run over the coset $\{hg^{-1}\} = \{h\}g^{-1}$.

   Given a finitely generated $RG$ module $M$ we call a $D$ which is of minimal order amongst all the subgroups $D'$ with $M \in mod(G,D')$ the *vertex* of $M$.

[1] A functor $F : \mathcal{C}' \longrightarrow \mathcal{C}$ is full, if it is surjective on the morphism sets.

2. (D. Higman) Let $M$ be a finitely generated $RG$–module and let $S$ be a subgroup of $G$. We claim that $M$ is a direct summand of $ind_S^G(L)$ for some finitely generated $RS$–module $L$ if and only if $M$ is a direct summand of $ind_S^G res_S^G(M)$.

   Proof. Clearly, if $M|ind_H^G res_S^G(M)$, then there is the $RS$–module $L = res_H^G M$ such that $M|ind_S^G(L)$.

   Conversely, let $L$ be a finitely generated $RS-$module such that $M|ind_S^G L$. Then, by Mackey's formula,

$$\begin{array}{rcl} ind_S^G res_S^G M &=& ind_S^G res_S^G ind_S^G L \\ &=& ind_S^G( \bigoplus_{HgH\in H\backslash G/H} ind_{H\cap\ {}^gH}^H res_{H\cap\ {}^gH}^H\ {}^gL) \\ &=& ind_H^G L \oplus \text{ others} \\ &=& M \oplus \text{ others} \end{array}$$

   How unique the vertices are is the subject of the following item.

3. We assume now that $R$ is a complete discrete valuation ring of characteristic 0 with $pR \neq R$ for a prime number $p$ or a field of characteristic $p$. Given an indecomposable $RG-$module $M$ we claim that vertices of $M$ are conjugate to each other.

   Proof. $M|ind_D^G V$ and $M|ind_{D'}^G W$ with $V \in Ob(RD-mod)$ and $W \in Ob(RD'-mod)$, $D$ and $D'$ being both vertices of $M$. But, using Mackey's formula,

$$res_{D'}^G ind_D^G V = \bigoplus_{DgD'} ind_{{}^gD\cap D'}^{D'} res_{{}^gD\cap D'}^{{}^gD}\ {}^gV$$

   and

$$res_{D'}^G ind_{D'}^G W = \bigoplus_{D'hD'} ind_{{}^hD\cap D'}^{D'} res_{{}^hD'\cap D'}^{{}^hD}\ {}^hW = W \oplus \text{ others }.$$

   $res_{D'}^G M$ is a direct summand of both modules. Direct summands $X$ in the above equation have vertices smaller than $D'$ or are isomorphic to $W$. If $X|res_{D'}^G M$ for $X \not\simeq W$ then $M|ind_{D''}^G$ for some smaller subgroup $D''$ of $D'$ and we reach a contradiction. Hence there must be some $g_0$ with

$$W|ind_{{}^{g_0}D\cap D'}^{D'} res_{{}^{g_0}D\cap D'}^{{}^{g_0}D}\ {}^{g_0}V$$

   Hence, $M|ind_{{}^{g_0}D\cap D'}^G res_{{}^{g_0}D\cap D'}^{{}^{g_0}D}\ {}^{g_0}V$ and by the minimality of $D$, we get ${}^{g_0}D \cap D' = D'$.

4. An indecomposable ring direct factor $B$ of $RG$ is called a *block* of $RG$. Of course, then $B$ is an $R(G\times G)$–module by putting $(g,h)\cdot m = gmh^{-1}$ where $(g,h) \in G\times G$ and $m \in B$.

   We claim that there is always a vertex of $B$ in $\{(g,g) \in G\times G | g \in G\} =: \Delta(G)$.

   Proof. We view $R(G\times G)$ as $RG$–right–module by letting $G$ act as $\Delta(G)$.

$$\begin{array}{rcl} R(G\times G)\otimes_{R\Delta(G)} R &\longrightarrow& RG \\ ((g,h)\otimes r) &\longrightarrow& grh^{-1} = rgh^{-1} \end{array}$$

for $r \in R$, $(g,h) \in G \times G$, is split by

$$\begin{array}{ccc} R(G\times G)\otimes_{R\Delta(G)} R & \longleftarrow & RG \\ (\sum_{g\in G} r_g(g,1))\otimes 1 & \longleftarrow & \sum_{g\in G} r_g g \end{array}$$

The splitting is a module homomorphism as one immediately verifies using that we tensor over $R\Delta(G)$.

The vertex of a block as $R(G \times G)$–module is called a *defect group* of the block.

We assume now that $R$ is a complete discrete valuation ring of characteristic 0 with $pR \neq R$ for a prime number $p$ or a field of characteristic $p$, then the defect groups $D$ are $p$-groups and the integer $log_p(|D|)$ is called the *defect of the block.*

5. Now we assume again that $R$ is a complete discrete valuation ring of characteristic 0 with $pR \neq R$ for a prime number $p$ or a field of characteristic $p$. If $B$ is a block of $G$ with defect group $D$ and if $M$ is an indecomposable $B$–module, then there is a vertex of $M$ contained in $D$.

Proof[2]. Since $B$ has defect group $D$, by 3. we see that

$$B|[R(G\times G)\otimes_{R\Delta(D)} B].$$

Hence,

$$\begin{array}{rcl} M & = & B\otimes_{RG} M|R(G\times G)\otimes_{R\Delta(D)} B\otimes_{RG} M \\ & = & RG\otimes_{RD} M \end{array}$$

With these preparations we shall illustrate the Green correspondence in the situation of Theorem 2.1.

*Example.* We fix a prime number $p$ and set $G := SL_2(p)$ the group of 2 by 2 matrices over the prime field of characteristic $p$ with determinant 1.

We look at the modular representations of $G$ over $k$ being the prime field of characteristic $p$.[3]

We set $GL_2(p)$ the group of invertible 2 by 2 matrices over the prime field of characteristic $p$. Then we get an exact sequence

$$1 \longrightarrow SL_2(p) \longrightarrow GL_2(p) \stackrel{det}{\longrightarrow} k^* \longrightarrow 1.$$

Now, $k^*$ has order $p-1$ and $GL_2(p)$ has order $(p^2-1)\cdot(p^2-p)$ since an invertible matrix is determined by its action on the 2-dimensional natural module and the first basis vector can be mapped to all of $k^2$ besides the zero element, the second basis vector can be mapped to $k^2$ besides the one dimensional space which is already spanned by the image of the first basis vector.

Hence, $|SL_2(p)| = (p-1)\cdot p\cdot(p+1)$.

The Sylow $p$–subgroup of $G$ is hence cyclic of order $p$. In fact, it is easy to find one explicitly:

$$D := \{\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} |y \in k\}$$

[2]This proof was pointed out to me by M.Linckelmann.

[3]This characteristic is commonly called the describing characteristic and in the theory of algebraic groups the describing characteristic always provides a hughe framework of techniques coming from algebraic geometry.

The normalizer $H$ of $D$ in $G$ is

$$H := \{\begin{pmatrix} x & y \\ 0 & x^{-1} \end{pmatrix} | x \in k^* \; ; y \in k\}$$

We shall illustrate the Green correspondence on the (natural) module

$$M = \begin{pmatrix} k \\ k \end{pmatrix}$$

on which $G$ acts by matrix multiplication.

Clearly, $M$ is indecomposable. By Theorem 2.1 we know that $res_H^G(M) \simeq f(M) \oplus P_H$ where $P_H$ is a projective $RH$–module.

**Claim 2.4.** *Let $P$ be a $p$–group and let $R$ be a complete discrete valuation ring of characteristic $0$ with residue field of characteristic $p$. Then, $RP$ is a local ring. Every projective $RP$–module is free.*

Proof. We have to show that $R/radR$ is the only simple $RP$–module. We proceed by induction on $|P|$.

The statement is true for the group with 1 element.

Let $P$ be arbitrary. Let $1 \neq c$ be a central element of $P$ of order $p$, which exists by the cojugacy class number formula, just counting the size of the conjugacy classes and observing that their order equals the index of the stabilizer of an element which is a subgroup, and let $V$ be a simple $RP$–module. Then, since $c$ is central, $V_1 := (c-1) \cdot V$ is also an $RP$–module. If $V_1 = 0$, then $V$ is an indecomposable $R(P/ < c >)$ module and $V$ is isomorphic to $R/radR$ by the induction hypothesis. Else, $V_1 = V$ by the simplicity of $V$. Hence,

$$V = (c-1) \cdot V = (c-1)^2 \cdot V = \cdots = (c-1)^p \cdot V = (c^p - 1) \cdot V = 0.$$

We reached a contradiction.

The above claim shows that a projective $kG$–module has $k$–rank at least order of a Sylow $p$–subgroup. In fact, the restriction of a projective $RG$–module to a Sylow $p$–subgroup is again projective, hence free.

This argument (or by elementary computations) shows, that the 2–dimensional $kH$–module $res_H^G(M)$ is indecomposable.

Conversely, let $N$ be the natural two dimensional $kH$–module. Then, $N = res_H^G(M)$. We look for its Green correspondent in $kG$. As is done above,

$$\begin{array}{rcl} ind_H^G N = ind_H^G res_H^G M & \longrightarrow & M \\ g \otimes m & \longrightarrow & gm \end{array}$$

is split and $M | ind_H^G N$. Hence, the Green correspondent of $N$ in $kG$ is $M = g(N)$.

But, since the index of $H$ in $G$ is $p+1$, $dim_k(ind_H^G N) = 2 \cdot (p+1)$ and $dim_k(ind_H^G(N)/M) = 2p$ and $ind_H^G(N)/M$ is a projective module of dimension $2p$. Observe that this matches our observation in Claim 2.4.

2.4. **The Green correspondence for adjoint pairs of functors.** We shall follow the paper [1].

All categories we deal with are assumed to be additive [4]. We start with three (additive) categories $\mathcal{D}$, $\mathcal{H}$ and $\mathcal{G}$ and functors

$$\mathcal{D} \xrightarrow{S'} \mathcal{H} \xrightarrow{S} \mathcal{G}$$

as well as

$$\mathcal{D} \xleftarrow{T'} \mathcal{H} \xleftarrow{T} \mathcal{G}$$

where $(S, T)$ and $(S', T')$ will form adjoint pairs.

In our later application to group theory these categories and functors will be specialized as follows.

Let $k$ be a field of characteristic $p \geq 0$, let $G$ be a finite group, and assume for simplicity that $kG$ is indecomposable as ring, let $D$ be the Sylow $p$–subgroup of $G$, let $H$ be a group with $D \leq H \leq G$. The results we get become non trivial only if we assume that in this example $H \geq N_G(D) := \{g \in G | gD = Dg\}$. One should think of

$$\mathcal{D} = kD - mod, \mathcal{H} = kH - mod, \mathcal{G} = kG - mod$$

and

$$S' = kH \otimes_{kD} -;\ S = kG \otimes_{kH} -;$$
$$T = res^G_H(-);\ T' := res^H_D(-)$$

where $res^G_H$ and $res^H_D$ are the restriction functors of $kG-$modules to $kH-$modules or of $kH$–modules to $kD$–modules respectively, and the adjointness is just Frobenius reciprocity as explained earlier in Subsection 2.2.

For technical reasons in later applications we fix isomorphisms, natural in both variables,

$$\alpha(N, M) : \mathcal{G}(SN, M) \longrightarrow \mathcal{H}(N, TM);\ \ \forall N \in Ob(\mathcal{H}), M \in Ob(\mathcal{G})$$

and

$$\gamma(L, N) : \mathcal{H}(S'L, N) \longrightarrow \mathcal{D}(L, T'N);\ \ \forall N \in Ob(\mathcal{H}), L \in Ob(\mathcal{D})\ .$$

**Throughout Section 2.4 we assume that**

$$TS = 1_{\mathcal{H}} \oplus U$$

**for an endofunctor $U$ of $\mathcal{H}$ and that the induced natural transformation**

$$\eta_I : 1_{\mathcal{H}} \xrightarrow{\eta} TS = 1_{\mathcal{H}} \oplus U \xrightarrow{proj} 1_{\mathcal{H}}$$

**is an isomorphism.**

**Notation 1.** • *All subcategories in Section 2.4 are meant to be full and additive. If $\mathcal{A}$ and $\mathcal{B}$ are full subcategories of the category $\mathcal{C}$, then we say that $\mathcal{A}$* divides *$\mathcal{B}$ if for all $M \in Ob(\mathcal{A})$ there is a $X \in Ob(\mathcal{B})$ such that $M|X$, i.e. $M$ is a direct summand of $X$. If $Ob(\mathcal{A})$ has only one element $M$, then we also say that $M$ divides $\mathcal{B}$. We use the notation $\mathcal{A}|\mathcal{B}$.*

[4] We remind the reader that a category is called additive if it has a zero object, there are finite products and coproducts, finite products over a set of objects and finite coproducts over this set are isomorphic by the natural map, and for every object $A$ there is an endomorphism $s_A$ of $A$ such that, denoting by $\Delta_A$ the diagonal mapping and by $\nabla_A$ the codiagonal mapping, $\Delta_A(1_A \oplus s_A)\nabla_A = 0$. In additive categories the set of morphisms carries a structure of an abelian group by setting $f + g = \Delta_A(f \oplus g)\nabla_B$ for $f, g \in Mor(A, B)$

- *Let $\mathcal{C}'$ be a subcategory of the category $\mathcal{C}$. We denote by $\mathcal{C}/\mathcal{C}'$ the category whose objects are the same as those of $\mathcal{C}$ and the morphisms are equivalence classes of morphisms of $\mathcal{C}$. Two morphisms are said to be equivalent if their difference factors through an object of $\mathcal{C}'$.*
- *Let $\mathcal{E}$ and $\mathcal{F}$ be categories and let $U : \mathcal{E} \longrightarrow \mathcal{F}$ be a functor. For any subcategory $\mathcal{Y}$ of $\mathcal{F}$ let $U^{-1}(\mathcal{Y})$ be the full additive subcategory of $\mathcal{E}$ generated by objects $M \in Ob(\mathcal{E})$ with $U(M)|\mathcal{Y}$.*

2.4.1. *The theorem in group theoretical terms.* We shall give the Green correspondence in the classical situation, before we turn to the more abstract setting.

**Theorem 2.5.** *(Green) Let $G$ be a finite group and let $R$ be a complete discrete valuation ring of characteristic 0 with residue field of characteristic $p > 0$ or let $R$ be a field of characteristic $p$. Let $D$ be a $p$-subgroup of $G$ and let $H \geq N_G(D)$. Set*

$$\mathcal{X} := \{X \leq D \cap gDg^{-1} | g \in G \setminus H\}$$

$$\mathcal{Y} := \{Y \leq H \cap gDg^{-1} | g \in G \setminus H\}$$

$$\mathcal{Z} := \{Z \leq D\}$$

*Set $mod(G, \mathcal{F})$ the category of finitely generated $RG-$modules with vertex in $\mathcal{F}$ for $\mathcal{F} \in \{\mathcal{X}, \mathcal{Y}, \mathcal{Z}\}$.*

*Then,*

$$ind_H^G : mod(H, \mathcal{Z})/mod(H, \mathcal{X}) \longrightarrow mod(G, \mathcal{Z})/mod(G, \mathcal{X})$$

*is an equivalence of categories and*

$$res_H^G : mod(G, \mathcal{Z})/mod(G, \mathcal{X}) \longrightarrow mod(H, \mathcal{Z})/mod(H, \mathcal{Y})$$

*is an equivalence of categories.*

*For every indecomposable object $M$ in $mod(H, \mathcal{Z}) \setminus mod(H, \mathcal{X})$ there is an indecomposable object $g(M)$ in $mod(G, \mathcal{Z}) \setminus mod(G, \mathcal{X})$ which is a direct summand of $ind_H^G(M)$.*

*For every indecomposable object $N$ in $mod(G, \mathcal{Z}) \setminus mod(G, \mathcal{X})$ there is an indecomposable object $f(N)$ in $mod(H, \mathcal{Z}) \setminus mod(H, \mathcal{Y})$ which is a direct summand of $res_H^G(N)$.*

We shall prove the theorem in the sequel.

2.4.2. *The general situation.* We are ready to state the most important theorem of this subsection. The remaining part deals with the particular situation of Krull–Schmidt categories. But even without this assumptions we are able to prove an equivalence of certain quotient categories. In the next subsection we shall explain how one can derive the usual Green correspondence from this rather abstract setting.

**Theorem 2.6.** [1]*(Green correspondence for adjoint functors) Let there be three additive categories $\mathcal{D}$, $\mathcal{H}$ and $\mathcal{G}$ and functors*

$$\mathcal{D} \xrightarrow{S'} \mathcal{H} \xrightarrow{S} \mathcal{G}$$

*as well as*

$$\mathcal{D} \xleftarrow{T'} \mathcal{H} \xleftarrow{T} \mathcal{G}$$

*where* $(S,T)$ *and* $(S',T')$ *will form adjoint pairs. Assume that* $TS = 1_{\mathcal{H}} \oplus U$ *and that for the unit* $\eta : 1_{\mathcal{H}} \longrightarrow TS$ *we get* $\eta_I := \eta \cdot proj_1$ *is an isomorphism. Let* $\mathcal{Y}$ *be a full, additive subcategory of* $\mathcal{H}$ *such that*

$$S'T'\mathcal{Y}|\mathcal{Y} \text{ and } S'T'\mathcal{Y}|U^{-1}\mathcal{Y}.$$

*Then,*

1. $S, T$ *induce functors*

$$\begin{array}{rcl} S : \mathcal{H}/S'T'\mathcal{Y} & \longrightarrow & \mathcal{G}/SS'T'\mathcal{Y} \\ T : \mathcal{G}/SS'T'\mathcal{Y} & \longrightarrow & \mathcal{H}/\mathcal{Y} \end{array}$$

2. *For* $\mathcal{Z} := (US')^{-1}\mathcal{Y}$*, the restrictions of the functors* $S$ *and* $T$

$$S : (addS'\mathcal{Z})/S'T'\mathcal{Y} \longrightarrow (addSS'\mathcal{Z})/SS'T'\mathcal{Y}$$

$$T : (addSS'\mathcal{Z})/SS'T'\mathcal{Y} \longrightarrow (addS'\mathcal{Z})/\mathcal{Y}$$

   *are equivalences of categories and*

$$TS : (addS'\mathcal{Z})/S'T'\mathcal{Y} \longrightarrow (addS'\mathcal{Z})/\mathcal{Y}$$

   *is isomorphic to the functor induced by the identity functor.*

Before we prove Theorem 2.6 we shall see what this says for the group theoretical situation.

