COMMENTS ON GENTLENESS OF ENDOMORPHISM ALGEBRAS

ALEXANDER ZIMMERMANN

Abstract. In a joint paper with Jan Schröer we have shown that a module M over a special biserial algebra A with $Ext^1_A(M,M)=0$ has gentle stable endomorphism algebra. In the present note we interpret this result and study the gentle algebras which occur as stable endomorphism algebras of modules over the alternating group of degree 4 in characteristic 2.

Introduction

The class of special biserial algebras is a very well studied class of algebras of tame representation type (see Skowroński and Waschbüsch [11]). These algebras occur naturally in many situations. By a result of Erdmann [5] blocks of group rings of finite groups over an algebraically closed field K of characteristic 2 and dihedral defect group are examples. Assem and Skowroński [1] introduced the class of gentle algebras as a subclass of special biserial algebras to axiomatize the properties of an iterated tilted algebra of a hereditary algebra of type $\tilde{A}_n$. They have remarkable properties, as they provide a sufficiently rich class of nevertheless quite well controlled behaviour. In recent work [10] with Jan Schröer we showed that the endomorphism ring, taken in the stable category, of a module M over a special biserial algebra A with $Ext^1_A(M,M)=0$ is a gentle algebra.

In the present note we present this result and we get a nice consequence. Slightly generalizing in Section 2 the original argument in [10], we show in the present note that an algebra whose derived category of bounded complexes of modules can be fully faithfully embedded as a triangulated category into the derived category of a gentle algebra is gentle again. Since the gentle algebras are combinatorially defined, it seems to be a nice project to furnish them with additional structure, and maybe single out particularly nice gentle algebras as those provided by the group theoretical situation. Moreover, the group theoretical properties of the module M should have an implication on the gentle algebras obtained this way. We study here in Section 3 the smallest non trivial case to show that this class of nice gentle algebras is non empty even for such a small group as the alternating group of degree 4. It appears that there are a finite number of algebras occuring this way, exactly 23 indecomposable algebras which appear to be of very symmetric structure.

The present note originated from my lecture in Luminy in June 2002 at the meeting "Théories homologiques, représentations et algèbres de Hopf" and I want to express my gratitude to the organisers for having invited me to this very interesting conference. Moreover, I want to thank the referee for his careful reading of the manuscript and various suggestions which lead to many improvements.

1. Quivers, special biserial and gentle algebras

Let Q be a quiver and let I be an admissible ideal of the quiver algebra KQ for an algebraically closed field K. Then $A=KQ/I$ is a finite dimensional K-algebra and, up to Morita equivalence, any finite dimensional K-algebra A arises this way. The pair (Q,I) is called *special biserial* if the following hold:

1) For any vertex v the set of lengths of the paths starting in v and not being in I is finite.
2) Each vertex in Q is the end point of at most 2 arrows and the starting point of at most 2 arrows.
3) For any arrow α there is at most one arrow β composable with α from the right and $\alpha\beta$ is not in I, and there is at most one arrow γ composable with α from the left and $\gamma\alpha$ is not in I.

A special biserial pair is called *gentle* if in addition:

4) There is a generating set of I as ideal consisting of paths of length 2;
5) For any arrow α there is at most one arrow β composable with α from the right and $\alpha\beta$ is in I, and there is at most one arrow γ composable with α from the left and $\gamma\alpha$ is in I.

Date: May 24, 2003.

A K-algebra A is called *gentle*, if it is Morita equivalent to an algebra KQ/I for (Q, I) gentle, and A is called *special biserial* if A is Morita equivalent to KQ/I for (Q, I) special biserial. In these cases, KQ/I is finite-dimensional if and only if the set of vertices Q_0 of Q is finite.

We shall concentrate in the sequel on the rôle of special biserial algebras in the representation theory of groups. It is known that blocks B of group rings KG of finite groups G over algebraically closed fields K are of tame representation type if and only if K is of characteristic 2 and the defect group of B is either dihedral, semidihedral or generalised quaternion. Karin Erdmann proved [5] that a block B of KG with defect group D is special biserial if and only if D is a dihedral 2-group. Moreover, Erdmann defined a small class of algebras by conditions on the quiver and on the relations as possible representatives of the Morita equivalence classes of blocks with dihedral defect group.

2. Linking gentle and special biserial algebras

By definition any gentle algebra is special biserial. There is a more intrinsic relation between special biserial algebras and gentle algebras. For an algebra A let RA be its repetitive algebra (cf e.g. [6]). Then, a result by Schröer [9] and Ringel [8] show that an algebra A is gentle if and only if RA is special biserial. Is it possible to derive from special biserial algebras other algebras which are always gentle?

