# Braid groups as self-equivalences of derived categories

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## 1. Introduction

Derived categories of algebras become more and more the appropriate working place of representation theorists. In the representation theory of groups this interest arose from various places notably from Broué's conjecture which states in its simplest form that for a big enough field $k$ of characteristic $p$ the derived categories of the principal block of the group ring $kG$ of the finite group $G$ and that of the principal block of the group ring $kN_G(P)$ of the normalizer $N_G(P)$ of an abelian $p$-Sylow subgroup $P$ of $G$ are equivalent as triangulated categories. Though this is our main motivation at the moment, there are more situations which we would like to recall. There is the development of character sheaves which was pushed very far by Lusztig, Shoji and others to describe generically the representation theory of algebraic groups. Moreover, many relationships between commutative or non commutative geometries to the representation theory of algebras has been discovered. This relationship is made precise and can be formulated by an equivalence between various corresponding derived categories.
In these notes we shall be concerned with consequences of the established or conjectured equivalences. In particular we work on symmetries of the representation theory which are implied by the existence of the derived category in the background. To be more precise for $R$ a commutative ring we shall study the group of self-equivalences of the derived category of an $R$-algebra which is projective as an $R$-module. This group often is quite rich as we established in joint work with Raphaël Rouquier in (RZ-96). In fact, braid groups seems to play an important rôle in the structure of these groups. Parallel developments by Khovanov, Lenzing, Meltzer, Miyachi, Polishchuk, Seidel, Thomas, Yekutieli and others also indicate in parts that for nice situations braid group symmetries arise.
In the present note we shall review some of these developments. Furthermore we shall compute the case of the principal block of the group ring of a group with Klein Four Sylow subgroup. There, a generalized braid group of type $\tilde{A}_2$ occurs. Moreover, we explain how these self-equivalences of the derived category of a principal block of a group ring $RG$ act on the cohomology ring $H^*(G,R)$ in a functorial way. Furthermore, we study this action for the case of the Klein Four Sylow subgroup, and in particular for the alternating group $\mathfrak{A}_4$.

## 2. Generalities on equivalences between derived categories

We recall some facts about equivalences between derived categories as is developed in Rickard (Ric-89; Ric-91; Ric-96) and Keller (Ke-93). For the notations we refer to (KZ-98).

### 2.1. The algebraic setup

Jeremy Rickard proved in his fundamental work a necessary and sufficient criterion for two derived categories of rings to be equivalent. In the situation we study here this criterion is as follows.

THEOREM 1 (Jeremy Rickard (Ric-91), Bernhard Keller (Ke-93)). *Let $S$ be a commutative ring and let $A$ be a finite dimensional $S$-algebra which is projective as $S$-module. Then, in case $B$ is another*

* These notes are an extended version of the talk I gave at the Euroconference "Rings, Modules and Representations" in Constanţa in August 2000. I acknowledge support by the CCCI.

*$S$-algebra and $D^b(A) \simeq D^b(B)$ as triangulated categories, there is a complex $X \in D^b(B \otimes_S A^{op})$ so that $F_X := X \otimes^{\mathbb{L}}_A -$ is a functor inducing an equivalence.*

We call functors *standard* if they are of the form $F_X$ and if they induce an equivalence $D^b(A) \simeq D^b(B)$. In case $F_X$ is a standard equivalence, we call $X$ a *twosided tilting complex*.

**(2.1)** Recall that an $S$-algebra $A$ for an integral domain $S$ with field of fractions $K$ is an $S$-*order* if first $A$ is finitely generated projective as $S$-module and second $K \otimes_S A$ is a semisimple $K$-algebra. An $S$-algebra $A$ is symmetric provided $Hom_S(A, S) \simeq A$ as $A \otimes_S A^{op}$-modules.
Suppose now $S$ is a field and $A$ an arbitrary $S$-algebra or $S$ is a Dedekind domain and $A$ is a symmetric $S$-order. Then if $D^b(A) \simeq D^b(B)$, also $B$ is projective as $S$-module (Z-99a) and a quasi-inverse of $F_X$ is standard as well.

**(2.2)** We keep the assumption that $A$ is an $S$-projective $S$-algebra. We define (RZ-96)

$$TrPic_S(A) := \{ \text{ isomorphism classes } [X] \mid X \in D^b(A \otimes_S A^{op}) \text{ is a twosided tilting complex} \}.$$

Then, $TrPic_S(A)$ is a group under the group law $[X] \cdot [Y] := [X \otimes^{\mathbb{L}}_A Y]$. This group contains the classical Picard group of $A$ (cf (Ba-68)), with it also the group of automorphisms of $A$ modulo inner automorphisms of $A$, and an infinite cyclic central subgroup generated by shift in degrees.