The question if the assumptions of the theorem are satisfied in the group theoretical situation will cover subsection 2.4.5 and we postpone this question until then.

Again let $R$ be a complete discrete valuation ring of characteristic 0 with residue field $k$ of characteristic $p > 0$ or let $R$ be a field of characteristic $p$. Let $G$ be a finite group and let $D$ be a $p-$ subgroup of $G$. Let $H$ be a subgroup of $G$ with $D \leq N_G(D) \leq H \leq G$. We set

$$\mathcal{G} := RG - mod^0, \mathcal{H} := RH - mod^0, \mathcal{D} := RD - mod^0$$

and

$$S = ind_H^G, S' = ind_D^H, T = res_H^G, T' = res_D^H.$$

Furthermore, we set

$$\mathcal{S} := \{V \leq G |\ \exists g \in G \setminus H : V \leq g \cdot D \cdot g^{-1} \cap H\}$$

and let $\mathcal{Y}$ be the full additive subcategory of $RH - mod^0$ whose objects are finite direct sums of indecomposable finitely generated $RH-$lattices which have vertex in $\mathcal{S}$.

We compute $S'T'\mathcal{Y}$. Let $V \in \mathcal{S}$ and $L \in Ob(RV - mod^0)$. A generating object of $S'T'\mathcal{Y}$ is of the form

$$\begin{array}{rcl} ind_D^H res_D^H ind_V^H L & = & ind_D^H\Big(\bigoplus_{VhD \in V\setminus H/D} ind_{{}^hV\cap D}^D res_{{}^hV\cap D}^{{}^hV}\ {}^hL\Big) \\ & = & \bigoplus_{VhD \in V\setminus H/D} ind_{{}^hV\cap D}^H res_{{}^hV\cap D}^{{}^hV}\ {}^hL \end{array}$$

If we now set

$$\mathcal{X} := \{V \leq D |\ \exists g \in G \setminus H : V \leq g \cdot D \cdot g^{-1} \cap D\}$$

we observe that the above generating modules are direct sums of modules which have vertices in $\mathcal{X}$. We set $\mathcal{U}$ the full additive subcategory of $RH - mod^0$ whose objects are finite direct sums of indecomposable finitely generated $RH$–lattices which have vertex in $\mathcal{X}$. Clearly, $\mathcal{H}/(S'T'\mathcal{Y}) = \mathcal{H}/\mathcal{U}$ since if a morphism factors through a direct summand it also factors through the whole direct sum.

In a similar vein, with somewhat more effort but still simply using Mackey's formula one proves that $\mathcal{Z}$ is the full additive subcategory of $RH - mod^0$ whose indecomposable objects are modules with vertex being a subgroup of $D$.

2.4.3. *The proof.* The proof of Theorem 2.6 will proceed in several steps. This will cover this subsubsection. The next subsubsection will deal with the special situation when we are given a Krull–Schmidt category.

The proof of Theorem 2.6 relies mainly on the following observation.

**Proposition 2.7.** *Let $\mathcal{Y}'$ be a subcategory of $\mathcal{H}$ and let $\mathcal{X}'$ be a subcategory of $\mathcal{G}$ such that $S\mathcal{Y}'|\mathcal{X}'$ and $T\mathcal{X}'|\mathcal{Y}'$. Then, $S, T$ extend naturally to functors between $\mathcal{H}/\mathcal{Y}'$ and $\mathcal{G}/\mathcal{X}'$ and $(S, T)$ form again an adjoint pair as functors between these quotient categories. The adjointness homomorphism $\alpha$ for the adjoint pair $(S, T)$ of functors between $\mathcal{H}$ and $\mathcal{G}$ induces an adjointness homomorphism for the adjoint pair $(S, T)$ of functors between $\mathcal{H}/\mathcal{Y}'$ and $\mathcal{G}/\mathcal{X}'$.*

Proof. First we prove that $S$ extends to the quotient categories. Let there be given two objects $M$ and $N$ in $\mathcal{H}$ and a morphism $f \in \mathcal{H}(M, N)$ which factors through an object $Y$ of $\mathcal{Y}'$. Then, there are $f_1 \in \mathcal{H}(M, Y)$ and $f_2 \in \mathcal{H}(Y, N)$ such that $f = f_1 f_2$. Therefore, $Sf = (Sf_1)(Sf_2)$ and $Sf$ factors through $SY$. But, $S\mathcal{Y}'|\mathcal{X}'$ and therefore, there is an $X \in Ob(\mathcal{X}')$ such that $SY|X$. Hence, $Sf$ factors through an object in $\mathcal{X}'$.

The argument that $T$ extends to the quotient categories is absolutely analogous.

We show that for any $N \in Ob(\mathcal{H}), M \in Ob(\mathcal{G})$ the mapping $\alpha(N, M)$ : $\mathcal{G}/\mathcal{X}'(SN, M) \longrightarrow \mathcal{H}/\mathcal{Y}'(N, TM)$ is an isomorphism. Let $f : N \longrightarrow TM$ be a morphism factoring through $Y \in \mathcal{Y}$. Then, there are $f_1 \in \mathcal{H}(M, Y)$ and $f_2 \in \mathcal{H}(Y, N)$ such that $f = f_1 f_2$. Now, $\alpha^{-1}(f) = \alpha^{-1}(f_1)\alpha^{-1}(f_2)$ with $Sf_1 = \alpha^{-1}(f_1) \in \mathcal{G}(SN, SY)$ and $\alpha^{-1}(f_2) \in \mathcal{G}(SY, M)$. However, $SY|X$ for an object $X \in \mathcal{X}'$. Hence, $\alpha^{-1}(f)$ factors through an object of $\mathcal{X}'$. Therefore, $\alpha^{-1}$ is defined over the quotient categories. Analogously, also $\alpha$ is defined over the quotient categories. It is clear that then $\alpha(N, M)$ is a natural isomorphism.

This proves the proposition.

**Corollary 2.8.** *Let $\mathcal{Y}$ be a subcategory of $\mathcal{H}$.*
*If $S'T'\mathcal{Y}|\mathcal{Y}$, then*

1. *$(S', T')$ is an adjoint pair as functors between $\mathcal{H}/\mathcal{Y}$ and $\mathcal{D}/T'\mathcal{Y}$. The isomorphisms $\gamma$ induce adjunctions also in the quotient categories.*
2. *$(S', T')$ is an adjoint pair as functors between $\mathcal{H}/S'T'\mathcal{Y}$ and $\mathcal{D}/T'\mathcal{Y}$. The isomorphisms $\gamma$ induce adjunctions also in the quotient categories.*
3. *The functor $1_{\mathcal{H}}$ induces a functor $1_{\mathcal{H}} : \mathcal{H}/S'T'\mathcal{Y} \longrightarrow \mathcal{H}/\mathcal{Y}$ and gives rise to an isomorphisms of bifunctors $(\mathcal{H}/S'T'\mathcal{Y})(S'-, -) \simeq (\mathcal{H}/\mathcal{Y})(S'-, -)$*

*If $T'TSS'T'\mathcal{Y}|T'\mathcal{Y}$, then*

4. *$(SS', T'T)$ is an adjoint pair between the categories $\mathcal{G}/SS'T'\mathcal{Y}$ and $\mathcal{D}/T'\mathcal{Y}$ with adjunction induced by $\gamma\alpha$.*
5. *If moreover $S'T'\mathcal{Y}|\mathcal{Y}$, then the inverse of $\alpha$ induce isomorphisms functorial in both variables $L \in Ob(\mathcal{D})$, $M \in Ob(\mathcal{G})$,*
$$\mathcal{H}/S'T'\mathcal{Y}(S'L, TM) \longrightarrow \mathcal{G}/SS'T'\mathcal{Y}(SS'L, M).$$

Proof:

Part 1. follows from Proposition 2.7 by just setting $\mathcal{X}' := T'\mathcal{Y}$ and $\mathcal{Y}' := \mathcal{Y}$.

Part 2. Set $\mathcal{X}' := S'T'\mathcal{Y}$ and $\mathcal{Y}' := T'\mathcal{Y}$. Then, $S'T'\mathcal{Y}|\mathcal{Y} \Longrightarrow T'S'(T'\mathcal{Y})|(T'\mathcal{Y})$ and Proposition 2.7 applies.

Part 3. We apply first 2. and then 1. to get the isomorphisms

$$\mathcal{H}/S'T'\mathcal{Y}(S'-,-) \overset{\gamma(-,-)}{\underset{\sim}{\longrightarrow}} \mathcal{D}/T'\mathcal{Y}(-,T'-) \overset{\gamma^{-1}(-,-)}{\underset{\sim}{\longrightarrow}} \mathcal{H}/\mathcal{Y}(S'-,-)$$

Part 4. This is just an application of Proposition 2.7 with $\mathcal{X}' := SS'T'\mathcal{Y}$ and $\mathcal{Y}' := T'\mathcal{Y}$ and as functors one takes just $T'T$.

Part 5. We have

$$\mathcal{G}/SS'T'\mathcal{Y}(SS'-,-) \overset{\gamma\alpha(-,-)}{\underset{\sim}{\longrightarrow}} \mathcal{D}/T'\mathcal{Y}(-,T'T-) \overset{\gamma^{-1}(-,-)}{\underset{\sim}{\longrightarrow}} \mathcal{H}/S'T'\mathcal{Y}(S'-,T-)$$

where the last part is due to 2. and the first is due to 4.

We come to the actual proof of Theorem 2.6. We need a lemma.

**Lemma 2.9.** *Under the assumptions of Theorem 2.6 we get the following.*

$$(S'T'\mathcal{Y}|\mathcal{Y} \text{ and } S'T'\mathcal{Y}|U^{-1}(\mathcal{Y})) \Longleftrightarrow TSS'T'\mathcal{Y}|\mathcal{Y}$$

Proof. $TS = 1_{\mathcal{H}} \oplus U \Longrightarrow TSS'T' = S'T' \oplus US'T'$ and inserting $Y \in \mathcal{Y}$ gives the result.

*We can now prove Part 1 of Theorem 2.6.* In fact, for $S$ the statement is clear and for $T$ it follows from Lemma 2.9.

**Lemma 2.10.** *Under the assumptions of Theorem 2.6 we get the following.*

1. *For all $L \in Ob(\mathcal{D}), B \in Ob(U^{-1}\mathcal{Y})$,*
$$S : (\mathcal{H}/S'T'\mathcal{Y})(S'L, B) \overset{\sim}{\longrightarrow} (\mathcal{G}/SS'T'\mathcal{Y})(SS'L, SB)$$
*gives an isomorphism.*
2. *For all $L \in Ob((US')^{-1}(\mathcal{D})), A \in \mathcal{G}$,*
$$T : (\mathcal{G}/SS'T'\mathcal{Y})(SS'L, A) \overset{\sim}{\longrightarrow} (\mathcal{H}/\mathcal{Y})(TSS'L, TA)$$
*gives an isomorphism.*

**Remark 2.11.** We notice at once that if $S'T'(US'\mathcal{D})|(US'\mathcal{D})$, then $\mathcal{Y} := (US'\mathcal{D})$ satisfies each of the equivalent conditions in part 1.

Proof of Lemma 2.10.

The conditions to apply Corollary 2.8 1.–5. are satisfied. We have $\eta : 1_{\mathcal{H}} \longrightarrow TS$ and $\eta_B \in \mathcal{H}(B, TSB)$. The following diagram is commutative.

$$\begin{array}{ccccc}
\frac{\mathcal{H}}{S'T'\mathcal{Y}}(S'L,B) & \overset{\eta_{B*}}{\longrightarrow} & \frac{\mathcal{H}}{S'T'\mathcal{Y}}(S'L,TSB) & \overset{\alpha^{-1}}{\longrightarrow} & \frac{\mathcal{G}}{SS'T'\mathcal{Y}}(SS'L,SB) \\
\downarrow 1_{\mathcal{H}} & & \simeq\downarrow 1_{\mathcal{H}} & & \\
\frac{\mathcal{H}}{\mathcal{Y}}(S'L,B) & \overset{\eta_{B*}}{\longrightarrow} & \frac{\mathcal{H}}{\mathcal{Y}}(S'L,TSB) = \frac{\mathcal{H}}{\mathcal{Y}}(S'L,B) & &
\end{array}$$

where the very left hand side vertical $1_{\mathcal{H}}$ is an isomorphism by Corollary 2.8. 3. $\alpha^{-1}$ is an isomorphism by Corollary 2.8. 5. The equality in the lower right corner follows from the fact that $TS = 1_{\mathcal{H}} \oplus U$ and $UB|\mathcal{Y}$. But, $\eta_{B*} \cdot pro_1 = \eta_I$ is an isomorphism. Therefore, going down, right, up we conclude that the upper $\eta_{B*}$ is an isomorphism.

Furthermore, $\forall h \in \mathcal{H}(S'L, B)$, we get

$$\alpha^{-1}(\eta_B \circ h) = \alpha^{-1}(\eta_B) \circ S(h) = 1_{SB} \circ S(h) = S(h)$$

where the first equation is just the functoriality, the second is the definition of $\eta$ by $\eta_N = \alpha(N, TSN)^{-1}(1_{SN})$. This shows the statement for $S$.

The statement for the functor $T$ is shown analogously.

We can now also prove Part 2 of Theorem 2.6. By Lemma 2.10 we see that the restriction of $S$ to *add* $S'\mathcal{Z}/S'T'\mathcal{Y}$ and of $T$ to *add* $SS'\mathcal{Z}/SS'T'\mathcal{Y}$ is full and faithful.

The restrictions of $S$ and of $T$ are dense to the image of a functor. This is precisely what we did.

Since $TSS'\mathcal{Z} = S'\mathcal{Z} \oplus US'\mathcal{Z}$ and since by the definition of $\mathcal{Z}$ for all $Z \in Ob(\mathcal{Z})$ we get $US'(Z)|\mathcal{Y}$, we see that $add\, TSS'\mathcal{Z}/\mathcal{Y} = add\; S'\mathcal{Z}/\mathcal{Y}$.

Moreover, $U$ makes all the occuring terms vanishing and henceforth by our assumption that $\eta_I$ is an isomorphism, $TS$ is just the natural projection.

This finishes the proof of the theorem.

2.4.4. *The Krull–Schmidt situation.*

**Notation 2.** *If $\mathcal{E}$ and $\mathcal{F}$ are subcategories of a common Krull–Schmidt category* [5] *$\mathcal{G}$, then $\mathcal{F}_{\mathcal{E}}$ denotes the full additive subcategory of $\mathcal{F}$ generated by objects $M \in Ob(\mathcal{F})$ such that no non zero direct summand of $M$ divides $\mathcal{E}$. One should think of $\mathcal{F}_{\mathcal{E}}$ as the part of $\mathcal{F}$ which has nothing to do with $\mathcal{E}$.*

**Lemma 2.12.** *Let $\mathcal{E}$ be a subcategory of a Krull–Schmidt category $\mathcal{F}$. Then, the identity functor induces a functor $\mathcal{F}_{\mathcal{E}} \longrightarrow \mathcal{F}_{\mathcal{E}}/\mathcal{E}$ which is full*[6]*, dense*[7] *and reflects isomorphisms*[8]*.*

Proof. $1_{\mathcal{F}}$ is clearly full and dense. Take an isomorphism $X \xrightarrow{f} Y$ in $\mathcal{F}_{\mathcal{E}}/\mathcal{E}$. Then, there is a $Y \xrightarrow{g} X$ with $gf = 1_X$ and $fg = 1_Y$ in $\mathcal{F}_{\mathcal{E}}/\mathcal{E}$. Take preimages $f_0$ and $g_0$ of $f$ and $g$ in $\mathcal{F}$. Then, $f_0g_0 = 1_X + k_X$ where $k_X$ is an endomorphism of $X$ which factors through an object of $\mathcal{E}$. No summand of $X$ divides $\mathcal{E}$ and therefore, $k_X \in radEnd(X)$. But, the Jacobson radical $radEnd(X)$ has the property that $1+radEnd(X)$ is a subgroup of the unit group. Similarly, $g_0f_0$ is invertible. Hence, $f_0$ is an isomorphism.