Denote by $A - proj$ the category of projective A-modules. The stable module category $A - \underline{mod}$ has objects the same as the category of A-modules, and the morphisms from an A-module M to a module N is the quotient of A-linear homomorphisms from M to N modulo those factoring through a projective module. The stable module category is cosuspended with the syzygy functor being the suspension functor. The same concept exists replacing projective modules by injective modules, and the stable category is then suspended and denoted by $A - \overline{mod}$. In case A is selfinjective, the two concepts coincide and the stable category is triangulated. We mention that the repetitive algebra RA is always selfinjective.

Theorem 1. [10] *Let A be a special biserial algebra over an algebraically closed field K and let M be an A-module. If $Ext^1_A(M, M) = 0$, then $\underline{End}_A(M)$ and $\overline{End}_A(M)$ are gentle K-algebras.*

The proof for this result uses heavily a basis given by Crawley-Boevey [4] for the vector space of homomorphisms $Hom_A(M, N)$ for two A modules M and N which behaves nicely under composition of mappings. Then, one carefully examinates when homomorphisms factor through projective modules, and in some critical cases, where they do not factor, they give rise to a non trivial extension class.

Let $D^b(A)$ be the derived categories of bounded complexes of A-modules. For details about this concept and notations see [7]. We get the following consequence.

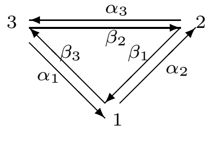
Corollary 2.1. *Let K be an algebraically closed field and let A and B be K-algebras. If the bounded derived category $D^b(B)$ can be fully faithfully embedded into $D^b(A)$ as triangulated categories, then B is a gentle algebra if A is a gentle algebra,.*

Proof. Let $E : D^b(B) \longrightarrow D^b(A)$ be a fully faithful embedding. Since B is projective, $Ext^1_B(B, B) = Hom_{D^b(B)}(B, B[1]) = 0$. Applying E and putting $T := E(B)$, one gets $Hom_{D^b(A)}(T, T[1]) = 0$. In [6] Happel proves that there is a fully faithful embedding $F_A : D^b(A) \hookrightarrow RA - \underline{mod}$ as triangulated categories. Since F_A is fully faithful, $0 = Hom_{D^b(A)}(T, T[1]) \simeq \underline{Hom}_{RA}(F_AT, F_A(T[1])) \simeq \underline{Hom}_{RA}(F_AT, \Omega^{-1}(F_AT)) \simeq Ext^1_{RA}(F_AT, F_AT)$.

Since RA is special biserial by the result of Schröer [9] and Ringel [8], $F_AT = F_A(E(B))$ satisfies the condition needed to apply the Theorem 1 to the RA-module F_AT. Moreover, $B = End_B(B) = End_{D^b(B)}(B) \simeq End_{D^b(A)}(EB) \simeq End_{D^b(A)}(T)$ since E is fully faithful. Hence, by Theorem 1, $B \simeq End_{D^b(A)}(T) \simeq \underline{End}_{RA}(F_AT)$ is gentle. ■

3. Defining gentle algebras group theoretically

Theorem 1 and the result of Karin Erdmann imply that the stable endomorphism ring of any module M over a block of a group ring KG with dihedral defect group is gentle, provided $Ext^1_{KG}(M, M) = 0$. Such a block has up to 3 simple modules. Let us consider in particular the alternating group $\mathfrak{A}_4$ of order 12 over an algebraically closed field K of characteristic 2. $K\mathfrak{A}_4$ is isomorphic to the following quiver algebra:



subject to the relations

$\alpha_i\alpha_{i+1} = \beta_i\beta_{i-1} = \alpha_i\beta_i - \beta_{i-1}\alpha_{i-1} = 0$ for all $i \in \mathbb{Z}/3\mathbb{Z}$