### 2.2. THE GEOMETRIC SETUP

There is a geometric analogue to this construction. Let $X$ and $Y$ be algebraic varieties of the same dimension and let $D^b(X)$ be the derived category of bounded complexes of coherent sheaves on $X$ and likewise $D^b(Y)$ be the derived category of bounded complexes of coherent sheaves on $Y$.

**(2.3)** Let $P$ be a bounded complex of coherent sheaves on $X \times Y$. Denote by $p_X$ the projection of $X \times Y$ on $X$ and by $p_Y$ the projection of $X \times Y$ on $Y$. Then, in case

$$F_P(-) := (p_X)_*(P \otimes^{\mathbb{L}}_{\mathcal{O}_{X\times Y}} (p_Y)^* -) : D^b(Y) \longrightarrow D^b(X)$$

is an equivalence of categories, this equivalence is called ((M-81; Brid-99; P-95)) *Fourier-Mukai transformation*. If $X$ is a point, then a Fourier-Mukai transformation is a standard equivalence.

**(2.4)** Tom Bridgeland gave in (Brid-99) some partial analogue to Rickard's theorem. For this suppose that $X$ and $Y$ are smooth projective varieties of the same dimension and that $P$ is a vector bundle on $X \times Y$. The bundle $P$ is called strongly simple over $Y$ if for each $y \in Y$ the bundle $P_y$ on $X$ is simple and if $Ext^i_X(P_{y_1}, P_{y_2}) = 0$ for any $(y_1, y_2) \in Y \times Y$ and any $i$.

THEOREM 2 (Tom Bridgeland (Brid-99)). *The functor $F_P$ is fully faithful if and only if $P$ is strongly simple over $Y$. The functor $F_P$ is an equivalence precisely when one also has $P_y \simeq P_y \otimes \omega_X$ for all $y \in Y$.*

Theorem 2 is some sort of analogue to the theorem of Rickard and Keller in case the twosided tilting complex is just a module. As far as I know there is no general theory like Rickard's for the geometric situation.

## 3. Some results on braid groups

We recall briefly some facts about braid groups.

**(3.1)** Let $W$ be a real reflection group generated by orthogonal reflections $\{s_1, \ldots, s_n\} =: S$ on some hyperplanes $\{H_1, \ldots H_n\}$ in some real euclidean space $E$. We associate to $(W, S)$ a graph $\Gamma_{(W,S)}$. The vertices of $\Gamma_{(W,S)}$ are the elements of $S$. There is an edge with weight $m_{i,j}$ between $s_i$ and $s_j$ if and only if $s_i$ and $s_j$ do not commute and the order of $s_i s_j$ is $m_{i,j} + 2$.
It is well known that $W$ is finite if and only if $\Gamma_{(W,S)}$ is a Dynkin diagram.

**(3.2)** Now, $W$ acts on the arrangement of hyperplanes and also on the complement. It is a result of Brieskorn (Brie-71) that $\pi_1((\mathbb{C}\otimes_{\mathbb{R}} E \setminus \bigcup_{i=1}^n \mathbb{C}\otimes_{\mathbb{R}} H_i)/W) =: Braid(\Gamma_{(W,S)})$ is a group with presentation $Braid(\Gamma_{(W,S)}) =< t_1,\ldots,t_n| \underbrace{t_it_jt_i\ldots}_{m_{i,j}+2 \text{ factors}} = \underbrace{t_jt_it_j\ldots}_{m_{i,j}+2 \text{ factors}} >$ . The case of Dynkin diagram $\Gamma_{(W,S)} = A_e$ is due to Arnold while the other cases are due to Brieskorn.

**(3.3)** This item was communicated to me by Claudio Procesi. We shall be concerned in particular with the braid group associated to the Dynkin diagram $A_2$ (which is just a graph with one edge and 2 vertices) and the affine Dynkin diagram $\tilde{A}_2$ (which is the unique graph with three edges and three vertices, each vertex being adjacent to two edges).
Then $Br(A_2) =< s,t|\ sts = tst >$ and $Br(\tilde{A}_2) =< s_1,s_2,s_3|\ s_is_js_i = s_js_is_j \forall\ i,j\in\{1,2,3\} >$ . The mappings $Br(A_2) \stackrel{\alpha}{\to} Br(\tilde{A}_2)$ defined by $\alpha(s) = s_1$ and $\alpha(t) = s_2$ as well as the mapping $Br(\tilde{A}_2) \stackrel{\beta}{\longrightarrow} Br(A_2)$ given by $\beta(s_1) = s$, $\beta(s_2) = t$ and $\beta(s_3) = t$ are group homomorphisms. We see immediately $\beta\circ\alpha = id_{Br(A_2)}$. Hence, $Br(\tilde{A}_2) = ker\beta \rtimes Br(A_2)$ .