**Proposition 2.13.** *Let $\mathcal{Y}$ be a subcategory of $\mathcal{H}$. Then $TS : \mathcal{H} \longrightarrow \mathcal{H}/\mathcal{Y}$ satisfies*

$$TS(U^{-1}\mathcal{Y}) \leq (U^{-1}\mathcal{Y})/\mathcal{Y}$$

*and*

$$TS : U^{-1}\mathcal{Y} \longrightarrow (U^{-1}\mathcal{Y})/\mathcal{Y} \text{ is isomorphic to } 1_{\mathcal{H}}.$$

*If $\mathcal{H}$ is a Krull–Schmidt category, then $TS : (U^{-1}\mathcal{Y})_{\mathcal{Y}} \longrightarrow (U^{-1}\mathcal{Y})_{\mathcal{Y}}/\mathcal{Y}$ is full, dense and reflects isomorphisms.*

[5] A Krull–Schmidt category is an additive category such that every object is a finite direct sum of indecomposable objects and endomorphism rings of indecomposable objects are local. It follows then that the decomposition into direct summands is unique up to isomorphisms.

[6] $F : \mathcal{C}' \longrightarrow \mathcal{C}$ is full, if it is surjective on the morphism sets.

[7] $F : \mathcal{C}' \longrightarrow \mathcal{C}$ is dense, if every object in $\mathcal{C}$ is of the form $FC'$ for a $C \in Ob(\mathcal{C}')$.

[8] $F : \mathcal{C}' \longrightarrow \mathcal{C}$ is reflects isomorphisms, if

$$F(f) \in Mor_{\mathcal{C}}(FC_1, FC_2) \text{ is an isomorphism iff } f \in Mor(C_1, C_2) \text{ is an isomorphism.}$$

Proof. We know that $\eta_I$ is an isomorphism.

$$\begin{aligned} B \in U^{-1}\mathcal{Y} &\Leftrightarrow U(B)|\mathcal{Y} \\ &\Leftrightarrow U(B) \simeq 0 \in Ob(\mathcal{H}/\mathcal{Y}) \\ &\Rightarrow \eta : 1_{\mathcal{H}}|_{U^{-1}\mathcal{Y}} \to TS|_{U^{-1}\mathcal{Y}} \text{ really is } \eta_I \text{ which is an isomorphism.} \end{aligned}$$

The second statement follows immediately from Lemma 2.12.

**Proposition 2.14.** *Let $\mathcal{Y}$ be a full additive subcategory of the Krull–Schmidt category $\mathcal{H}$.*

1. $$S : U^{-1}(\mathcal{Y})_{\mathcal{Y}} \longrightarrow S(U^{-1}(\mathcal{Y})_{\mathcal{Y}})/T^{-1}\mathcal{Y}$$
   *is dense, reflects isomorphisms, and $N$ is indecomposable in $U^{-1}(\mathcal{Y})_{\mathcal{Y}}$ if and only if $SN$ is indecomposable in $\mathcal{G}/T^{-1}\mathcal{Y}$.*
2. $$T : S(U^{-1}(\mathcal{Y})_{\mathcal{Y}})/T^{-1}\mathcal{Y} \longrightarrow U^{-1}(\mathcal{Y})_{\mathcal{Y}}/\mathcal{Y}$$
   *is full, dense, reflects isomorphisms and $M$ is indecomposable in $S(U^{-1}(\mathcal{Y})_{\mathcal{Y}})/T^{-1}\mathcal{Y}$ if and only if $TM$ is indecomposable in $\mathcal{H}/\mathcal{Y}$.*

Proof.
$S$ is dense by definition.
$SN_1 \simeq SN_2 \Rightarrow TSN_1 \simeq TSN_2 \Rightarrow N_1 \simeq N_2$ since $TS$ reflects isomorphisms by Proposition 2.14.
$T$ is dense since $TS$ is dense.
$T$ is full since given $TX \xrightarrow{f} TY$, then there is a $X', Y'$ mapping to $X, Y$ by $S$ such that given $TSX' \xrightarrow{f} TSY'$. But, $TS$ is full, again using Proposition 2.14, hence, $f = TSf'$ for $X' \xrightarrow{f'} Y'$ and $T(Sf') = f$ and $Sf$ is a preimage.
$TM_1 \simeq TM_2 \Rightarrow \exists_{N_1,N_2} M_i = SN_i \Rightarrow TSN_1 \simeq TSN_2 \Rightarrow N_1 \simeq N_2 \Rightarrow M_1 \simeq SN_1 \simeq SN_2 \simeq M_2$.
$N$ is decomposable in $U^{-1}(\mathcal{Y})_{\mathcal{Y}} \Rightarrow$ take $N_1|N \Rightarrow SN_1|SN$. But, $SN_1 = 0 \Rightarrow N_1 = 0$. Therefore, $SN$ is decomposable.
Let $M$ be decomposable in $S(U^{-1}(\mathcal{Y})_{\mathcal{Y}})/T^{-1}\mathcal{Y}$. Take $0 \neq M_1|M \Rightarrow TM_1|TM$. But, $TM_1 = 0 \in U^{-1}(\mathcal{Y})_{\mathcal{Y}}/\mathcal{Y} \Rightarrow TM_1|\mathcal{Y} \Rightarrow M_1 \in T^{-1}\mathcal{Y} \Rightarrow M_1 = 0 \in S(U^{-1}(\mathcal{Y})_{\mathcal{Y}})/T^{-1}\mathcal{Y}$. Hence, $TM$ is decomposable.
$SN$ decomposable$\Rightarrow TSN$ decomposable in $U^{-1}(\mathcal{Y})_{\mathcal{Y}}/T^{-1}\mathcal{Y} \Rightarrow N$ is decomposable since $TS$ is full, dense and reflects isomorphisms.
$TM$ is decomposable with $M \in S(U^{-1}(\mathcal{Y})_{\mathcal{Y}})/T^{-1}\mathcal{Y} \Rightarrow \exists_N M \simeq SN \Rightarrow TSN \simeq TM$ is decomposable $\Rightarrow N$ is decomposable since $TS$ is full, dense, and reflects isomorphisms $\Rightarrow M = SN$ is decomposable.
Putting all the pieces together, we have proved the proposition.

**Corollary 2.15.** *Let $\mathcal{H}$ and $\mathcal{G}$ be Krull–Schmidt categories and let $\mathcal{Y}$ be a full additive subcategory of $\mathcal{H}$.*

1. *For all indecomposable objects $N \in U^{-1}(\mathcal{Y})_{\mathcal{Y}}$ the object $SN$ has exactly one indecomposable summand $g(N)$ which is not contained in $T^{-1}\mathcal{Y}$.*
2. *For all indecomposable objects $M \in (add\ S(U^{-1}(\mathcal{Y})_{\mathcal{Y}}))_{T^{-1}\mathcal{Y}}$ the object $TM$ has exactly one indecomposable summand $f(M)$ that does not divide $\mathcal{Y}$.*

3. $f(g(N)) \simeq N$
4. $g(f(M)) \simeq M$

**Remark 2.16.** One should note that this establishes one part of the Green correspondence, namely, the bijective correspondence between parts of the two module categories of the group rings.

However, one should read this carefully. If we wanted to apply this to our group theoretic situation, one does not need for the statement that $H \geq N_G(D)$. It will become clear that if this is not the case, then *there is no indecomposable object as required.*

Proof of Corollary 2.15.

1. $SN = M_1 \oplus \cdots \oplus M_s$ for indecomposables $M_i$ in $\mathcal{G}$ with $i = 1, ..., s$ and $s \in \mathbb{N}$. By Lemma 2.12, $M_j$ is indecomposable or zero in $\mathcal{G}/T^{-1}\mathcal{Y}$.
   By Proposition 2.14 $SN$ is indecomposable in $\mathcal{G}/T^{-1}\mathcal{Y}$.
   Hence, there is exactly one $j_0$ with $M_{j_0}$ not being contained in $T^{-1}\mathcal{Y}$.
2. There is an indecomposable $N \in U^{-1}(\mathcal{Y})_{\mathcal{Y}}$ and some $M \in \mathcal{G}$ with $SN = M \oplus M'$. Since $M$ is not contained in $T^{-1}\mathcal{Y}$ we get $SN \simeq M$ in $\mathcal{G}/T^{-1}\mathcal{Y}$.
   Let $TSN = TM = N_1 \oplus \cdots \oplus N_t$ for indecomposable objects $N_i$ in $\mathcal{H}$ and $i = 1, ..., t$ and $t \in \mathbb{N}$. But we know that $TSN$ is indecomposable in $\mathcal{H}/\mathcal{Y}$ by Proposition 2.14.
   Hence there is exactly one $i_0$ where $N_{i_0}$ does not divide $\mathcal{Y}$.
3. $g(N) = SN$ in $\mathcal{G}/T^{-1}\mathcal{Y}$. $f(g(N)) \simeq TSN \simeq N$ in $\mathcal{H}/\mathcal{Y}$ since $TS \simeq 1$ as functor $U^{-1}\mathcal{Y} \longrightarrow (U^{-1}\mathcal{Y})/\mathcal{Y}$.
4.
$$\begin{array}{lcl} fgf(M) \simeq f(M) \text{ by 3.} & \Rightarrow & Tgf(M) \simeq TM \text{ in } \mathcal{H}/\mathcal{Y} \\ & \overset{Prop.2.14.2}{\Longrightarrow} & gf(M) \simeq M \text{ in } \mathcal{G}/T^{-1}\mathcal{Y} \\ & \overset{Lemma 2.12}{\Longrightarrow} & gf(M) \simeq M \text{ in } \mathcal{G}. \end{array}$$

Now we combine Theorem 2.6 and Corollary 2.15 to state the result.

**Corollary 2.17.** *Assume we are in the situation of Theorem 2.6 and let in addition be $\mathcal{G}$ and $\mathcal{H}$ Krull–Schmidt categories. Then*[9] *,*

1. $\forall N \in ind(add\ S'\mathcal{Z})_{S'T'\mathcal{Y}}$*, the object $SN$ has precisely one indecomposable summand $g(N)$ not dividing $SS'T'\mathcal{Y}$.*
2. $\forall M \in ind(add\ SS'\mathcal{Z})_{SS'T'\mathcal{Y}}$*, the object $TM$ has precisely one indecomposable summand $f(M)$ not dividing $\mathcal{Y}$.*
3. $f(g(N)) \simeq N$
4. $g(f(M)) \simeq M$

2.4.5. *The situation for group rings.* Again we shall follow [1] closely.

We shall apply Theorem 2.6 to the case mentioned at the beginning of Subsection 2.4. We fix the following setting.

1. Let $R$ be a commutative Noetherian ring.
2. Let $G$ be a finite group and let $D \leq H < G$. [10]
3. $\forall_{F \leq G}$ $ind_F^G := RG \otimes_{RF} - : RF - mod \longrightarrow RG - mod$ ;
   $res_F^G : RG - mod \longrightarrow RF - mod$ is the restriction functor.

[9] $ind\mathcal{C}$ means the class of indecomposable objects in the category $\mathcal{C}$.

[10] To avoid technical difficulties we assume that $H \neq G$. Otherwise we shall have to deal with unpleasant exceptions arising from empty set discussions in our formulas.

4. $$\begin{aligned} mod(G,F) &= add(ind_F^G(RF-mod)) \\ &= \text{'direct summands of } RG\text{–modules induced from } F\text{'}. \end{aligned}$$
The objects are called relatively $F$–projective modules.
5. If $\mathcal{F}$ is a set of subgroups of $G$, then $ind_{\mathcal{F}}^G$ is the least full additive subcategory of $RG-mod$ containing all modules of the form $ind_F^G(V)$ with $V \in Ob(RF-mod)$ and $F \in \mathcal{F}$. $mod(G,\mathcal{F})$ is the least full additive subcategory of $RG-mod$ containing $mod(G,F)$ with $F \in \mathcal{F}$.
6. If $g \in G$ and $F \leq G$. Then, ${}^gF = gFg^{-1}$ and for $M \in RF-mod$ one forms the $R\,{}^gF$–module ${}^gM$ by $f \cdot m =: gfg^{-1}m$ for all $m \in M, f \in F$.
7. We take a disjoint union
$$G = \bigcup_{i=1}^{s} Hg_iH \text{ for } g_i \in G; g_1 = 1$$
Then, as $RH$–$RH$–bimodule,
$$RG = \bigoplus_{i=1}^{s} RHg_iH.$$
8. $p : RG \longrightarrow RH \cdot 1 \cdot RH$ is $RH-RH$–linear and an epimorphism.
9. $i : RH \longrightarrow RH \cdot 1 \cdot RH$ is $RH-RH$–linear and a monomorphism.
10. Set $\mathcal{G} := RG-$mod, $\mathcal{H} := RH-mod$, $\mathcal{D} := RD-$mod.
$S := ind_H^G$, $S' := ind_D^H$, $T := res_H^G$, $T' := res_D^H$.
11. For all $N \in Ob(RH-mod), M \in Ob(RG-mod)$ set
$$\alpha(N,M) : Hom_{RG}(ind_H^G(N), M) \quad \overset{\sim}{\longrightarrow} \quad Hom_{RH}(N, res_H^G M)$$
by Frobenius' reciprocity, as explained in Section 2.2.
12. $$\eta : 1_{RH-mod} \longrightarrow res_H^G \circ ind_H^G$$
by means of
$$\eta_N = \alpha(N, ind_H^G N)(1_{ind_H^G(N)}) = (n \longrightarrow 1 \otimes n)).$$
with $N \in RH-mod, n \in N$.
13. $U := \bigoplus_{i=2}^{s} RHg_iRH \otimes_{RH} -$.
14. Since $pi = 1_{RH}$, we get $TS = 1_{RH} \oplus U$.
15. $\eta_I$ is the identity, hence an isomorphism.

To apply the theorem, one has to try to produce a subcategory $\mathcal{Y}$ of $RH-mod$ such that
$$S'T'\mathcal{Y}|\mathcal{Y} \text{ and } S'T'\mathcal{Y}|U^{-1}\mathcal{Y}.$$

**Notation 3.** *We fix for any set $\mathcal{S}$ of subgroups of $H$*
$$\mathcal{X} := \{D \cap Y | Y \in \mathcal{S}\} \text{ and } \mathcal{Y} := ind_{\mathcal{S}}^H.$$

**Remark 2.18.** Since $ind$ and $res$ result to isomorphic modules when passing to conjugate subgroups, one may assume that $\mathcal{S}$ is closed under conjugation. Since $res$ and $ind$ are transitive, one may furthermore assume that $\mathcal{S}$ is closed under subgroups.

**Proposition 2.19.** 1. *$S'T'\mathcal{Y}$ is a subcategory of $ind_{\mathcal{X}}^H$. Furthermore, $ind_{\mathcal{X}}^H|S'T'\mathcal{Y}$.*

2. *$RH-mod/S'T'\mathcal{Y} = RH-mod/ind_{\mathcal{X}}^H$*
3. *$S'T'\mathcal{Y}$ is a subcategory of $\mathcal{Y}$.*

Proof.
Part 1. Let $N \in \mathcal{Y}$. Then, $N \simeq ind_V^H W$ with $W \in Ob(RV-mod); V \in \mathcal{S}$. We apply Mackey's formula to obtain

$$T'N = res_D^H ind_V^H W = \bigoplus_{VgD\in V\backslash G/D; g_1=1} ind^D_{D\cap\ {}^gY} res^{{}^gY}_{D\cap\ {}^gY}\ {}^gW \in Ob(ind_{\mathcal{X}}^D)$$

Since $ind_D^H ind_{\mathcal{X}}^D = ind_{\mathcal{X}}^H$, we get the first statement.
Let $W \in Ob(D\cap Y-mod), Y \in \mathcal{Y}$, then again using Mackey's formula,

$$res_D^H ind_{D\cap Y}^H W = ind_{D\cap Y}^D W \oplus \bigoplus_{D\neq DgD\in D\backslash H/D} ind^H_{{}^g(D\cap Y)\cap H} res^{{}^g(D\cap Y)}_{{}^g(D\cap Y)\cap H}\ {}^gW$$

Therefore,

$$ind_{D\cap Y}^D W | res_D^H ind_{D\cap Y}^H W = res_D^H ind_{D\cap Y}^D W = T'(ind_{D\cap Y}^D W)$$

and with the transitivity of *ind*, one gets that

$$ind_{D\cap Y}^H W | S'T'(ind_{D\cap Y}^H W).$$

Part 2. follows from 1. since by the first inclusion, every morphism factoring through an object of $S'T'\mathcal{Y}$ factors also through an object of $ind_{\mathcal{X}}^H$. On the other hand side, by the second statement, a morphism factoring through an object of $ind_{\mathcal{X}}^H$ factors also through an object of $S'T'\mathcal{Y}$, this having the object from before as direct summand.
Part 3. follows from the transitivity of the induction and part 1.