We denote by P_1, P_2 and P_3 the projective indecomposable $K\mathfrak{A}_4$-modules for K being a field containing $\mathbb{F}_4$. We call $S_i := P_i/rad(P_i)$. We have $rad(P_i)/soc(P_i) \simeq S_{i-1} \oplus S_{i+1}$ where we take $i \in \mathbb{Z}/3\mathbb{Z}$. The indecomposable representations over $K\mathfrak{A}_4$ are classified (see e.g. [5, II.7.4], [2, pp 192 ff]).
a) three indecomposable projective modules P_1, P_2, P_3,
b) n-th syzygies $\Omega^n(S_i)$ of the simple modules S_1, S_2, S_3 for $n \in \mathbb{Z}$,
c) certain modules $W_{n,\lambda}(S_i)$.,
d) modules $\Omega^n(S_i)/S_j$, for certain i and j and $n < 0$. The projective resolutions of these modules are periodic of period 3, and for any dimension $2d$, $d \in \mathbb{N}$, there are, up to isomorphism, six indecomposable modules $M_d(i)$ and $N_d(i)$ for $i \in \mathbb{Z}/3\mathbb{Z}$ of composition length $2d$ and with periodic resolution.

Lemma 3.1. $Ext^1_{K\mathfrak{A}_4}(W_{n,\lambda}(S_i), W_{n,\lambda}(S_i)) \neq 0$.

Proof. $W_{n,\lambda}(S_i) := V_{n,\lambda} \uparrow^{K\mathfrak{A}_4}_{K(C_2\times C_2)} \otimes_K S_i$ for a certain $K(C_2\times C_2)$-module $V_{n,\lambda}$ with $\Omega(V_{n,\lambda}) = V_{n,\lambda}$. We get $Ext^1_{K(C_2\times C_2)}(V_{n,\lambda}, V_{n,\lambda}) \neq 0$ and by Frobenius reciprocity $Ext^1_{K\mathfrak{A}_4}(W_{n,\lambda}(S_i), W_{n,\lambda}(S_i)) \neq 0$. ■

Lemma 3.2. *Let M be a $K\mathfrak{A}_4$-module with $Ext^1_{K\mathfrak{A}_4}(M, M) = 0$. If $\Omega^n(S_i) \oplus \Omega^m(S_j)$ is a direct factor of M, then $n = m$ and $i = j$.*

Proof. Since $K\mathfrak{A}_4$ is a symmetric algebra, $Ext^1_{K\mathfrak{A}_4}(\Omega^n S_i, \Omega^m S_j) \simeq \underline{Hom}_{K\mathfrak{A}_4}(\Omega^{n+1-m} S_i, S_j) \simeq \underline{Hom}_{K\mathfrak{A}_4}(\Omega^{n+1-m} S_1, S_{j-i+1})$. The question now is to determine when $\underline{Hom}_{K\mathfrak{A}_4}(\Omega^{n+1} S_1, S_j) = 0 = \underline{Hom}_{K\mathfrak{A}_4}(S_j, \Omega^{n-1} S_1)$ for $n \in \mathbb{Z}$ and $j \in \mathbb{Z}/3\mathbb{Z}$. But, since S_j is simple and $\Omega^m(S_1)$ is indecomposable non projective for any $m \in \mathbb{Z}$, one gets $\underline{Hom}_{K\mathfrak{A}_4}(\Omega^m S_1, S_j) = Hom_{K\mathfrak{A}_4}(\Omega^m S_1, S_j)$. But now, the first condition implies that n is non positive and the second implies that n is non negative. Hence, $Hom_{K\mathfrak{A}_4}(\Omega^{n+1} S_1, S_j) = 0 = \underline{Hom}_{K\mathfrak{A}_4}(S_j, \Omega^{n-1} S_1)$ if and only if $n = 0$ and $j = 1$. This implies the statement. ■

Lemma 3.3. *Let $M_n(i,j) = \Omega^{-n}(S_i)/S_j$ be an indecomposable module for $n \leq -1$. Then, $d \geq 3 \Rightarrow Ext^1_{K\mathfrak{A}_4}(M_n(i,j), M_n(i,j)) \neq 0$.*

Proof. The modules $M_n(i,j)$ are special string modules coming from walking in the quiver. So, up to a cyclic permutation of the indices, $M_n(i,j)$ is given by the following string: $1 \leftarrow 2 \rightarrow 3 \leftarrow 1 \rightarrow 2 \leftarrow 3 \; \cdots \; \text{-n} \rightarrow \text{2-n}$
The module $N_n(i,j)$ is given by turning the opposite direction of the quiver. Since $\Omega(M_n(i,j))$ is given by starting the string one sink later and ending it at one source later, so that $\Omega(M_n(i,j))$ is given by the following string: $2 \leftarrow 3 \rightarrow 1 \leftarrow 2 \rightarrow 3 \leftarrow 1 \; \cdots \; \text{-n} \leftarrow \text{-n+1}$ so that as soon as $n \geq 3$ the submodule given by the substring of the left $n-2$ arrows is identical to the substring of the $n-2$ right arrows of $M_n(i,j)$. Since at least one of the modules contains composition factors of the top of $M_n(i,j)$ and of $\Omega(M_n(i,j))$, the morphism induced by this identification cannot factor through a projective module. This proves the statement. ■

Summarizing we get

Corollary 3.4. *Let M be a $K\mathfrak{A}_4$-module with $Ext^1_{K\mathfrak{A}_4}(M, M) = 0$, then there is an $m \in \mathbb{Z}$ so that any indecomposable direct factor of $\Omega^m(M)$ is one of the following modules:*
1) indecomposable modules $M_{i,j} \in Ext^1_{K\mathfrak{A}_4}(S_i, S_j)$ where $i \neq j$ and $i, j \in \mathbb{Z}/3\mathbb{Z}$,
2) indecomposable modules $M_4(i)$ and $N_4(i)$, where we define $M(2) := \Omega^{-2}(S_1)/S_3$, $M(3) := \Omega(M(2))$ and $M(1) := \Omega(M(3))$ as well as $N(3) := \Omega^{-2}(S_1)/S_2$, $N(2) := \Omega(N(3))$ and $N(1) := \Omega(N(2))$,
3) simple modules S_i for $i \in \mathbb{Z}/3\mathbb{Z}$. Moreover, if $S_i \oplus S_j$ is a direct factor of M, then $i = j$.
4) $\Omega^\varepsilon(S_i)$ for $i \in \mathbb{Z}/3\mathbb{Z}$ and $\varepsilon \in \{+1, -1\}$. Moreover, if $\Omega^{\varepsilon_1}(S_i) \oplus \Omega^{\varepsilon_2}(S_j)$ is a direct factor of M, then $\varepsilon_1 = \varepsilon_2$ and $i = j$. Similarly, $\Omega^{\varepsilon_1}(S_i) \oplus S_j$ is not a direct factor of M for any $\varepsilon \in \{+1, -1\}$, i and j.

The stable endomorphism rings of any of these indecomposable modules is one-dimensional for modules of type 1,3,4, and the algebra of dual numbers $K[X]/X^2$ for type 2 modules.

3.1. Determining the possible M without a direct factor of type 2.

3.1.1. *Suppose M has two direct factors.* We see that $End_{K\mathfrak{A}_4}(M_{1,2} \oplus S_1) \simeq A_1$ where A_1 is the quiver algebra of $\bullet \longrightarrow \bullet$, and $Ext^1_{K\mathfrak{A}_4}(M_{1,2} \oplus S_1, M_{1,2} \oplus S_1) = 0$. By direct inspection which we leave to the reader, none of the other indecomposable gentle algebras with two vertices is the stable endomorphism ring of a module M with $Ext^1_{K\mathfrak{A}_4}(M, M) = 0$, and without a direct factor of type 2.

3.1.2. *Suppose M has three direct factors.* A module of type 3 and a module of type 4 cannot be simultaneously a direct factor of M. So, applying the syzygy functor if necessary, we may assume that no module of type 4 is a direct factor of M.

Lemma 3.5. *There is an integer m and $j \in \mathbb{Z}/3\mathbb{Z}$ so that $S_j \otimes_K \Omega^m(M) \simeq S_1 \oplus M_{i,j} \oplus M_{k,l}$ for $(i,j,k,l) \in \{(1,2,1,3),(1,2,2,1),(1,3,3,1),(2,1,3,1)\}$. Any of these modules M has non self-extensions and the stable endomorphism algebra of any of these modules are $\bullet \longrightarrow \bullet \longrightarrow \bullet$ or $\bullet \longleftarrow \bullet \longrightarrow \bullet$ or $\bullet \longrightarrow \bullet \longleftarrow \bullet$. Any of these algebras occur as stable endomorphism algebras.*