Just to be able to write the result in a more concise form we introduce a new

**Notation 4.** *Let $\mathcal{F}$ be a set of subgroups of $H$. We set*

$$\mathcal{F}' := \{H\cap\ {}^gF | g\in G\setminus H \text{ and } F\in\mathcal{F}\}$$

**Lemma 2.20.** *$\forall F\in\mathcal{F}$ with $F\leq H$ is*

$$U(ind_{\mathcal{F}}^H) \subseteq ind_{\mathcal{F}'}^H | U(ind_{\mathcal{F}}^H)$$

Proof. The 'source' of the lemma is entirely Mackey's formula. This fact makes the proof somewhat unpleasantly technical.
Of course, it is sufficient to prove the statements for $\mathcal{F} = \{F\}$, a set with cardinality 1. We start with proving this statement and hence assuming that we are given a $V\in RF-mod$ and set $N := ind_F^H V$.
By definition,

$$U(N) = \bigoplus_{i=2}^{s} RHg_iH \otimes_{RH} N.$$

It is hence enough to prove that

$$RHgH\otimes_{RH} N \in ind_{\mathcal{F}'}^H \text{ for any } g\in G\setminus H.$$

We first discuss what is meant by $RHgH$. We see that $gRH$ is isomorphic to the $R({}^gH)-RH-$bimodule ${}_gRH$ which is $RH$ as $R-$module and on which from the left ${}^gh \in\ {}^gH$ acts by multiplication by ${}^gh$ on $RH$. Now, precisely those objects

$hgh' \in HgH$ belong to $\{1\}gH$ for which $h \in H \cap {}^gH$. Therefore, $RHgH = RH \otimes_{R(H\cap {}^gH)} {}_gRH$ as bimodule. We compute

$$\begin{aligned} RHgH \otimes_{RH} N &= RH \otimes_{R(H\cap {}^gH)} {}_gRH \otimes_{RH} N \\ &= RH \otimes_{R(H\cap {}^gH)} {}^gN \\ &= ind^H_{H\cap {}^gH} res^{{}^gH}_{H\cap {}^gH} {}^gN \\ &= ind^H_{H\cap {}^gH} res^{{}^gH}_{H\cap {}^gH} ind^{{}^gH}_{{}^gF} {}^gV \\ &= ind^H_{H\cap {}^gH} \bigoplus_{(H\cap {}^gH)t\ {}^gF} ind^{[{}^gH\cap H]}_{[{}^{tg}F\cap H\cap {}^gH]} res^{[{}^{tg}F]}_{[{}^{tg}F\cap H\cap {}^gH]} {}^{tg}V \end{aligned}$$

Furthermore, for $n := g^{-1}tg$,

$$H\cap {}^gH\cap {}^{tg}F = H\cap {}^{gn}F$$

since $x \in H\cap {}^{gn}F \Rightarrow x \in H$ and $x \in {}^{gn}F$ and ${}^{gn}F = \{gn\cdot f\cdot n^{-1}g^{-1}\} \subseteq {}^gH$, taking into account that $t \in {}^gH \Rightarrow n \in H$. Hence,

$$\begin{aligned} RHgH \otimes_{RH} N &= \bigoplus_n ind^H_{H\cap {}^{gn}F}(res^{{}^{gn}F}_{[H\cap {}^{gn}F]} {}^{gn}V) \\ &\in ind^H_{\mathcal{F}'} \end{aligned}$$

We hence proved the first statement.

We have to prove the second statement. Let $g \in G$ and $n \in H$, let $V \in Ob(RF-mod)$ and $N := ind^H_F V$. Then, by the above calculation,

$$ind^{[{}^gH\cap H]}_{[{}^{tg}F\cap H\cap {}^gH]} res^{[{}^{tg}F]}_{[{}^{tg}F\cap H\cap {}^gH]} {}^{tg}V \mid RHgH \otimes_{RH} N.$$

We just have to show that a $W = ind^H_{H\cap {}^gF}Q$ for $Q \in R(H\cap {}^gF)-mod$ is a direct summand of a module of the above form.

Set $ind^F_{g^{-1}H\cap F} {}^{g^{-1}}Q =: V \in Ob(RF-mod)$. Then again, applying Mackey's formula,

$$\begin{aligned} ind^H_{H\cap {}^gF} res^{{}^gF}_{H\cap {}^gF}(ind^{{}^gF}_{H\cap {}^gF}Q) &= ind^H_{H\cap {}^gF}(Q\oplus \text{modules from lower subgroups}) \\ &= W \oplus \text{modules from lower subgroups} \end{aligned}$$

We hence have shown that $W$ is a direct summand of a module of the above type which in turn divides $U(ind^H_F)$. We have shown the lemma.

We remind the reader that we are given the set of subgroups $\mathcal{S}$ and we have set $\mathcal{Y} := ind^H_{\mathcal{S}}$ and $\mathcal{X} := \{V\cap D | V \in \mathcal{S}\}$.

**Corollary 2.21.** *If $\mathcal{X}' \subseteq \mathcal{S}$, then $S'T'\mathcal{Y}|\mathcal{Y}$ and $S'T'\mathcal{Y}|U^{-1}\mathcal{Y}$.*

Proof.

- We have just to show that $U(ind^H_{\mathcal{X}})|ind^H_{\mathcal{S}}$, since by Proposition 2.19, $S'T'\mathcal{Y}$ is a subcategory of $\mathcal{Y}$ and hence obviously $S'T'\mathcal{Y}|\mathcal{Y}$ automatically. Also, $S'T'\mathcal{Y}|ind^H_{\mathcal{X}}$. Hence, $U(S'T'\mathcal{Y})|U(ind^H_{\mathcal{X}})$ and if we could show that $U(ind^H_{\mathcal{X}})|ind^H_{\mathcal{S}} = \mathcal{Y}$, then we also had the second condition.
- But, $U(ind^H_{\mathcal{X}}) \subseteq ind^H_{\mathcal{X}'}$ by Lemma 2.20.
- $ind^H_{\mathcal{X}'}|U(ind^H_{\mathcal{X}})$ by Lemma 2.20.
- $ind^H_{\mathcal{X}'} \subseteq ind^H_{\mathcal{S}}$ since $\mathcal{X}' \subseteq \mathcal{S}$.
- Hence, $ind^H_{\mathcal{X}'}|ind^H_{\mathcal{S}}$ and even $U(ind^H_{\mathcal{X}})|ind^H_{\mathcal{S}}$.

**Remark 2.22.** We immediately check two situations where we may verify the condition in Corollary 2.21.

1. If $\mathcal{S} = \{V|$ there is a $g \in G \setminus H : V \leq H \cap {}^gD\}$, then $\mathcal{X} = \{X|$ there is a $g \in G \setminus H : X \leq D \cap {}^gD\}$ and $\mathcal{X}' \subseteq \mathcal{S}$.
2. If $E$ is a normal subgroup of $H$ and $D \cap E$ is a normal subgroup of $G$, set $\mathcal{S} = \{V|V \leq E\}$. Then, $\mathcal{X} = \{X|X \leq D \cap E\}$ and $\mathcal{X}' \leq \mathcal{S}$.

The first situation leads to the classical Green correspondence contrary to the second which is a new application and leads to a theorem due to Auslander–Kleiner [1].

Summarizing the results we just apply Theorem 2.6 to the above situation with the knowledge we have obtained for it up to now.

**Theorem 2.23.** *Let $\mathcal{Z}$ be the largest set of subgroups of $D$ such that $\mathcal{Z}' \subseteq \mathcal{S}$. If $\mathcal{X}' \subseteq \mathcal{Y}$, then*

1. $$ind_H^G : \frac{mod(H, \mathcal{Z})}{mod(H, \mathcal{X})} \xrightarrow{\sim} \frac{mod(G, \mathcal{Z})}{mod(G, \mathcal{X})}$$
   *is an equivalence of categories.*
2. $$res_H^G : \frac{mod(G, \mathcal{Z})}{mod(G, \mathcal{X})} \xrightarrow{\sim} \frac{mod(H, \mathcal{Z})}{mod(H, \mathcal{S})}$$
   *is an equivalence of categories.*
3. $res_H^G \circ ind_H^G$ *is induced by the identity functor on $RH - mod$.*

Corollary 2.17 now translates to

**Corollary 2.24.** *Assume we are in the situation of Theorem 2.23 and furthermore assume that $RG - mod$ and $RH - mod$ are Krull–Schmidt categories. Given $M \in mod(G, \mathcal{Z}) \setminus mod(G, \mathcal{X})$ indecomposable and $N \in mod(H, \mathcal{Z}) \setminus mod(H, \mathcal{X})$ indecomposable.*
*Then,*

1. $ind_H^G(N)$ *has a unique indecomposable direct summand $g(N)$ in $mod(G, \mathcal{Z}) \setminus mod(G, \mathcal{X})$.*
2. $res_H^G(M)$ *has a unique indecomposable direct summand $f(M)$ in $mod(H, \mathcal{Z}) \setminus mod(H, \mathcal{S})$.*
3. $fg(N) \simeq N$.
4. $gf(M) \simeq M$.

An application is the definition of a Brauer correspondent. Assume that we are in the situation of Corollary 2.24.

1. The syzygy-operator $\Omega_G$ on the stable category of $RG - mod$ and the syzygy-operator $\Omega_H$ on the stable category of $RH - mod$ commutes with $g$ and commutes with $f$. More precisely:
   $$\Omega_G g \simeq g\Omega_H \text{ and } \Omega_H f \simeq f\Omega_G.$$
   This follows since $ind_H^G$ and $res_H^G$ are exact and send projective modules to projective modules, hence a projective resolution to a projective resolution. Then, applying Schanuel's Lemma, we realize that syzygies are well defined up to projective direct summands. This gives the result.

2. In the situation of the first part of Corollary 2.21 we look at the various sets of subgroups of $G$ more closely.
$$\begin{aligned} \mathcal{S} &= \{V|\,\exists\, g \in G \setminus H : V \leq H \cap\ {}^{g}D\} \\ \mathcal{X} &= \{X|\,\exists\, g \in G \setminus H : X \leq D \cap\ {}^{g}D\} \\ \mathcal{Z} &= \{Z \leq D|\,\exists\, g, g' \in G \setminus H :\ {}^{g'}Z \cap H \leq H \cap\ {}^{g}D\} \end{aligned}$$
Since $D \leq H$, we always get $D \in \mathcal{Z}$ choosing $g = g'$. If $N_G(D) \setminus H \neq \emptyset$, then there is a $g \in N_G(D) \setminus H$. Taking this $g$, we conclude that $D \in \mathcal{X}$. But then $\mathcal{Z} = \mathcal{X} = \mathcal{S}$ and Theorem 2.23 establishes a bijection between the empty sets and in Corollary 2.24 there is no indecomposable module satisfying the assumptions. If $H \geq N_G(D)$, then trivially this never happens and the theorem is non trivial.
3. Let $B$ be a block of $RG$ with defect group $D$. Let $H \geq N_G(D)$. There is a Green correspondence for $\mathcal{G} = R(G \times G)$ and $\mathcal{H} = R(H \times H)$ and $\mathcal{D} = R(D \times D)$ since $N_{G\times G}(D \times D) = N_G(D) \times N_G(D)$. Furthermore, as usual the functors $S, S'$ are the induction functors. $T, T'$ are the corresponding restriction functors. Now,
$$res^{G\times G}_{H\times H} :\ RG \longrightarrow [\bigoplus_{HgH\in H\backslash G/H} RHgH] = [RH \oplus \bigoplus_{HgH\in H\backslash G/H;\ g\notin H} RHgH]$$
and so a block $B$ of $RG$ with defect group $D$ has a Green correspondent $b$ which is a direct summand of the right hand side. It is now easy to see that the Green correspondent of $B$ is a direct summand of $RH$. The Green correspondent $f(B)$ of $B$ is a block of $RH$ and is called the *Brauer correspondent* of $B$.

# 3. Classical theory of blocks with cyclic defect groups and Green's walk around the Brauer tree

In this section we shall present the results of Green [5], Dade [2] and Michler [13, 14].

Throughout this section we use the following notations.

1. $R$ is a complete discrete valuation ring of characteristic 0 with residue field $k$ of characteristic $p$. The field of fractions of $R$ is $K$.
2. $G$ is a finite group.
3. $B$ is a block of $RG$ with defect $d$.
4. $D$ is a defect group of $B$ with order $q = p^d$.
5. $D_1$ is the subgroup of $D$ of order $p$.
6. $H = N_G(D_1) \geq N_G(D)$.
7. $B'$ is the Brauer correspondent of $B$ in $RH$.
8. $C_G(D_1) =: C$.

## 3.1. The theory of Dade on blocks with cyclic defect group.

**Definition 3.1.** (Michler [13, 14]; Feit [3])

- The number $e$ of isomorphism classes of simple $B'$–modules is called the inertial index of $G$.
- There is a finite Galois extension $\hat{K}$ of $K$ such that for the ring of integers $\hat{R}$ in $\hat{K}$ over $R$ all the primitive $|G|^{th}$ roots of unity are contained in $\hat{R}/rad\ \hat{R}$. Let $\hat{B}$ be one indecomopsable factor of $\hat{R} \otimes_R B'$. (The others are Galois conjugate to this.) The number $\hat{e}$ is defined to be the number of isomorphism classes of simple $\hat{B}$–modules.

Michler shows [13, 14] that $e$ divides $p - 1$.

Set $I := \{0, 1, \ldots, (e - 1)\}$.

The main theorem of Dade describes the structure of the composition series of projective $k \otimes_R B$–modules in terms of combinatorial data, a *Brauer tree.* Janusz and independently Kupisch [8], [10, 11] prove that not only the composition series of the indecomposable projective modules are determined but also those of all indecomposable modules. We do not need this description for Roggenkamp's description of blocks of cyclic defect group and so we refrain from presenting this theory as well.

The theory of Dade on the structure of blocks with cyclic defect groups is one of the most beautiful in the theory of blocks. *It is a complete answer to the questions on the module structure of blocks with cyclic defect groups in terms of a very ingenious combinatorial description.* One of the key tools is the Green correspondence.

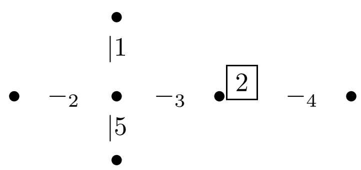
**Theorem 3.2.** *(Dade [2]) We assume that $k$ contains all $|G|$–th roots of unity. There is a set $\Lambda$ of simple $KG$–modules, called the* exceptional $k \otimes_R B$–modules *with the following properties.*

1. *The graph which consists of the following data is a tree:*
   - *The vertices of the graph are the isomorphism classes of the non exceptional simple $K \otimes_R B-$modules and an additional vertex; representing the set of exceptional modules, called the exceptional vertex.*
   - *There is an edge between two vertices $v, w$ if and only if there is an indecomposable projective $B$–module $P$ such that $K \otimes_R P$ has the modules which correspond to the vertices $v$ and $w$ as direct summands.*

*The graph is called a* Brauer tree *and in case $frac\ R$ is a splitting field for $B$, the cardinality of $\Lambda$ is called the* multiplicity *of the exceptional vertex and equals $\mu = (|D| - 1)/e$.*

2. *Let $P$ be a projective indecomposable $k \otimes_R B$–module. Then, $rad\ P/soc\ P$ is a direct sum of two uniserial modules $S_P$ and $T_P$.*
3. *There is an embedding of the Brauer tree in the plane*[11] *such that one can get the composition series of $S_P$ and $T_P$ by the following algorithm. $S_P$ (and $T_P$) correspond to a vertex $v$ (and $w$), say. By symmetry we describe the algorithm just for $S_P$. Since the tree is embedded into the plane, one has an ordering of the $n_P$ projective $RG-$modules $Q$ such that $KQ$ has composition factor $KS_P$ by a counterclockwise numeration of the edges adjacent to $v$. Now, $rad^i(S_P)/rad^{i+1}(S_P) \simeq Q_i/rad(Q_i)$ for $i = 1, 2, \ldots, n(P) \cdot e(P) - 1$ where $Q_i$ is the projective indecomposable module which is $i$ positions after $P$ in the counterclockwise ordering. If $v$ is the exceptional vertex, then $e(P) = e$ and if $v$ is not the exceptional vertex then $e(P) = 1$.*

We describe the algorithm by a simple example.
We are given the following Brauer tree




We impose to the second right vertex a multiplicity 2, as indicated in by a box. We shall give the composition series of the projective indecomposable modules for this example.

$$P_1 = \begin{pmatrix} 1 \\ 2 \\ 5 \\ 3 \\ 1 \end{pmatrix}; P_2 = \begin{pmatrix} 2 \\ 5 \\ 3 \\ 1 \\ 2 \end{pmatrix}; P_5 = \begin{pmatrix} 5 \\ 3 \\ 1 \\ 2 \\ 5 \end{pmatrix}; P_4 = \begin{pmatrix} 4 \\ 3 \\ 4 \\ 3 \\ 4 \end{pmatrix}; P_3 = \begin{pmatrix} & 3 & \\ 1 & & 4 \\ 2 & & 3 \\ 5 & & 4 \\ & 3 & \end{pmatrix}.$$

3.2. **Green's walk around the Brauer tree.** After Dade's paper, Green proved the following theorem. This theorem is of fundamental importance not only for the proof of Roggenkamp's theory of Green orders.

Let $W_i$; $i = 0, \ldots, (e-1)$ be the projective indecomposable $B$–modules.

**Theorem 3.3.** *(Green [5]) We assume that $k$ contains all $|G|$–th roots of unity. Let $G$ be a finite group, let $B$ be an $RG-block$, let $D$ be a cyclic defect group of $B$ and let $\Gamma$ be the Brauer tree of $B$.*

1. *There is a family $(A_n)_{n\in\mathbb{Z}}$ of $RG$–lattices and a permutation $\delta$ of $I = \{0, \ldots, (e-1)\}$ such that there exist short exact sequences of $RG$–modules*

$$E_{2i} : 0 \longrightarrow A_{2i+1} \longrightarrow W_{\delta(i)} \longrightarrow A_{2i} \longrightarrow 0$$

[11]This is just another way of saying that one imposes to each vertex $v$ of the tree a cyclic ordering of the edges $v \overset{e}{-} w$ which are incident to the vertex $v$.

$$E_{2i+1} : 0 \longrightarrow A_{2i+2} \longrightarrow W_i \longrightarrow A_{2i+1} \longrightarrow 0$$

*with* $W_i \simeq W_{i+e}$ *and* $A_i \simeq A_{i+2e}$ *for all* $i \in \mathbb{Z}$.