Proof. If no module of type 3 is a direct summand of M, we have to deal with M being the sum of three modules of type 1. Since the syzygy functor has two orbits of length 3 on the modules of type 1, if M is a direct sum of three modules of type 1, at least two of them belong to the same orbit under the syzygy functor. But then, the sum of these two has a non trivial Ext^1 since there cannot be four non-isomorphic modules in a single orbit under the syzygy functor. Hence, this case is excluded. Since $Ext^1_{K\mathfrak{A}_4}(S_i,S_j) \neq 0$ for $i \neq j$, we only need to consider the case $M = S_1 \oplus M_{i,j} \oplus M_{k,l}$. Since $Ext^1_{K\mathfrak{A}_4}(S_1,M_{2,3}) \neq 0 \neq Ext^1_{K\mathfrak{A}_4}(S_1,M_{3,2})$, we only have to consider four cases: $(i,j,k,l) \in \{(1,2,1,3),(1,2,2,1),(1,3,3,1),(2,1,3,1)\}$, since $(1,2,3,1)$ and $(1,3,2,1)$ leads to a non trivial $Ext^1(M,M)$. Now, the stable endomorphism algebra of $S_1 \oplus M_{1,2} \oplus M_{1,3}$ is $\bullet \longrightarrow \bullet \longleftarrow \bullet$, the stable endomorphism algebra of $S_1 \oplus M_{2,1} \oplus M_{3,1}$ is $\bullet \longleftarrow \bullet \longrightarrow \bullet$, and the stable endomorphism algebra of $M_{1,3} \oplus M_{3,1} \oplus S_1$ (or $M_{1,2} \oplus M_{2,1} \oplus S_1$) is $\bullet \longrightarrow \bullet \longrightarrow \bullet$ We leave the (tedious) verification to the reader that indeed these modules M have $Ext^1(M,M)=0$. ■

3.1.3. *Suppose M has four direct factors.* Since if M has four direct summands, it also has three direct summands, and we need to add another module $M_{x,y}$ to $S_1 \oplus M_{i,j} \oplus M_{k,l}$ for $(i,j,k,l) \in \{(1,2,1,3),(1,2,2,1),(1,3,3,1),(2,1,3,1)\}$. It follows that $(x,y) \in \{(1,2),(2,1),(3,1),(1,3)\}$ and any of these choices leads to a non trivial self-extension. We proved

Lemma 3.6. *If M is a $K\mathfrak{A}_4$-module with $Ext^1(M,M)=0$ and with four pairwise non isomorphic direct factors, then a module of type 2 is a direct factor of M.*

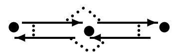
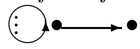
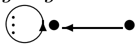
3.2. **Determining the possible M with a direct factor of type 2.** Since $\underline{Hom}_{K\mathfrak{A}_4}(N(1),N(3)) \neq 0$ at most one of the modules $N(i)$ can be a direct factor of M, in order to get $Ext^1_{K\mathfrak{A}_4}(M,M)=0$. Similarly at most one of the modules $M(i)$ can be a direct factor of M. Nevertheless, there may be $N(i) \oplus M(j)$ being a direct factor of M. We basically have to distinguish the cases $i=j$ and $i \neq j$.

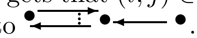
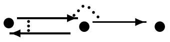
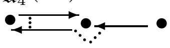
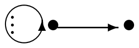
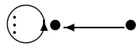
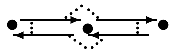
3.2.1. *Suppose M has two direct factors.* If a module of type 4 and no module of type 3 is a direct factor of M, by applying the syzygy functor if necessary, we may reduce to the case that no module of type 4 but a module of type 3 is a direct factor of M.

Lemma 3.7. *Let M be a module with direct factor $M(i)$ or $N(i)$ for an $i \in \mathbb{Z}/3\mathbb{Z}$ and so that $Ext^1(M,M)=0$, then $\underline{End}_{K\mathfrak{A}_4}(M)$ is one of the following algebras:*
$K[X]/X^2 \times K[X]/X^2$, $K \times K[X]/X^2$, $\bullet \rightleftarrows \bullet$, $\circlearrowleft\bullet \longrightarrow \bullet$, $\circlearrowleft\bullet \longleftarrow \bullet$.

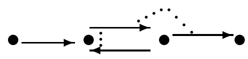
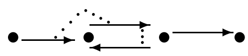
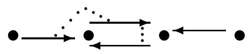
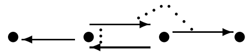
Proof. Now, $\underline{End}_{K\mathfrak{A}_4}(M(i) \oplus N(j)) \simeq K[X]/X^2 \times K[X]/X^2$. Moreover, $Ext^1_{K\mathfrak{A}_4}(S_j \oplus M(i), S_j \oplus M(i)) \neq 0$ unless $i=j$, and likewise for $N(i)$ instead of $M(i)$. Now, $\underline{End}_{K\mathfrak{A}_4}(M(1) \oplus S_1)$ is the quiver algebra of the quiver $\bullet \rightleftarrows \bullet$. By duality we get the same result for $\underline{End}_{K\mathfrak{A}_4}(N(1) \oplus S_1)$. We have to look at the stable endomorphism ring of a module of type $M(i) \oplus M_{l,j}$. By duality we get the same results for $N(i) \oplus M_{l,j}$. Now, we may assume that $i=1$ and then only $M = M(1) \oplus M_{3,2}$ has non trivial $Ext^1(M,M)$. Moreover, $\underline{End}_{K\mathfrak{A}_4}(M(1) \oplus M_{l,j}) \simeq K \times K[X]/X^2$ for $(l,j) \in \{(1,2),(3,1),(2,3)\}$. Now, $\underline{End}_{K\mathfrak{A}_4}(M(1) \oplus M_{2,1})$ is isomorphic to the quiver algebra $\circlearrowleft\bullet \longrightarrow \bullet$ and $\underline{End}_{K\mathfrak{A}_4}(M(1) \oplus M_{1,3})$ is isomorphic to the quiver algebra $\circlearrowleft\bullet \longleftarrow \bullet$. ■

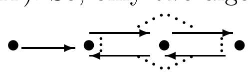
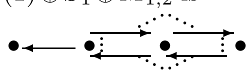
3.2.2. *Suppose M has three direct factors.* Replacing M by $\Omega^m(M)$ if necessary we may assume that the only direct factors of M are of type 1, 2 and 3.

Lemma 3.8. *Let M be a direct sum of three modules of type 1,2 and 3. Then, $\underline{End}_{K\mathfrak{A}_4}(M)$ is one of the following algebras: $\bullet \rightleftarrows \bullet \longleftarrow \bullet$, $\bullet \rightleftarrows \bullet \longrightarrow \bullet$, $\bullet \rightleftarrows \bullet \longleftarrow \bullet$, $\bullet \rightleftarrows \bullet \longrightarrow \bullet$. If M is a direct sum of one module of type 2 and two modules of type 1, then $\underline{End}_{K\mathfrak{A}_4}(M)$ decomposes as $K \times A$ where A is Morita equivalent to either $\circlearrowleft\bullet \longrightarrow \bullet$ or $\circlearrowleft\bullet \longleftarrow \bullet$. If M has at least two direct summands* *of type 2, then* $\underline{End}_{K\mathfrak{A}_4}(M)$ *is Morita equivalent to one of the following algebras:*  *or* $K[X]/X^2 \times A$ *where* A *is Morita equivalent to either*  *or* .

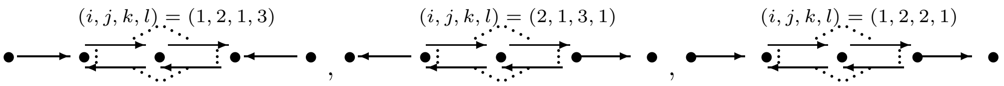
Again, for M being a sum of a module of type 2, type 3 and type 1, we get $M = N(1) \oplus S_1 \oplus M_{i,j}$ up to syzygy, tensor product with a simple module and duality. Examining the cases one gets that $(i,j) \in \{(1,2),(2,1),(1,3),(3,1)\}$. Choosing $(i,j) = (1,2)$ we get $\underline{End}_{K\mathfrak{A}_4}(M)$ is isomorphic to . If $(i,j) = (2,1)$, then $\underline{End}_{K\mathfrak{A}_4}(M)$ is isomorphic to , and finally $(i,j) = (1,3)$ yields $\underline{End}_{K\mathfrak{A}_4}(M)$ is isomorphic to , as well as $(i,j) = (3,1)$ gives $\underline{End}_{K\mathfrak{A}_4}(M)$ is isomorphic to . We still have to study the case of one module of type 2 and two modules of type 1. But there, again one may assume that $N(1)$ is a direct factor of M. We get 6 combinations of two modules of type 1, namely $(M_{1,2}, M_{2,1})$, $(M_{1,2}, M_{1,3})$, $(M_{1,2}, M_{3,2})$, $(M_{3,1}, M_{2,1})$, $(M_{1,3}, M_{3,1})$ and $(M_{3,2}, M_{3,1})$, the other possibilities inducing self-extensions. Any of the endomorphism rings decomposes into $K \times A$ where A is one of the two endomorphism rings  or . Finally, we have to study the case of two modules of type 2 and another module of type 1, 3 or 4. Here the two modules of type 2 are necessarily $M(i) \oplus N(j)$, otherwise one would get a non vanishing Ext^1. We first study the case $i = j$, where we may assume $i = 1$. Then, the only possible module of type 3 is S_1. Then, $\underline{End}_{K\mathfrak{A}_4}(M(1) \oplus N(1) \oplus S_1)$ is the quiver algebra for the quiver . Applying the syzygy functor if necessary and changing i to another index, the analogous argument yields the same endomorphism algebra for any of the modules of type 4. We have to consider the case of one module of type 1 as direct summand of M. Then, as it is easily seen, the stable endomorphism algebra decomposes as a direct product of $K[X]/X^2$ and one of the algebras A with one loop and one arrow, as above. In case $i \neq j$, no simple direct module can be a direct summand of M, otherwise there would be a non trivial $Ext^1(M,M)$. Applying the syzygy functor, the same is true for a direct factor of type 4. We are left with direct factors of type 3. There, by direct inspection the same algebras occur as for $M = M(1) \oplus N(1) \oplus M_{i,j}$. We leave to the reader the tedious, but straight forward verification that for any of the given M one has $Ext^1_{K\mathfrak{A}_4}(M,M) = 0$. This gives the statement. ■