2. *The* $A_0, A_1, \ldots, A_{2e-1}$ *are mutually non isomorphic.*
3. $KA_n$ *is a vertex of* $\Gamma$.

We should remind the reader that it is possible to reconstruct the Brauer tree from the permutation $\delta$. One forms a path

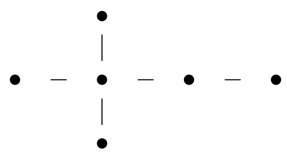
$$\bullet \xrightarrow{1} \bullet \xrightarrow{\delta(1)} \bullet \xrightarrow{2} \cdots \xrightarrow{\delta(e-1)} \bullet \xrightarrow{e} \bullet \xrightarrow{\delta(e)} \bullet$$

which closes to an oriented circle. Then, one glues $\bullet \xrightarrow{i} \bullet$ with $\bullet \xleftarrow{\delta(j)} \bullet$ if $i = \delta(j)$ and the result is the Brauer tree. Obviously it is a *graph* but by Dade's theorem, this is in fact a *tree* and looking at the isomorphism classes of the vertices even the *Brauer tree*.

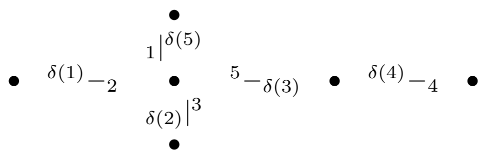
One may define conversely a permutation $\delta$ of the set of edges for every *embedded* tree out of which it is possible to reconstruct the tree in the above way. This permutation depends not only on the tree but also on a starting point:

1. One starts at a certain edge $e$ and declares this edge to be 1.
2. Take a vertex $v$ which is adjacent to the edge taken. The edge following $e$ in the circular ordering at $v$ is defined to be $\delta(1)$.
3. The other extremity (not the vertex $v$) of $\delta(1)$ is $w$.
4. The edge following $\delta(1)$ in the circular ordering at $w$ is 2.
5. To find $\delta(2)$ one proceeds as in 2.
6. One stops after having determined $\delta(e)$.

In our example

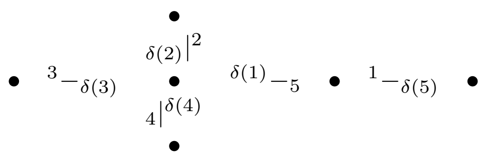


starting with the upper vertical edge one gets the assignment




and the permutation (1 2 3 5).

If one starts with the right most edge, one gets the assignment




and the permutation (1 5).

Michler generalized Theorem 3.3 to the case where there is a ring $R$ as in the introduction, *without assuming that the residue field is large enough.*

**Theorem 3.4.** *(Michler [13, 14]) Let $R$ be as in the introduction to this section. Let $G$ be a finite group, let $B$ be an $RG-$ block and let $D$ be a cyclic defect group of $B$.*

1. *There are precisely $e$ pairwise non isomorphic indecomposable $k\otimes_R B$–modules $M_i$ with source $k$, the trivial $kD$–module.*
2. *Let, for all $i = 1, \ldots e$, $\overline{P}_i$ be the $k \otimes_R B$–projective cover of $M_i$ and let*
$$0 \longrightarrow \overline{\Omega} M_i \longrightarrow \overline{P}_i \longrightarrow M_i \longrightarrow 0$$
*be exact. Then, $(1-\alpha)kD$ is the source of $\overline{\Omega} M_i$.*
3. *Let, for all $i = 1, \ldots e$, $\overline{Q}_i$ be the $k \otimes_R B$–projective cover of $\overline{\Omega} M_i$ and let*
$$0 \longrightarrow \overline{\Omega}^2 M_i \longrightarrow \overline{Q}_i \longrightarrow \overline{\Omega} M_i \longrightarrow 0$$
*be exact. Then, one can find a numbering for the $M_i$ such that $\overline{\Omega}^2 M_i \simeq M_{i+1}$.*
4. *There are $e$ pairwise non isomorphic indecomposable $B$–lattices $W_i$ with source $R$, the trivial $RD$–lattice and $k \otimes_R W_i = M_i$. For each $i$ there is, up to isomorphism, only one $B$–lattice with these properties.*
5. *Let, for all $i = 1, \ldots e$, $P_i$ be the projective cover of $W_i$ and let*
$$0 \longrightarrow \Omega W_i \longrightarrow P_i \longrightarrow W_i \longrightarrow 0$$
*be exact. Then, $(1-\alpha)RD$ is the source of $\Omega W_i$.*
6. *Let, for all $i = 1, \ldots e$, $Q_i$ be the projective cover of $\Omega W_i$ and let*
$$0 \longrightarrow \Omega^2 W_i \longrightarrow Q_i \longrightarrow \Omega W_i \longrightarrow 0$$
*be exact. Then, one can find a numbering for the $W_i$ such that $\Omega^2 W_i = W_{i+1}$ where the indices are taken modulo $e$, and $k \otimes_R \Omega W_i = \overline{\Omega} M_i$.*
7. *In the set $\{\Omega^k W_1 | k \in \mathbb{N}\}$ a maximal subset of pairwise non isomorphic modules has cardinality $2e$.*

Following Feit [3, Chapter VII Remark after Theorem 2.11] we define the property

> $(*)$ The number of characters of the group $H$ which are afforded by irreducible $frac(R) \otimes_R B$–modules is equal to $(q-1)/\hat{e}$.

As is proved in Feit [3, Chapter VII, Corollary 6.8], the definition for a Brauer tree as in Theorem 3.2 works also for a more general $R$ satisfying condition $(*)$. Feit gives also an example that it is in general not enough to adjoin all $q^{th}$ roots of unity.

As is proved in Feit [3, Chapter VII Theorem 10.6] one can prove a theorem which is analogous to Theorem 3.3 also for more general $R$ *satisfying the assumption* $(*)$ from above.

Green shows Theorem 3.3 by first showing Theorem 3.5 below. He applies Green correspondence with $\mathcal{G} = kG - mod, \mathcal{H} = kH - mod$ and $\mathcal{D} = kD - mod$. Clearly, $D_1$ is the only minimal subgroup of $D$ and one chooses $H$ such that each $g \in G \setminus H$ satisfies ${}^g D_1 \cap D_1 = \{1\}$, hence, $\mathcal{X} = \{1\}$. The block $B'$ of $kH$ is the Brauer correspondent of $B$. The simple $B'$–modules are called $S_1, \ldots, S_e$.

**Theorem 3.5.** 1. *$B$ contains $e$ simple $kG$-modules $V_i$; $i \in I$ such that every simple $kG$–module in $B$ is isomorphic to exactly one $V_i$. Let $\overline{W}_i$ be the projective cover of $V_i$ as $kG$–module for all $i \in I$.*

2. *There is a numbering of the $V_i$ such that*

$$Hom_{kH}(fV_j, S_i) \simeq Hom_{kG}(V_j, gS_i) = \begin{cases} k & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

*and there is a permutation $\delta$ of $I$ such that*

$$Hom_{kH}(S_i, fV_j) \simeq Hom_{kG}(gS_i, V_j) = \begin{cases} k & \text{if } \delta(i) = j \\ 0 & \text{if } \delta(i) \neq j \end{cases}$$

3. *For all $i \in I$ there are non split exact sequences*

$$F_{2i} : 0 \longrightarrow \Omega_G(gS_i) \longrightarrow \overline{W}_{\delta(i)} \longrightarrow gS_i \longrightarrow 0$$

*and*

$$F_{2i+1} : 0 \longrightarrow gS_{i+1} \longrightarrow \overline{W}_i \longrightarrow \Omega_G(gS_i) \longrightarrow 0$$

We should take some few lines to interpret Theorem 3.5. The theorem says in other words that the permutation $\delta$ can be read off from the Green correspondents of the simple $B'$–modules. In fact, the Green correspondent $g(S_i)$ has the property

$$top(g(S_i)) = V_{\delta(i)} \text{ and } soc(g(S_i)) = V_i \text{ for all } i = 1, \dots e$$

of course after a renumeration. By the discussion of the permutation $\delta$ one gets the tree back from the permutation. Therefore, the Brauer tree as abstract tree is determined by the Green correspondence.

We shall prove Theorem 3.3 in detail in the following subsections.

3.3. **Dade's description for blocks with normal cyclic defect groups.** As illustration on the degree of completeness of the description of the module structure as well as preparation for the proof of Theorem 3.3 we give Dade's results for the special case of a normal cyclic defect group $D$ of of the block $B$ in this subsection.

Then, the Brauer tree is a star and the exceptional vertex is in the centre. This is the subject of the following Lemma.

We introduce some notation before. As above, the Brauer correspondent of $B$ in $kH$ is called $B'$.

**Lemma 3.6.** *(Dade) We assume that $k$ contains all $|G|$–th roots of unity. $B'$ contains $e$ simple modules $S_0, \dots, S_{e-1}$ with projective covers $T_0, \dots, T_{e-1}$.*

1. *There is a multiplicative isomorphism $\overline{\phantom{x}} : D \longrightarrow$ Centre(kC) such that taking a generator $\alpha$ of $D$ and defining $a := \overline{\alpha} - 1$ the only composition series, which is also the radical series of each $T_i$ with $i = 1, \dots e$ is*

$$T_i > T_i \cdot a > T_i \cdot a^2 > \dots > T_i a^q = 0 \ .$$

2. *Every indecomposable $kH-$module is isomorphic to one of the $T_{i,\nu} = T_i/(T_i a^\nu)\ ; i = 0, 1, \dots, (e-1); \nu = 0, 1, \dots, (q-1)$.*
3. *There is a $kC_G(D_1)$–block $b$ such that $kH \otimes_{kC_G(D_1)} b = B'$ and all such blocks are conjugate in $H$. Moreover, the stabilizer of $b$ in $H$ is of the form $C_G(D_1) \rtimes E$ for a subgroup $E$ of $N_G(D)$. The group $E$ operates on $D_1$ by conjugation and $\overline{\phantom{x}}$ is $E$-linear.*

We take $\alpha_1 = \alpha^{p^{d-1}}$.

Then, $D_1 =< \alpha_1 >$. Since $H = N_G(D_1)$ for all $h \in H$ there is a number $n(h)$ defined uniquely modulo $p$ such that,

$$h^{-1} \cdot \alpha_1 h = \alpha_1^{n(h)}.$$

$$\begin{array}{rcl}\psi : H & \longrightarrow & (k \setminus \{0\}, \cdot) \\ h & \longrightarrow & n(h)\end{array}$$

is a homomorphism and gives rise to one dimensional module. Since $C = C_G(D_1)$, we have $C \in ker(\psi)$ and hence $\psi^{|H:C|} = 1$.

We do similar computations with $E$ and $D$. We define $\forall_{z \in E} z^{-1} \alpha z = \alpha^{n(z)}$. We use the same symbols for $D_1$ as well as for $D$ since we used a compatible choice for the generators of $D$ and $D_1$.

We compute

$$\overline{\alpha}^z \overset{by\ 3.}{=} \overline{\alpha^z} \overset{by\ Def.}{=} \overline{\alpha^{n(z)}} \overset{by\ 3.}{=} \overline{\alpha}^{n(z)}.$$

**Lemma 3.7.** *(Green)*

1. $S_{i,\nu} := T_{i,\nu}/T_{i,(\nu+1)} \simeq \psi^\nu \otimes_k S_i$
2. *For all* $n \in \mathbb{Z}$ *set* $S_n := S_{0,n}$ *and then* $\{S_0, S_1, \ldots, S_{e-1}\}$ *is a complete set of representators of isomorphism classes of simple* $kH-$*modules.*
3. *The composition factors of* $T_i$ *are* $S_i, S_{i+1}, \ldots, S_{i+q-1} \simeq S_i$.

Proof.

Part 1. $\forall_{t \in T_i; z \in E} : t \cdot a^\nu \cdot z = t \cdot z \cdot (\overline{\alpha}^{n(z)} - 1)^\nu = t \cdot z \cdot a^\nu \cdot (1 + \overline{\alpha} + \overline{\alpha}^2 + \cdots + \overline{\alpha}^{n(z)-1})^\nu \equiv t \cdot z \cdot a^\nu \cdot n(z)^\nu$ since $(1 + \overline{\alpha} + \overline{\alpha}^2 + \cdots + \overline{\alpha}^{n(z)-1}) \equiv n(z)$ modulo $T_1 \cdot a$.

Part 2. Since $\psi^{|H:C|} = 1 \Rightarrow S_m \simeq S_n$ if $m \equiv n \bmod |H : C|$. But, $|H : C| \mid (p-1)$, then follows by part 1 that all $S_n$ are composition factors of $T_0$. Hence, $S_n$ all belong to $B'$.

Let $S$ be a simple $B'-$module. Then, there is a sequence $i_0, i_1, \ldots, i_r \in I$ such that $S_{i_0} \simeq S_0$ and $S_{i_r} \simeq S$ and $S_{i,j}$ is a composition factor of $T_{i_{j-1}}$.[12] We know that $S_{i_j} \simeq \psi^{\nu_j} \otimes S_{i_{j-1}}$ for all $j$. Hence, there is an integral number $x$ such that $S \simeq \psi^x \otimes S_0$. We know, that there are precisely $e$ simple modules and therefore we found all of them.

Part 3. follows from Part 1, Part 2. and Lemma 3.6.

**Corollary 3.8.** *(Green) Let* $i \in I$ ; $\nu \in \{1, \ldots, q\}$.

1. $T_{i,\nu}$ *is projective if and only if* $\nu = q$.
2. *There are non split exact sequences*

$$0 \longrightarrow T_{i+1,q-\nu} \longrightarrow T_{i,q} \longrightarrow T_{i,\nu} \longrightarrow 0$$

3. $\forall_{1 \leq \nu \leq q-1} \Omega_H(T_{i,\nu}) \simeq T_{i+\nu,q-\nu}$.
4. $\Omega_H(\Omega_H S_i) \simeq S_{i+1}$

The proof is clear.

3.4. **Definition of the 'walk'** $\delta$. In this subsection we follow closely Green [5]. We shall prove in this subsection Theorem 3.5.

In the following we first examine the situation over $k$ and pass then, in the next subsection, over to $R$.

For the proof of Theorem 3.5 we proceed in several lemmata.

Let $\{V_j\}$ be a complete set of simple $kG$–modules. For proving Part 1 of the theorem we have to show that there is a bijection between $I$ and $J$.

---

[12] This is an alternative method of describing blocks. In fact, we need only the necessity. If there was not such a sequence, then we can divide the projective indecomposables into two disjoint sets $\mathcal{P}_1$ and $\mathcal{P}_2$ such that for all $P_1 \in \mathcal{P}_1$ and all $P_2 \in \mathcal{P}_2$ $Hom(P_1, P_2) = Hom(P_2, P_1) = 0$ and hence $B' = End(\oplus_{P_1 \in \mathcal{P}_1} P_1 \oplus \oplus_{P_2 \in \mathcal{P}_2} P_2) = End(\oplus_{P_1 \in \mathcal{P}_1} P_1) \oplus End(\oplus_{P_2 \in \mathcal{P}_2} P_2)$ decomposes.

**Claim 3.9.** *$fV_j$ is indecomposable and non projective and belongs to $B'$. $gS_i$ is indecomposable and non projective and belongs to $B$.*

Proof. The only thing one has to show is that $fV_j$ and $gS_i$ belong to the blocks as claimed. Since $fg(N) = N$ and $gf(M) = M$ for all $N$ and $M$, we just have to show one of the statements.

We have the Green correspondence with $\mathcal{G} = kG - mod$ and $\mathcal{H} = kH - mod$ and $\mathcal{D} = kD - mod$ on the level of the modules. The Brauer correspondence is a Green correspondence with $\mathcal{G} = k(G \times G) - mod$ and $\mathcal{H} = k(H \times H) - mod$ and $\mathcal{D} = k(D \times D) - mod$. Since $G \times G \longrightarrow G \times 1 \simeq G$ is an epimorphism, we can view each $kG$–module as $k(G \times G)$–module. The analogous holds for $H$ and $D$. The Green correspondent for a $kG$–module $V$ is the same as the Green correspondent of $V$ as $k(G \times G)-$module. This proves the statement since belonging to a block means for a module that the corresponding idempotent of the block acts as identity on the module. Then using the functoriality gives the statement.

We now turn to prove Part 2 of the theorem.

For this purpose we prove that

$$Hom_{kH}(S_i, fV_j) = (Hom_{kH}/mod(H, 1))(S_i, fV_j)$$

and similarly

$$Hom_{kG}(gS_i, V_j) = (Hom_{kG}/mod(G, 1))(gS_i, V_j)\ .$$

More generally, let $X$ be a non projective indecomposable module, then

$$Hom_{kH}(S_i, X) = (Hom_{kH}/mod(H, 1))(S_i, X)$$

Let $\phi : S_i \longrightarrow X$ be a map which is zero on the right side of the equation. Then, $\phi$ factors through a projective module. However, group rings are selfinjective algebras[13]. Since $S_i$ is simple, the projective module over which the mapping factors has as direct summand the injective hull $P$ of $S_i$ and the mapping actually factors over $P$. But, $soc(P) = S_i$ and therefore, if the mapping is not zero, it is injective. However, then the injective module $P$ is a submodule and hence even a direct summand of $X$. This gives a contradiction.