3.2.3. *Suppose* M *has four direct factors.* Suppose first that there is only one direct factor of M of type 2. Then, the other three factors have to occur in the above list of modules M without a factor of type 2. All of them have a direct factor S_1. So, the only possibility for the module of type 2 is $M(1)$ or $N(1)$. We get the following result.

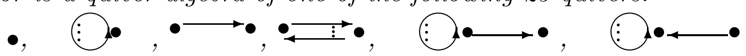
M	$\underline{End}_{K\mathfrak{A}_4}(M)$
$(N(1) \oplus S_1 \oplus M_{1,2} \oplus M_{2,1})$ or $(N(1) \oplus S_1 \oplus M_{1,3} \oplus M_{3,1})$	
$(M(1) \oplus S_1 \oplus M_{1,2} \oplus M_{2,1})$ or $(M(1) \oplus S_1 \oplus M_{1,3} \oplus M_{3,1})$	
$(N(1) \oplus S_1 \oplus M_{1,2} \oplus M_{1,3})$ or $(M(1) \oplus S_1 \oplus M_{1,2} \oplus M_{1,3})$	
$(N(1) \oplus S_1 \oplus M_{2,1} \oplus M_{3,1})$ or $(M(1) \oplus S_1 \oplus M_{2,1} \oplus M_{3,1})$	

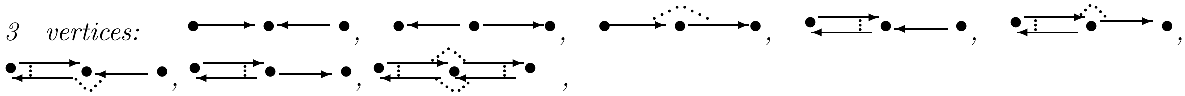
We need to deal with the case of two direct factors $M(i) \oplus N(j)$ being a direct summand of M. We have to distinguish the cases $i = j$ and $i \neq j$. If $i = j$, we may suppose $i = 1$. Then, if we have two additional summands of M of type 1, we get in all cases that the stable endomorphism algebra decomposes into a direct product of two two-vertex algebras with one loop and one arrow, as displayed in the case occuring above. All four possibilities occur. The other possibility is that $M = M(1) \oplus N(1) \oplus S_1 \oplus M_{i,j}$. Then, $(i,j) \in \{(1,2),(1,3),(2,1),(3,1)\}$ to get $Ext^1(M,M) = 0$. Moreover, we see that $M(1) \oplus N(1) \oplus S_1 \oplus M_{i,j} = Hom_K(M(1) \oplus N(1) \oplus S_1 \oplus M_{j,i}, K)$. So, only two algebras actually occur. The stable endomorphism ring of $M(1) \oplus N(1) \oplus S_1 \oplus M_{1,2}$ is  and the stable endomorphism ring of $M(1) \oplus N(1) \oplus S_1 \oplus M_{2,1}$ is . Now, if $M(1) \oplus N(i)$ for $i \neq 1$ is a direct factor of M, then the other two direct factors can only be of type 1. We observe that the stable endomorphism rings are as in the case of $i = j$ and two supplementary factors of type 1.