We proved the

[13]An $R$–algebra $A$ is called selfinjective if each projective $A$–module is injective. Group rings are selfinjective since there is a linear map $\lambda : A \longrightarrow R$ such that $ker\lambda$ contains no non zero left nor right ideal and $\forall_{a,b\in A}\lambda(ab) = \lambda(ba)$. Such algebras are called symmetric, which is a slightly stronger condition. A group algebra $RG$ is symmetric since we put $\lambda(\sum_{g\in G} r_g g) := r_1$. Since $\lambda((\sum_{g\in G} r_g g)g^{-1}) = r_g$ there is no ideal in $ker\lambda$. Taking

$$\begin{array}{ccc} {}_AA & \longrightarrow & Hom_R(A_A, R) \\ a & \longrightarrow & b \longrightarrow \lambda(ab)) \end{array}$$

we realize that this mapping is injective since an element in the kernel would induce an ideal in the kernel of $\lambda$ generated by this element. Going to the residue field of $R$ we see that this mapping is also surjective, hence an isomorphism. Injective modules are hence also projective and vice versa. Projective modules for symmetric artinian algebras over a field have the property that the socle and the head are isomorphic.

**Lemma 3.10.**

$$\begin{array}{rcl} Hom_{RH}(S_i, fV_j) & \overset{above}{=} & (Hom_{RH}/mod(H,1))(S_i, fV_j) \\ & \overset{Greencorr.}{=} & (Hom_{RG}/mod(G,1))(gS_i, V_j) \\ & \overset{above}{=} & Hom_{RG}(gS_i, V_j) \end{array}$$

*and analogously*

$$Hom_{RH}(fV_j, S_i) = Hom_{RG}(V_j, gS_i).$$

**Lemma 3.11.** *There is a bijection $h : J \longrightarrow I$ such that*

$$\forall_{i\in I, j\in J} h(j) = i \Longleftrightarrow Hom_{RH}(fV_j, S_i) \neq 0$$

Proof. $fV_j \simeq T_{h(j),\nu(j)}$ for $h(j) \in I, \nu(j) \in \{1,\ldots,q-1\}$ by its indecomposability. But then, it is even uniserial and

$$Hom_{kH}(fV_j, S_i) = \left\{ \begin{array}{ll} k & \text{if } h(j) = i \\ 0 & \text{if } h(j) \neq i \end{array} \right.$$

by Schur's Lemma.[14]

Given $i \in I$ and $S|soc(gS_i)$. Since $S$ is a $B$–module, there is a $j \in J$ such that $V_j \simeq S$. Hence, $Hom_{kG}(V_j, gS_i) \neq 0$ and therefore, $Hom_{kH}(fV_j, S_i) \neq 0$ by Lemma 3.10. This proves that $h(i) = j$ and $h$ is surjective.

Given $j, j' \in J$ with $h(j) = h(j') = i$. Then, we may assume without loss of generality, interchanging $j$ and $j'$ if necessary that $f(V_j) = T_{i,\nu}$ and $f(V_{j'}) = T_{i,\nu'}$ for some $1 \leq \nu' \leq \nu \leq q-1$. Hence, there is an epimorphism

$$T_{i,\nu} \longrightarrow\!\!\!\!\!\rightarrow T_{i,\nu'}$$

If this mapping would factor through a projective module it would factor through $T_i$ which is the projective cover of $T_{i,\nu'}$. Hence,[15] $top(T_{i,\nu})$ is mapped to a subquotient of $rad(T_{i,\nu})$ unless $\nu = 0$ what we excluded. Therefore, the mapping was not surjective and this gives a contradiction. We conclude

$$\begin{array}{rcl} 0 \neq (Hom_{kH}/mod(H,1))(fV_j, fV_{j'}) & \overset{Green\ corr.}{=} & (Hom_{kG}/mod(G,1))(V_j, V_{j'}) \\ & \overset{Schur}{=} & Hom_{kG}(V_j, V_{j'}) \\ & \Rightarrow & j = j'. \end{array}$$

Hence, $h$ is also injective which finishes the proof of Lemma 3.11.

From now on, we take $I = J$ and $h = id_I$ and have $fV_j = T_{j,\nu(j)}$ for all $j \in I$ and certain $\nu(j) \in \{1,\ldots,q-1\}$.

Now we use the same proof as in the lemma in the situation $Hom_{kH}(S_i, fV_j)$ instead of $Hom_{kH}(fV_j, S_i)$ to obtain a bijection $\delta : I \longrightarrow I$, which is a permutation, such that $Hom_{kH}(S_i, fV_j) = k$ if and only if $\delta(i) = j$ and 0 else.

This completes the proof of Part 2. of the theorem.

We are going to show Part 3.

[14] Schur's Lemma says that given a ring $A$ and simple $A$–modules $S$ and $T$, then

$$Hom_A(S,T) = \left\{ \begin{array}{cc} \text{a skewfield} & \text{if } S \simeq T \\ 0 & \text{else} \end{array} \right. .$$

The proof is easy since a kernel and an image under an $A$–isomorphism are ideals which are either 0 or the whole module by the simplicity of $S$ and $T$.

[15] We use the terminus 'top' synonymous to 'head'.

By Part 2. we get $soc(gS_i) \simeq V_i$ and $top(gS_i) \simeq V_{\delta(i)}$. Therefore, there is a short exact sequence

$$0 \longrightarrow \Omega gS_i \longrightarrow \overline{W}_{\delta(i)} \longrightarrow gS_i \longrightarrow 0.$$

Since $\overline{W}_{i+1}$ is also injective and since $soc(gS_i) \simeq V_i$ there is a short exact sequence

$$0 \longrightarrow gS_{i+1} \longrightarrow \overline{W}_i \longrightarrow V \longrightarrow 0$$

with some $kG$–module $V$. But,

$$gS_{i+1} \simeq g\Omega^2 S_i \simeq \Omega^2 gS_i$$

and we see that there is a non split exact sequence

$$0 \longrightarrow gS_{i+1} \longrightarrow \overline{W} \longrightarrow \Omega gS_i \longrightarrow 0$$

with a projective $kG$–module $\overline{W}$. Applying Schanuel's Lemma gives $V \simeq \Omega gS_i$.

We have also proven Part 3. of the theorem.

**Remark 3.12.** The same proof, and statement, works for a stable equivalence between two selfinjective $k$–algebras $A$ and $B$ such that $A$ is serial.

3.5. **Turning to characteristic** 0**.** We now prove the main result of this section. For the reader's convenience we state it here again.

**Theorem 3.13.** *(Green) Let $G$ be a finite group, let $B$ be an $RG-$block, let $D$ be a cyclic defect group of $B$ and let $\Gamma$ be the Brauer tree of $B$.*

1. *There is a family $(A_n)_{n\in\mathbb{Z}}$ of RG–lattices and a permutation $\delta$ of $I = \{0,\ldots,(e-1)\}$ such that there exist short exact sequences of RG–modules*
$$E_{2i} : 0 \longrightarrow A_{2i+1} \longrightarrow W_{\delta(i)} \longrightarrow A_{2i} \longrightarrow 0$$
$$E_{2i+1} : 0 \longrightarrow A_{2i+2} \longrightarrow W_i \longrightarrow A_{2i+1} \longrightarrow 0$$
*with $W_i \simeq W_{i+e}$ are projective indecomposable $B$–modules and $A_i \simeq A_{i+2e}$ for all $i \in \mathbb{Z}$.*
2. *The $A_0, A_1, \ldots, A_{2e-1}$ are mutually non isomorphic.*
3. *$KA_n$ is a vertex of $\Gamma$.*

Proof. We can lift the projective indecomposable modules $\overline{W}_i$ to projective indecomposable $RG$–modules $W_i$ such that $k \otimes_R W_i \simeq \overline{W}_i$ for all $i \in I$. We may extend this definition to $i \in \mathbb{Z}$ by requiring that $W_i \simeq W_{i+e}$.

In the situation of Theorem 3.5 we define,

$$B_{2i} := gS_i \text{ and } B_{2i+1} := \Omega gS_i.$$

**Lemma 3.14.** *Let $m \in \mathbb{Z}$ and $M$ be an $RG-$lattice such that $k\otimes_R M \simeq B_m$. Then there are $RG-$lattices $A_n$ with $A_m \simeq M$ and short exact sequences $E_n$ such that $k \otimes_R E_n \simeq F_n$ for all $n \in \mathbb{Z}$.*

$$E_n : 0 \longrightarrow A_{n+1} \longrightarrow W_n \longrightarrow A_n \longrightarrow 0$$

Proof. Start with $A_m = M$.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_{m+1} & \longrightarrow & W_m & \longrightarrow & A_m & \longrightarrow & 0 \\ & & \downarrow \phi & & \downarrow k\otimes_R - & & \downarrow k\otimes_R - & & \\ 0 & \longrightarrow & B_{m+1} & \longrightarrow & \overline{W}_m & \longrightarrow & B_m & \longrightarrow & 0 \end{array}$$

with $A_{m+1}$ is just defined to be the kernel of $W_m \longrightarrow A_m$. $\phi$ is defined by the universal property of the kernel. Hence, there is an $A_{m+1}$ which lifts $B_{m+1}$ and inductively one gets all $A_n$ for $n \geq m$.

For $n \leq m$ one uses the following strategy. We apply $Hom_R(-,R)$ to the first row and $Hom_k(-,k)$ to the second. We get

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A^*_{m-1} & \longrightarrow & Hom_R(W_{m-1},R) & \longrightarrow & Hom_R(A_m,R) & \longrightarrow & 0 \\ & & \downarrow \phi & & \downarrow k\otimes_R - & & \downarrow k\otimes_R - & & \\ 0 & \longrightarrow & Hom_k(B_{m-1},k) & \longrightarrow & Hom_k(\overline{W}_{m-1},k) & \longrightarrow & Hom_k(B_m,k) & \longrightarrow & 0 \end{array}$$

with $A^*_{m-1}$ is just defined to be the kernel of $Hom_R(W_{m-1},R) \longrightarrow Hom_R(A_m,R)$ and $\phi$ is defined by the universal property of the kernel. Dualizing again,

$$0 \longrightarrow A^*_{m-1} \longrightarrow Hom_R(W_{m-1},R) \longrightarrow Hom_R(A_m,R) \longrightarrow 0$$

we get that

$$0 \longrightarrow A_m \longrightarrow W_{m-1} \longrightarrow Hom_R(A_{m-1},R) \longrightarrow 0$$

is exact and inductively we get the statement.

**Lemma 3.15.** *If $KM$ is a vertex of $\Gamma$, then all $KA_n$ are vertices and $KA_n \simeq KA_{n+2e}$ for all $n \in \mathbb{Z}$ and $KA_n \simeq KA_{n+2e}$.*

Proof. We get that

$$KW_n = KA_n \oplus KA_{n+1} \;\; \forall\, n \in \mathbb{Z}$$

Hence, all $KA_n$ are vertices. In the Grothendieck group $K^0(KG)$ we take

$$\sum_{i=n}^{n+2e} (-1)^i [KW_i] = 0 = (-1)^n \cdot ([KA_n] - [KA_{n+2e}]).$$

**Lemma 3.16.** *Let $A_n$ and $M$ be as above, then $A_n \simeq A_{n+2e}$.*

Proof. Given $n \in \mathbb{Z}$. We have a unique decomposition $KW_n = Y_n(1) \oplus Y_n(2)$ where $Y_n(1)$ and $Y_n(2)$ are both vertices of $\Gamma$. Define

$$X_n(i) := W_n \cap Y_n(i)\,;\, i = \{1,2\}.$$

These are $R$-pure submodules of $W_n$ since

$$W_n/(W_n \cap Y_n(i)) \simeq (W(n) + Y_n(i))/Y_n(i) \leq KW(n)/Y_n(i)$$

the latter of which is $R$–torsion free. Furthermore, they are the only two $R$–pure submodules $\tilde{X}$ of $W_n$ with $K\tilde{X}$ is a vertex in $\Gamma$. In fact, let $\tilde{X}$ be a counterexample with $K\tilde{X} = Y_n(i)$. Then surely $\tilde{X} \leq X_n(i)$. The following diagram is then commutative with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \tilde{X} & \longrightarrow & W_n & \longrightarrow & Q_n & \longrightarrow & 0 \\ & & \downarrow & & \| & & \downarrow & & \\ 0 & \longrightarrow & X_n(i) & \longrightarrow & W_n & \longrightarrow & R_n & \longrightarrow & 0 \end{array}$$

where by the serpent lemma the right most vertical map is surjective. Also, we assumed $Q_n$ to be $R$–free and again by serpent lemma also the cokernel of $\tilde{X} \longrightarrow X_n(i)$ is torsion free. But then, $K\tilde{X} \neq X_n(i)$ and we reached a contradiction.

This proves the lemma.

Now the problem is just reduced to trying to find an $M$ to start with. For the principal block we just take $M = R$, the trivial module.

The general case is more involved and uses a construction of Dade.

It remains to prove Part 2.

We have:

$$k \otimes_R A_{2n+1} \simeq \Omega g S_n$$

$$k \otimes_R A_{2n} \simeq gS_n$$

If $q = |D| \geq 2$ then $S_i = T_{i,1} \not\simeq T_{i,q-1} = \Omega S_i$ and clearly $S_i \not\simeq S_j$ for $i \neq j \ mod \ e$. If $q = 2$ then $A_0 \simeq A_1$ and since $e|(q-1)$ we get $e = 1$. But, $KW_0 \simeq Y_0(1) \oplus Y_0(2)$ which are non isomorphic and also $KW_0 \simeq KA_0 \oplus KA_1$ where both are vertices of $\Gamma$. This gives a contradiction.

Hence, we proved now the whole theorem.

## 4. Blocks with cyclic defect groups are Green orders

We maintain the notation of Section 3.

In this section we shall define 'Green orders' due to Roggenkamp and prove that a block of the group ring $RG$ with cyclic defect group $D$ is a Green order. Moreover, we shall describe the structure of Green orders in great detail.

We follow the exposition in [16] and mention that all the material is contained in [16].

4.1. **A small example.** Let $p$ be a rational prime number. We shall discuss the integral group ring of the dihedral group $D_p$ of order $2p$. We remark that $D_p$ fits well in Green's framework at the prime $p$, where we mean that $\mathbb{Z}_pD_p$ is a group ring which satisfies all the assumptions of Theorem 3.3.

Let $D_p =< a, b|a^p = b^2 = baba = 1 >$ be a presentation of $D_p$. Then, $< a >$ is a cyclic normal subgroup of index 2 in $D_p$. Hence, we get a surjective ring homomorphism

$$\mathbb{Z}D_p \longrightarrow \mathbb{Z}C_2.$$

This is induced by multiplication by the central idempotent

$$e = \frac{1}{p}\sum_{i=1}^{p} a^i \in \mathbb{Q}D_p.$$

Hence, one gets a pullback diagram

$$\begin{array}{ccc} \mathbb{Z}D_p & \stackrel{e\cdot}{\longrightarrow} & \mathbb{Z}D_pe \\ \downarrow \cdot(1-e) & & \downarrow \\ \mathbb{Z}D_p(1-e) & \longrightarrow & \mathbb{Z}D_pe/(\mathbb{Z}D_pe \cap \mathbb{Z}D_p) \end{array}$$

which becomes

$$\begin{array}{ccc} \mathbb{Z}D_p & \stackrel{e\cdot}{\longrightarrow} & \mathbb{Z}C_2 \\ \downarrow \cdot(1-e) & & \downarrow \\ \Lambda & \longrightarrow & \mathbb{F}_pC_2 \end{array}$$

where $\mathbb{F}_p$ is the prime field of characteristic $p$ and the right hand vertical mapping is just reduction modulo $p$. In fact, $pe \in \mathbb{Z}D_p$ and $\mathbb{Z}D_pe \cap \mathbb{Z}D_p = p\mathbb{Z}D_pe$.

We have to determine $\Lambda$. Multiplication of $\mathbb{Z}D_p$ by $(1-e)$ amounts to saying that $a$ acts on $\mathbb{Z}[\zeta_p]$ as multiplication by $\zeta_p$, where $\zeta_p$ is a primitive $p^{th}$ root of unity. In fact, $1 + a + a^2 + \cdots + a^{p-1}$ acts as 0 and this is the only relation among the elements of $\mathbb{Z} < a >$. The $b$ however, it inverts the $a$ and acts therefore as Galois automorphism $\zeta_p \longrightarrow \zeta_p^{-1}$. The element $a$ acts as $\zeta_p$ and this means that over the fixed ring $\mathbb{Z}[\zeta_p + \zeta_p^{-1}]$ the element $a$ satisfies the minimal polynomial

$$X^2 - (\zeta_p + \zeta_p^{-1})X + 1$$

and with basis $\{1, \zeta_p\}$ the representation can be described by the accompanying matrices

$$a \longrightarrow \begin{pmatrix} 0 & -1 \\ 1 & \zeta_p + \zeta_p^{-1} \end{pmatrix} \text{ and } b \longrightarrow \begin{pmatrix} 1 & \zeta_p + \zeta_p^{-1} \\ 0 & -1 \end{pmatrix}.$$

Conjugating by $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ from the left one gets the representations

$$a \longrightarrow \begin{pmatrix} \zeta_p + \zeta_p^{-1} - 1 & 1 \\ \zeta_p + \zeta_p^{-1} - 2 & 1 \end{pmatrix} \text{ and } b \longrightarrow \begin{pmatrix} -1 & 0 \\ \zeta_p + \zeta_p^{-1} - 2 & 1 \end{pmatrix}$$

where $\pi_p := \zeta_p + \zeta_p^{-1} - 2$ generates the unique prime ideal above $p$ in $\mathbb{Z}[\zeta_p + \zeta_p^{-1}]$ (see [6]).

After constructing the standard idempotents in the matrix ring, one gets

$$\Lambda = \begin{pmatrix} \mathbb{Z}[\pi_p] & \mathbb{Z}[\pi_p] \\ \pi_p\mathbb{Z}[\pi_p] & \mathbb{Z}[\pi_p] \end{pmatrix}$$

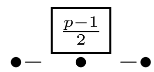
and the mapping to $\mathbb{F}_pC_2$ equals reduction modulo

$$J = \begin{pmatrix} \pi_p\mathbb{Z}[\pi_p] & \mathbb{Z}[\pi_p] \\ \pi_p\mathbb{Z}[\pi_p] & \pi_p\mathbb{Z}[\pi_p] \end{pmatrix}$$

We write down

$$\begin{aligned} \mathbb{Z}D_p = \{(u, \begin{pmatrix} x & y \\ z & w \end{pmatrix}, v) \in \mathbb{Z} \times \begin{pmatrix} \mathbb{Z}[\pi_p] & \mathbb{Z}[\pi_p] \\ \pi_p\mathbb{Z}[\pi_p] & \mathbb{Z}[\pi_p] \end{pmatrix} \times \mathbb{Z} \quad | \quad & x - u \in \pi_p\mathbb{Z}[\pi_p] \text{ and} \\ & u - w \in \pi_p\mathbb{Z}[\pi_p] \text{ and} \\ & u - v \in 2\mathbb{Z}\} \end{aligned}$$

Localizing at the prime $p$ this gives the Brauer tree




with exceptional vertex with multiplicity $(p-1)/2$ in the centre. We shall show that this structure has a feature which is common for all blocks of finite groups with cyclic defect groups.

4.2. **Defining Green orders.** We shall define a class of orders with a structure like in the above example. These orders are introduced by Roggenkamp [16] and who called them *Green orders.*

Throughout this subsection let $R$ be a local Dedekind domain.

- Let $\Gamma$ be a tree[16] embedded in the plane[17].
- Choose a local $R$–torsion $R$–algebra $k$ finitely generated as $R$–module.
- Associate to each vertex $v$ of $\Gamma$ a pair $(\Omega_v, f_v)$ where $\Omega_v$ is a local $R$–order in a semisimple algebra $A_v$, and where $f_v$ is a surjective ring homomorphism $f_v : \Omega_v \longrightarrow k$ with kernel being a principal ideal $a_v\Omega_v$.
- If $v \overset{e}{-} w$ is an edge, then put $\nu_e(v) := w$ and $\nu_e(w) := v$ the mapping giving the other extremity of the edge $e$. Of course, $\nu_e$ is an involution.
- If $e$ is an edge incident to a vertex $v$ of the graph $\Gamma$, then set $\alpha_v(e)$ the edge which follows $e$ in the cyclic ordering at the vertex $v$. We set $\alpha_v^i(e) = \alpha_v^{i-1}(\alpha_v(e))$ and $\alpha_v^1 = \alpha_v$.
- $n_v := \#\{\alpha_v^i(e) | i \in \mathbb{N}; e \text{ is an edge incident to } v\}$ for any vertex $v$ of the graph $\Gamma$.

  A leaf of $\Gamma$ is a vertex $v$ with $n_v = 1$.

---

[16] A tree is understood to be a finite, connected, undirected graph without cycles.

[17] This is equivalent to saying that one imposes to each vertex a cyclic ordering of the edges incident to the vertex.

- Attach to each vertex $v$ of $\Gamma$ the order

$$\Lambda_v := \begin{pmatrix} \Omega_v & \dots & \dots & \dots & \Omega_v \\ (a_v) & \Omega_v & & & \vdots \\ \vdots & (a_v) & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ (a_v) & \dots & \dots & (a_v) & \Omega_v \end{pmatrix}_{n_v \times n_v}$$

- We denote the pullback

$$(*) \quad \begin{array}{ccc} \Omega_v - \Omega_w & \longrightarrow & \Omega_v \\ \downarrow & & \downarrow f_w \\ \Omega_w & \xrightarrow{f_v} & k \end{array} .$$

- We form the iterated pullback of the orders $\Lambda_v$ for each vertex $v$ under the following iterative procedure. Set $\Lambda_g := \bigoplus_{v \in \Gamma_{vertex}} \Lambda_v$
  1. Fix a leaf[18] $v$. The edge leaving $v$ is $e$ and $w := \nu_e(v)$. Form the pullback $(*)$ between[19] $(\Lambda_w)_{(1,1)}$ and $\Omega_v$. Set $\Lambda_g$ the subring of the old $\Lambda_g$ given by this pullback.
  2. For each $i = 2, \dots, n_w$ form the subring of $\Lambda_g$ by the pullback $(*)$ between $(\Lambda_w)_{(i,i)}$ and $(\Lambda_{\nu_{\alpha_w^{i-1}(e)}(w)})_{(1,1)}$. Put $\Lambda_g$ the new subring formed by these pullbacks. Call the vertices $\nu_{\alpha_w^{i-1}(e)}(w)$ reached. Call $w$ saturated.
  3. If there is no vertex which is not yet saturated, then we define the *generic Green order to the tree* $\Gamma$ *with data* $(\Omega_v, f_v)$ *to be* $\Lambda_g$. Stop the algorithm!
  4. Else there is a vertex $v$ which is reached and not saturated. Since $v$ is reached, $\Lambda_g$ contains a pullback between $(\Lambda_v)_{(1,1)}$ and a $\Omega_w$. Set $e$ to be the edge $v - w$. Proceed with 2.

**Definition 4.1.** The resulting order $\Lambda_g$ which occurs after the algorithm executed point 3 in the algorithm is called *generic Green order to the tree* $\Gamma$ *with data* $(\Omega_v, f_v)_{v \in \Gamma_{vertex}}$.

The reader might like to construct the generic Green order to the tree in Section 3.1.

We remark that the isomorphism type of the generic Green order depends only on the embedded graph and the data. This can be proved since $\Lambda_v$ contains an automorphism conjugation by

$$\begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ \vdots & 0 & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & & & \ddots & 1 \\ a_v & 0 & \dots & \dots & 0 \end{pmatrix}_{n \times n}$$

This induces a cyclic permutation of the diagonal entries, the last becoming the first.

In the following section we shall elaborate on the orders $\Lambda_v$.

[18]A leaf of a graph is a vertex $v$ ith $n_v = 1$.

[19]The notation $M_{(i,i)}$ means the $(i,i)-$entry of the matrix $M$.

**Definition 4.2.** (Roggenkamp [16]) Let $R$ be a Dedekind domain with field of fractions $K$. An $R$–order $\Lambda$ in a separable $K$–algebra $A$ is called a *Green order* if there is a finite connected tree with vertices $\{v_i\}_{i=0}^n$ and edges $\{e_k\}_{k=1}^n$.

1. The vertices $\{v_i\}_{i=0}^n$ correspond to (not necessarily primitive) central idempotents $\{\eta_i\}_{i=1}^n$ of $A$ with $1 = \sum_{i=0}^n \eta_i$.
2. The edges $\{e_k\}_{k=1}^n$ correspond to a full set of indecomposable projective $\Lambda$–lattices $\{P_k\}_{k=1}^n$.
3. The tree and a starting vertex determines[20] a permutation $\delta$ of $\{1, \ldots, e\}$ and there is a set of $\Lambda$–lattices $\{A_i\}_{i=1}^n$ such that
   (a) $KA_i \simeq A\eta_i$ for all $i = 0, 1, \ldots, n$
   (b) for all $i = 0, \ldots, n$ there are short exact sequences

$$E_{2i} : \ 0 \longrightarrow A_{2i+1} \longrightarrow P_{\delta(i)} \stackrel{\eta_{\delta(i)}\cdot}{\longrightarrow} A_{2i} \longrightarrow 0$$

$$E_{2i+1} : \ 0 \longrightarrow A_{2i+2} \longrightarrow P_i \stackrel{\eta_i\cdot}{\longrightarrow} A_{2i+1} \longrightarrow 0 \ .$$

The term *generic Green order* is used since in Theorem 4.3 (see also the proof of Lemma 3.16) it will be proven that all Green orders are Morita equivalent to generic Green orders.

*Example.* Let $G$ be a finite group and let $R$ be a complete discrete valuation ring of characteristic 0 with residue field of characteristic $p$ containing all the $|G|^{th}$ roots of unity. Let $B$ be a block of $RG$ with cyclic defect group $D$. By Theorem 3.3 $B$ is a Green order.

**Theorem 4.3.** *(Roggenkamp)* [16] *Let $\Lambda$ be a Green order with tree $\Gamma$. Then $\Lambda$ is Morita equivalent to a generic Green order with tree $\Gamma$.*

We shall give Roggenkamp's proof of Theorem 4.3 in the sequel. For this purpose we shall introduce in the next section another type of orders, which Roggenkamp calls in [17] *isotypic orders.* These are the orders $\Lambda_v$ in the definition for a generic Green order.

4.3. **The rational components; isotypic orders.** Throughout this subsection let $R$ be a Dedekind domain with field of fractions $K$. Let $\Lambda$ be an $R$–order in a separable $K$–algebra $A$.

**Definition 4.4.** (Roggenkamp) [16] The order $\Lambda$ is called *isotypic order* provided there is a twosided $\Lambda$–ideal $J$ such that

1. $K \cdot J = A$
2. J is projective as left $\Lambda$–module.
3. $\Lambda/J$ is a direct product of local $R$–algebras.
4. $\Lambda$ is nilpotent modulo the Higman ideal $H(\Lambda)$.[21]

Then, $J$ is called associated to $\Lambda$ or defining ideal of the isotypic order.

One first property is almost immediate:
$\Lambda$ is isotypic if and only if $\hat{R}_\wp \otimes \Lambda$ is isotypic for all prime ideals $\wp$ of $R$. Here we denote by $\hat{R}_\wp$ the completion of $R$ at $\wp$.

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[20] in the sense described in the discussion in Section 3.2.

[21] The Higman ideal of an $R$–order $\Lambda$ is the $R$–annihilator of $Ext^1_{\Lambda \otimes_R \Lambda^{op}}(\Lambda, -)$. For orders $\Lambda$ in a separable algebra we have $K \cdot H(\Lambda) = 0$ . [18, V. 3.5]

Proof. If $\Lambda$ is isotypic, then $\hat{R}_\wp \otimes \Lambda$ is isotypic. In fact, 1. and 3. are clear. 2. and 4. are consequences of the 'change of rings' theorem.

If $\hat{R}_\wp \otimes \Lambda$ is isotypic for all prime ideals $\wp$ of $R$, then we use the following general property for orders $\Lambda$ in a separable algebra $A$ over a Dedekind domain $R$.

If $M$ and $L$ are full $R$–lattices in the $K$–vector space $V$, then $M_\wp = L_\wp$ almost everywhere. Furthermore, for each $\wp$ let there be given for each $\wp$ a full $R_\wp$–lattice $X(\wp)$ such that $X(\wp) = M_\wp$. Then $N := \bigcap X(\wp)$ has the property that $N_\wp = X(\wp)$ for all $\wp \in Spec(R)$.

We apply this to $J$. We have ideals $J(\wp)$ for all $\wp$ by the definition of isotypic orders. Set $J(\wp) = \hat{\Lambda}_\wp$ whenever $\Lambda_\wp$ is a maximal order. We form $J := \bigcap_{\wp \in Spec(R)} J(\wp)$. Now, $J$ is projective since $J(\wp) = J_\wp$ is projective for all $\wp$. (This is observed most easily by seeing that $Hom_\Lambda(J, -)$ is exact. This in turn is seen by the 'change of rings theorem'. $KJ = A$ since this holds locally and $\Lambda_\wp/J_\wp$ is $R_\wp$–torsion since $K\Lambda_\wp = KJ_\wp$. Hence,

$$\Lambda/J \simeq \prod_\wp R_\wp \otimes \Lambda/J \simeq \prod_\wp (R_\wp \otimes \Lambda)/(R_\wp \otimes J) \simeq \prod_\wp \Lambda_\wp/J_\wp$$

the latter of which was assumed to be a direct product of local algebras.

This finishes the proof of the observation.

*We may assume by this observation, in clarifying the structure of isotypic orders, that $R$ is a complete discrete valuation domain with residue field $\mathbb{F}$ and radical $\pi R$.*

Assuming this, the Higman ideal of $\Lambda$ is a power of $\pi R$. By a general property of Jacobson radicals for artinian algebras gives us that $J$ is nilpotent modulo $\pi R$, if and only if $J \leq rad\ \Lambda$. However, $J$ is nilpotent modulo $\pi R$ if and only if $J$ is nilpotent modulo a certain power of $\pi R$. (Just multiply the nilpotency degree by the power which was fixed at the beginning.) So Condition 4. translates to $J \leq rad\ \Lambda$.

**Theorem 4.5.** *(Roggenkamp [16]) Let $R$ be a complete discrete valuation domain. Assume that $\Lambda$ is a basic[22] isotypic $R$–order with associated ideal $J$. Let furthermore $\Lambda$ be indecomposable as ring.*

*Then, there is a local $R$–order $\Omega$ and a regular non unit $a \in \Omega$ such that $a\Omega = \Omega a =: (a)$ and $\Omega/(a\Omega)$ is a local algebra, and a natural number $n$ such that*

$$\Lambda \simeq \Lambda_0 = \Lambda_0(\Omega, a, n) := \begin{pmatrix} \Omega & \dots & \dots & \dots & \Omega \\ (a) & \Omega & & & \vdots \\ \vdots & (a) & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ (a) & \dots & \dots & (a) & \Omega \end{pmatrix}_{n \times n}$$

*Conversely, every such order is isotypic.*

[22] A Noetherian ring is basic if under each decomposition into a direct sum of indecomposable left projective modules $\Lambda = \oplus_{i\in I} P_i$, no two projective summands $P_i$ and $P_j$ with $i \neq j; i, j \in I$ are isomorphic.

**Remark 4.6.** 1. For $\Lambda_0$ one can take the associated ideal

$$J = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ \vdots & 0 & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & & & \ddots & 1 \\ a & 0 & \dots & \dots & 0 \end{pmatrix}_{n\times n} \cdot \begin{pmatrix} \Omega & \dots & \dots & \dots & \Omega \\ (a) & \Omega & & & \vdots \\ \vdots & (a) & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ (a) & \dots & \dots & (a) & \Omega \end{pmatrix}_{n\times n}$$

which is principal. We call the generating element above $\omega$.

2. The product of local algebras as in 3. in the definition of 'isotypic order' ranges over a set of pairwise isomorphic local algebras $\Omega/(a)$.

Proof.
$\Lambda_0$ is isotypic. In fact,

- $a$ is regular, hence Part 1. of the Definition of an isotypic order.
- $J$ is principal, generated by a regular element, hence free and Part 2. follows.
- $\Lambda/J = \prod_{i=1}^n \Omega/(a)$ is a direct product of local orders, hence we get also Part 3.

$a$ is not a unit. Since $\Omega$ is local, $J$ is contained in the radical of $\Lambda$. The radical of $\Lambda$ is the ideal with $rad\Omega$ in the main diagonal and in the lower triangular matrix, $\Omega$ in the rest of the entries.

We have to prove the converse. We may order a complete set of projective indecomposable modules $P_1, P_2, \dots, P_n$ such that $rank_R P_i \leq rank_R P_{i+1}$ for all $i = 1, 2, \dots, n-1$.

Since for any $i$ the module $P_i$ is projective, there is a $Q_i$ and an integer $m(i)$ such that $P_i \oplus Q_i \simeq \Lambda^{m(i)}$. But then, $J \otimes_\Lambda P_i \oplus J \otimes_\Lambda Q_i \simeq J^m$ which is projective since $J$ is projective. Hence, $J \otimes_\Lambda P_i$ is projective. Then, for any $i$ the set

$$\mathcal{P}_i := \{ \text{ isomorphism classes of the modules } J^j \otimes_\Lambda P_i | j \in \mathbb{N}\}$$

consists of projective modules. The cardinality of this set is $k(i)$.

**Claim 4.7.** *$J^{k(1)} \otimes_\Lambda P_1 \simeq P_1$. Furthermore, $k(1) = n$ and hence $\mathcal{P}_1$ is a complete set of isomorphism classes of projective modules.*

Proof. Since $KJ = A$, we get $rank_R P_1 = rank_R J^j \otimes_\Lambda P_1$ and we may order the projective indecomposable modules such that $J^j \otimes_\Lambda P_1 = P_{1+j}$. Let $Q = \{Q_1, Q_2, \dots, Q_s\}$ be a set of representatives of isomorphism classes of indecomposable projective modules such that the isomorphism class of no element of $Q$ is contained in $\mathcal{P}_1$.

Since $\Lambda$ is indecomposable, there is a $Q_i$ with $Hom_\Lambda(Q_i, P_{1+j}) \neq 0$. If not, the endomorphism ring of $\Lambda$ would be the direct product of the endomorphism rings of the direct sum of modules in $Q$ and that of $\oplus_{i=0}^{k(1)} P_{1+i}$.