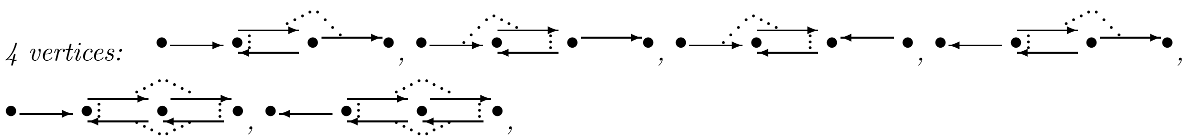
3.2.4. *Suppose M has five direct factors.* The case of five direct summands of M is the last we have to consider. In fact, there has to be a direct factor $M(j) \oplus N(i)$. Indeed the case of four direct factors of M occurs only with either two direct factors of type 2, or two direct factors of type 1, one of type 2 a simple module, and three direct factors of type 1 in M cannot occur. Moreover, $i = j$ in this case since otherwise we have to have three summands of type 1, which implies that $Ext^1(M, M) \neq 0$. Therefore, we may assume that $i = 1$, and that $M = S_1 \oplus N(1) \oplus M(1) \oplus M_{i,j} \oplus M_{k,l}$. Again $(i, j, k, l) \in \{(1, 2, 1, 3), (1, 2, 2, 1), (1, 3, 3, 1), (2, 1, 3, 1)\}$. There the cases $(1, 2, 2, 1)$ and $(1, 3, 3, 1)$ are analogous, whereas the cases $(1, 2, 1, 3)$ and $(2, 1, 3, 1)$ are dual. So, at most three endomorphism algebras occur.

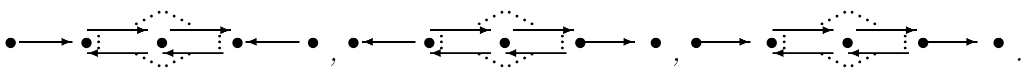
We get 

Theorem 2. *Let k be a field of characteristic 2 with at least 4 elements, and let $\mathfrak{A}_4$ be the alternating group of order 12. Let M be a $K\mathfrak{A}_4$-module with $Ext^1_{K\mathfrak{A}_4}(M, M) = 0$. Then, M has at most 5 non isomorphic direct factors, $\underline{End}_{K\mathfrak{A}_4}(M)$ has at most two indecomposable direct factors, and any indecomposable factor is a quiver algebra of one of the following 23 quivers:*

1 or 2 vertices: 

3 vertices: 

4 vertices: 

5 vertices: 

Moreover, each of these quivers occurs.

Remark. The next step should be to deal with dihedral defect blocks in general, give a group theoretical interpretation of the gentle algebras which occur this way, and maybe more interestingly those which do not occur this way.

References

[1] *I. Assem, A. Skowroński*, Iterated tilted algebras of type $\tilde{\mathbb{A}}_n$, Math. Z. **195** (1987), 269–290.
[2] *D. Benson*, Modular Representation Theory: New trends and methods, SLNM 1081 (1984).
[3] *I.N.Bernstein, I.M.Gelfand and S.I.Gelfand,* Algebraic bundles over $\mathbb{P}^n$ and problems of linear algebra, Funct. Anal. and Appl. **12** (1978) 212-214.
[4] *W. Crawley-Boevey*, Maps between representations of zero-relation algebras, J. Algebra **126** (1989), 259–263.
[5] *K. Erdmann,* Blocks of tame representation type and related algebras, SLNM 1428 (1990).
[6] *D. Happel,* Triangulated categories in the representation theory of finite dimensional algebras, London Math. Soc. Lecture Notes 119 (1988).
[7] *S. König and A. Zimmermann*, Derived equivalences for group rings, SLNM 1685 (1998).
[8] *C.M. Ringel*, The repetitive algebra of a gentle algebra, Bol. Soc. Mat. Mexicana (3) **3** (1997) 235-253.
[9] *J. Schröer*, On the quiver with relations of a repetitive algebra, Arch. Math. (Basel) **72** (1999), 426–432.
[10] *J. Schröer and A. Zimmermann*, Stable endomorphism algebras of modules over special biserial algebras, to appear in Math Z.
[11] *A. Skowroński and J. Waschbüsch*, Representation-finite biserial algebras, J. reine und angewannte Mathematik **345** (1983) 172-181.

Université de Picardie, Faculté de Mathématiques et UMR 6140 du CNRS, 33 rue St Leu, F-80039 Amiens Cedex 1, France
E-mail address: alexander.zimmermann@u-picardie.fr
URL: http://www.mathinfo.u-picardie.fr/alex/azim.html