We now reduce to artinian algebras. Take $0 \neq \phi \in Hom_\Lambda(Q_i, P_{1+j})$. Then, there is a $\nu \in \mathbb{N}$ such that the composition

$$Q_i \xrightarrow{\phi} P_{1+j} \longrightarrow (P_{1+j})/(\pi^\nu P_{1+j})$$

is non zero. Since $J \leq rad\ \Lambda$, there is a $\mu$ such that $J^\mu \leq \pi^\nu \Lambda$. But, for all $k$, $J^k/J^{k+1}$ is a projective $\Lambda/J$ module. However, this is a direct product of local algebras. Hence,

$$J^k \cdot P_{1+j}/J^{k+1} \cdot P_{1+j}$$

is a local $\Lambda/J$–module and has all composition factors isomorphic to $P_{1+j}/rad\ P_{1+j}$. Hence, $P_{1+j}/\pi^\nu P_{1+j}$ has composition factors all isomorphic to $P_k/rad\ P_k$ for $k = 1, \dots, k(1)$. But, the top of $Q_i$ is isomorphic to one of them, hence the isomorphism class of $Q_i$ is in $\mathcal{P}_1$, a contradiction and $Q$ is empty.

It remains to prove that $J^{k(1)} \otimes_\Lambda P_1 \simeq P_1$.

If $J^{k(1)} \otimes_\Lambda P_1 \simeq P_k$ for some $k$, then one forms the set $\mathcal{P}_k$ instead of $\mathcal{P}_1$ and $\{P_1, \dots, P_{k-1}\}$ would be in $Q$. The analogous arguments as above give a contradiction and we are done.

By Morita theory, $\oplus_{i=1}^n P_i$ gives a Morita bimodule inducing a Morita equivalence between $\Lambda$ and $End_\Lambda(\oplus_{i=1}^n P_i)$. Since we have the Krull–Schmidt theorem for $\Lambda$ and since $\Lambda$ is basic, $\Lambda \simeq End_\Lambda(\oplus_{i=1}^n P_i)$.

**Claim 4.8.** *$End_\Lambda(\oplus_{i=1}^n P_i) \simeq \Lambda_0$ for $\Omega := End_\Lambda(P_1)$ and $(a) := Hom_\Lambda(P_1, J^n \otimes_\Lambda P_1)$.*

Proof. Since $KP_1$ is simple by construction, the isomorphism $P_1 \longrightarrow J^n P_1$ is multiplication by a regular element $a \in \Omega$. Hence,

$$Hom_\Lambda(P_1, J^n \otimes_\Lambda P_1) = \Omega \cdot a.$$

Any endomorphism $\phi \in \Omega$ can be extended to an endomorphism of $J^i \otimes_\Lambda P_1$ by $id_{J^i} \otimes \phi$. Hence,

$$\Omega \leq End_\Lambda(P_2) \leq \cdots \leq End_\Lambda(P_{n-1}) \leq End_\Lambda(P_n) \leq End_\Lambda(J^n \otimes_\Lambda P_1) = a^{-1} \cdot \Omega \cdot a.$$

But then, all of the endomorphism rings are equal, $\Omega$ being noetherian.

Since when tensored by $K$ over $R$, all the $P_i$ are isomorphic and simple, any non zero mapping $\phi : P_i \longrightarrow P_{i+j}$ for $i = 1, \dots, n$ and $j = 1-i, \dots, n-i$ is injective. Looking at the tops of the modules $P_{i+j}/J^k P_{i+j}$ by the arguments given in the proof of the preceding claim, $Hom_\Lambda(P_i, P_j) = \Omega$ if $j \leq i$ and $Hom_\Lambda(P_i, P_j) = \Omega \cdot a$ if $j > i$.

This proves the claim and also the theorem.

## 4.4. **Structure theorem for Green orders.** 

Proof of Theorem 4.3. We assume that $\Lambda$ is basic. The proof will be done in several steps.

**Lemma 4.9.** *Let $\Lambda$ be a basic Green order with connected tree $\Gamma$ and let $v$ be a leaf. Let $e$ be the edge joining $v$ with some vertex $w$. The projective indecomposable module associated with $e$ is $P_0$. Then, $End_\Lambda(\Lambda/P_0)$ is a Green order with tree $\Gamma'$ where $\Gamma'_{vertex} = \Gamma_{vertex} \setminus \{v\}$ and $\Gamma'_{edge} = \Gamma_{edge} \setminus \{e\}$.*

Proof. Without loss of generality we may set $A_0 = \Lambda\eta_0$. Set $P := \Lambda/P_0$. Apply $F := Hom_\Lambda(P, -)$ to $E_n$ for all $n = 0, \dots 2n-1$. Since $P$ is projective, this functor is exact. However, since there is only one projective indecomposable module, namely $P_0$, $F(A_0) = 0$. But then $F(E_{2i})$ for $i = 1, \dots, n$ and $F(E_{2i+1})$ for $i = 0, \dots, n-1$ are the required exact sequences.

This finishes the proof of the lemma.

**Lemma 4.10.** *Let $\Lambda$ be a basic $R$–order in the separable algebra $A$. Let $e$ be a central idempotent of $A$ and let $P|\Lambda$ with $P \cdot e = 0$. Put $End_\Lambda(\Lambda/P) =: \Lambda_0$ as a subring of $\Lambda$ (not with the same unit!). Then,*

$$\Lambda \cdot e = \Lambda_0 \cdot e \text{ and } \Lambda_0 \cdot e \cap \Lambda = \Lambda_0 \cdot e \cap \Lambda_0.$$

Proof. To prove the first statement one observes that

$$\begin{array}{rcl}\Lambda e & = & e\Lambda e \\ & = & eEnd_\Lambda((\Lambda/P)\oplus P)e \\ & = & e\begin{pmatrix} End_\Lambda(\Lambda/P) & Hom_\Lambda((\Lambda/P),P) \\ Hom_\Lambda(P,\Lambda/P) & End_\Lambda(P)\end{pmatrix}e \\ & = & \begin{pmatrix} eEnd_\Lambda(\Lambda/P)e & 0 \\ 0 & 0\end{pmatrix} \\ & = & \Lambda_0 e\end{array}$$

The second statement is proved as follows:
By the first statement,

$$\Lambda_0\cap(\Lambda_0 e)=\Lambda_0\cap(\Lambda e)$$

and clearly

$$\Lambda\cap(\Lambda e)\supseteq\Lambda_0\cap(\Lambda e)$$

Hence we have to show that

$$\Lambda\cap(\Lambda e)\subseteq\Lambda_0\cap(\Lambda e)$$

This is done by the following computation:

$$\begin{array}{rcl}\Lambda\cap(\Lambda e) & = & \begin{pmatrix} End_\Lambda(\Lambda/P) & Hom_\Lambda((\Lambda/P),P) \\ Hom_\Lambda(P,\Lambda/P) & End_\Lambda(P)\end{pmatrix} \\ & & \cap e\begin{pmatrix} End_\Lambda(\Lambda/P) & Hom_\Lambda((\Lambda/P),P) \\ Hom_\Lambda(P,\Lambda/P) & End_\Lambda(P)\end{pmatrix}e \\ & = & \begin{pmatrix} End_\Lambda(\Lambda/P) & Hom_\Lambda((\Lambda/P),P) \\ Hom_\Lambda(P,\Lambda/P) & End_\Lambda(P)\end{pmatrix} \\ & & \cap\begin{pmatrix} eEnd_\Lambda(\Lambda/P)e & 0 \\ 0 & 0\end{pmatrix} \\ & = & \begin{pmatrix} End_\Lambda(\Lambda/P) & 0 \\ 0 & 0\end{pmatrix}\cap\begin{pmatrix} eEnd_\Lambda(\Lambda/P)e & 0 \\ 0 & 0\end{pmatrix} \\ & = & \Lambda_0\cap(\Lambda_0 e)\end{array}$$

**Claim 4.11.** *Let $\Lambda$ be a basic Green order with tree $\Gamma$. For the idempotent $\eta$ corresponding to a vertex $v$ the ring $\Lambda/(\Lambda\cap\Lambda\eta)$ is a direct product of local algebras.*

Proof. We use induction on the number of vertices.

If the Green order has only 2 vertices, then the statement is clear since there is just one projective indecomposable module and the Green order is local.

Let then $v$ be a leaf of the tree $\Gamma$. Let $e$ be the edge of the tree that links $v$ with the rest of the tree. Let $P_0$ be the projective indecomposable which corresponds to $e$. Set $P=\Lambda/P_0$ and $\Lambda_0=End_\Lambda(P)$. Then, by Lemma 4.10 the tree $\Gamma'$ defined by

$$\Gamma_{vertex}\setminus\{v\}=:\Gamma'_{vertex};\Gamma_{edge}\setminus\{e\}=:\Gamma'_{edge}$$

defines a Green order structure on $\Lambda_0$.

Let $v\overset{e}{-}w$ and let $v'$ with $w\neq v'\neq v$ be a vertex of $\Gamma$. Then, for $v'$ the statement is true by induction.

We need a proof for $w$ and its central idempotent $\eta_w$ only. Let $\eta_v$ be the central idempotent corresponding to $v$.

$\epsilon := 1 - \eta_v - \eta_w$. We get the following pullback diagram.

$$\begin{array}{ccc} \Lambda & \longrightarrow & \Lambda(\epsilon + \eta_v) \\ \downarrow & & \downarrow \\ \Lambda\eta_w & \longrightarrow & \Lambda(\epsilon + \eta_v)/(\Lambda \cap \Lambda(\epsilon + \eta_v)) \end{array}$$

We see that

$$\Lambda(\epsilon + \eta_v) = \Lambda\epsilon \oplus \Lambda\eta_v$$

and

$$\Lambda(\epsilon + \eta_v)/(\Lambda \cap \Lambda(\epsilon + \eta_v)) = [\Lambda\epsilon/\Lambda \cap \Lambda\epsilon] \oplus [\Lambda\eta_v/\Lambda \cap \Lambda\eta_v].$$

But, by Lemma 4.10

$$\Lambda\epsilon/(\Lambda \cap \Lambda(\epsilon)) = \Lambda_0\epsilon/(\Lambda_0 \cap \Lambda_0(\epsilon))$$

and since $\epsilon$ is also a central idempotent of $K\Lambda_0$, which gives rise to a pullback diagram itself with quotient $\mathcal{A} := \Lambda_0\epsilon/(\Lambda_0 \cap \Lambda_0(\epsilon))$, the ring $\mathcal{A}$ is a direct product of local $R$–algebras by induction.

Now we use the fact that $v$ is a leaf. If $\Lambda\eta_v$ was not local, it had two non isomorphic simple modules. Since $\Lambda\eta_v$ is an image of $\Lambda$, $\Lambda$ itself has these two non isomorphic modules. However, there is only one simple $\Lambda$–module, namely the top of $P_0$, on which $Hom_\Lambda(P_0, -)$ is non zero. The two simple $\Lambda$–modules constructed above however have this property, and hence they cannot exist. We conclude, $\Lambda\eta_v$ is local.

Since $\Lambda\eta_v$ is local, $\Lambda\eta_v/(\Lambda\eta_v \cap \Lambda)$ is local.

**Lemma 4.12.** *Let $\Lambda$ be a basic Green order with tree $\Gamma$. Let $v$ be a vertex of $\Gamma$ with corresponding central idempotent $\eta_v$. Then, $\Lambda\eta_v \cap \Lambda$ is a free $\Lambda\eta_v$–module.*

Proof. Again we use induction on the number of vertices of $\Gamma$.

Assume that there are only two vertices. Hence, there are short exact sequences

$$0 \longrightarrow A_1 \longrightarrow \Lambda \stackrel{\eta_v}{\longrightarrow} A_0 \longrightarrow 0$$

$$0 \longrightarrow A_0 \longrightarrow \Lambda \stackrel{1-\eta_v}{\longrightarrow} A_1 \longrightarrow 0.$$

Thus, $\Lambda(1 - \eta_v) \cap \Lambda \simeq A_0 \simeq \Lambda\eta_v$.

Assume we have more than two vertices.

If $v$ is a leaf with idempotent $\eta_v$, let $v \stackrel{e}{-} w$ be the edge of the tree linking $v$ with the rest of $\Gamma$. The idempotent associated with $w$ is denoted by $\eta_w$. Hence, there is an indecomposable projective $\Lambda$–module $P_0$, and there are short exact sequences

$$0 \longrightarrow A_0 \longrightarrow P_0 \stackrel{\eta_w}{\longrightarrow} A_{-1} \longrightarrow 0$$

$$0 \longrightarrow A_1 \longrightarrow P_0 \stackrel{\eta_v}{\longrightarrow} A_0 \longrightarrow 0.$$

Let $P_0 = \Lambda e_0$ for an idempotent $e_0$ of $\Lambda$. Since $v$ is a leaf, $\eta_v P_0 = \eta_v\Lambda$; the proof is the same as that of proving that $\Lambda\eta_v$ is local. Now, since $\eta_w P_0 = (1 - \eta_v)P_0$,

$$\Lambda\eta_v \cap \Lambda = \eta_v P_0 \cap \Lambda = \eta_v P_0 \cap P_0 \simeq A_0 = \eta_v P_0$$

If $v_2$ is a vertex different from $v$ and $w$, then let $\eta_{v_2}$ be the central idempotent associated to $v_2$. Then, since

$$\Lambda\eta_{v_2} \cap \Lambda = \Lambda_0\eta_{v_2} \cap \Lambda_0 \text{ and } \Lambda\eta_{v_2} = \Lambda_0\eta_{v_2},$$

as usual for $\Lambda_0 = End_\Lambda(\Lambda/P_0)$, by Lemma 4.10, the statement is true by the induction hypothesis.

We have to prove the statement for $w$. Let $\{e_1, e_2, \ldots, e_n\}$ be the set of edges adjacent to $w$. Let $\{P_1, P_2, \ldots, P_n\}$ be the corresponding projective indecomposable modules. Set $\Delta := End_\Lambda(\bigoplus_{i=1}^n P_i)$. Then, again by Lemma 4.10

$$\Delta\eta_w = \Lambda\eta_w \text{ and } \Delta\eta_w \cap \Delta = \Lambda\eta_w \cap \Lambda$$

Hence we may prove the statement for $\Gamma$ being a star and $w$ in the centre and number the edges in their cyclic ordering. Let $\{\eta_1, \eta_2, \ldots, \eta_n\}$ be the idempotents corresponding to the vertices being the end of $\{e_1, e_2, \ldots, e_n\}$ unequal to $w$. The idempotent corresponding to $w$ is $\eta$. By the definition of a Green order we get short exact sequences

$$\begin{array}{c} 0 \longrightarrow A_1 \longrightarrow P_1 \xrightarrow{\eta_1\cdot} A_0 \longrightarrow 0 \\ 0 \longrightarrow A_2 \longrightarrow P_2 \xrightarrow{\eta\cdot} A_1 \longrightarrow 0 \\ 0 \longrightarrow A_3 \longrightarrow P_2 \xrightarrow{\eta_2\cdot} A_2 \longrightarrow 0 \\ 0 \longrightarrow A_4 \longrightarrow P_3 \xrightarrow{\eta\cdot} A_3 \longrightarrow 0 \\ 0 \longrightarrow A_5 \longrightarrow P_3 \xrightarrow{\eta_3\cdot} A_4 \longrightarrow 0 \\ \ldots\ldots \\ 0 \longrightarrow A_{2n-1} \longrightarrow P_n \xrightarrow{\eta_n\cdot} A_{2n-2} \longrightarrow 0 \\ 0 \longrightarrow A_0 \longrightarrow P_1 \xrightarrow{\eta\cdot} A_{2n-1} \longrightarrow 0 \end{array}$$

For computing $\Lambda\eta \cap \Lambda$ we have to sum up every second kernel and get

$$\Lambda\eta \cap \Lambda \simeq \bigoplus_{i=1}^n A_{2i} = \bigoplus_{i=1}^n \eta P_i = \eta \bigoplus_{i=1}^n P_i$$

is a progenerator. If we furthermore assume that the Green order is basic, $\Lambda\eta \cap \Lambda$ is a free $\Lambda\eta$–module.

This proves the lemma.

To prove the theorem we just have to paste the different pieces together.

We see that for each vertex $v$ the order $\Lambda_v := \Lambda\eta_v$ is an isotypic order with defining ideal $J_v := \Lambda\eta_v \cap \Lambda$.

In fact,

- $K \otimes_R (\Lambda\eta_v \cap \Lambda) = K \otimes_R \Lambda\eta_v$,
- $J_v$ is a free $\Lambda\eta_v$–ideal by Lemma 4.12,
- and $\Lambda\eta_v/(\Lambda\eta_v \cap \Lambda)$ is a direct product of local $R$–algebras by Lemma 4.9.
- The ideal $J_v$ for all vertices $v$ is contained in the radical of $\Lambda_v$: Let $S$ be a simple $\Lambda_v$–module. One has to show that $J_v$ acts as 0 on $S$. Since $\Lambda \longrightarrow\!\!\!\!\rightarrow \Lambda_v$, each simple $\Lambda_v$–module is also a simple $\Lambda$–module. But, we get $n$ simple $\Lambda$–modules just by the local algebras which we get in the quotients $\Lambda_v/J_v$, where $n$ is the number of edges of $\Gamma$. All these are annihilated by all the $J_v$. $S$ is one of them.

We get the structure of $\Lambda_v$ by Theorem 4.5. The $\Omega_v$ in the main diagonals of the matrices correspond to the projective indecomposable modules, as one sees from the proof of Theorem 4.5, and these are for Green orders just the $\eta_v P$ for $\eta_v$ being the idempotent corresponding to $v$ and for $P$ being a projective indecomposable module corresponding to an edge incident to $v$. The pullbacks linking the different $\Lambda_v$ are as constructed in the example since this is the way the exact sequences corresponding to the tree are built.

**Remark 4.13.** In [19, 9] it is proven that two Green orders having the same data $(\Omega_v, f_v)$ but not necessarily the same underlying tree have equivalent bounded derived module categories. An explicit twosided tilting complex which provides this derived equivalence is given in [20].

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