**(3.4)** Independently of the existence of a Weyl group we define for any weighted graph $\Gamma$ a generalized braid group $Braid(\Gamma)$. The case of complex reflection groups $(W,S)$ is studied by Broué, Malle, Rouquier (BMR-98) and even more involved types of diagrams and associated braid groups have been introduced.

## 4. Some results on self-equivalences of derived categories

### 4.1. THE ABELIAN SITUATION

The group $TrPic_S(A)$ was first studied in (RZ-96) (see also (Z-96)). As one of the main results we got the following description.

THEOREM 3 (Raphaël Rouquier and Alexander Zimmermann (RZ-96)). *Let $k$ be a prime field and let $B_e$ be an indecomposable Brauer tree algebra with no exceptional vertex for a Brauer tree with $e$ edges. Then, there is a group homomorphism $\varphi_e : Braid(A_e) \longrightarrow TrPic_k(B_e)$ . Moreover, $\varphi_2$ is injective and the image of $\varphi_2$ is normal of index* 8*. For any $e > 1$ the image of a standard generator is infinite cyclic.*

**(4.1)** Injectivity is proven by identifying $Braid(A_2)/Z(Braid(A_2)) \simeq PSL_2(\mathbb{Z})$ and one then identifies the images of projective indecomposable modules with the matrix coefficients of elements in $PSL_2(\mathbb{Z})$. Actually, let $P_+$ and $P_-$ be the two indecomposable projective $B_2$-modules. For $[X] \in TrPic_k(B_2)$ let $T_+ := X\otimes_{B_2} P_+$ and $T_- := X\otimes_{B_2} P_-$. Replace $T_+$ by the representative in the isomorphism class of $T_+$ with the smallest $k$-dimension and likewise for $T_-$. Then, $T_+ = P_+^{n_+}\oplus P_-^{n_-}$ and $T_- = P_+^{m_+}\oplus P_-^{m_-}$ as modules, forgetting the differentials. Then one proves that the image of $X$ in $PSL_2(\mathbb{Z})$ is the matrix $\begin{pmatrix} \pm n_+ & \pm m_+ \\ n_- & m_- \end{pmatrix}$. This description immediately gives the statement.
Surjectivity is proven by using that there are only *two* simple modules up to isomorphism and any complex has *two* ends. For more ample details the reader should consult (RZ-96).

**(4.2)** There is some analogue for orders over a complete discrete valuation domain. There, so-called Green orders take the rôle of Brauer tree algebras and one gets the same situation but replacing $B_e$ by the corresponding order $\Lambda_e$. The interest is that tensoring with the field of fractions gives a very different situation to tensoring obtained by tensoring with the residue field. Some nice arithmetic structure comes into the play. For more ample details the reader might consult (Z-00a).

**(4.3)** Later, independently, Mike Khovanov and Paul Seidel discovered this homomorphism $\varphi_e$.

THEOREM 4 (M. Khovanov and P. Seidel (KS-00; ST-00; HK-00)). *$\varphi_e$ is injective for all $e > 2$.*

A more general class of algebras, so-called zig-zag algebras, is covered. The proof is very involved and uses sophisticated methods related to Mirror symmetry. In particular one of the key steps is to establish an equivalence of categories between the derived category of the algebra under consideration and some object coming from symplectic geometry. Using Floer cohomology the authors are able to prove injectivity.

**(4.4)** In the geometrical setting we get some braid group picture as well.

THEOREM 5 (Alexander Polishchuk (P-95)). *Let $A$ be a connected abelian variety of dimension $g$ over a field $k$ endowed with a principal symmetric polarisation. Then, $Braid(A_2)$ acts on $D^b(A)$ by Fourier-Mukai transformations.*

The symmetric polarisation is used to identify the variety with its dual. The construction goes along the following lines. One finds some sort of involution and a complex inducing a Fourier-Mukai transformation. Conjugating the complex by the involution one gets a second complex and these two complexes are the braid group generators.

There is one more example in this direction using Fourier-Mukai transformation actions on derived categories.

THEOREM 6 (Raphaël Rouquier (Rou-98)). *Let $G$ be a semisimple complex algebraic group with Weyl group $W$ and let $\mathcal{B}$ be the variety of Borel subgroups of $G$. Then, the braid group $B_W$ associated to $W$ acts on $D^b(\mathcal{B})$ by Fourier-Mukai transformations.*

## 4.2. NON ABELIAN STATEMENTS

**(4.5)** Amnon Yekutieli is interested in non communicative geometry and was interested for this reason in algebras with finite global dimension.

THEOREM 7 (Jun-Ichi Miyachi and Amnon Yekutieli (MY-99)). *Let $A$ be a hereditary basic finite dimensional $k$-algebra of finite representation type with quiver $\Delta$. Then, there is a group isomorphism $TrPic_k(A) \simeq Aut_k(\overrightarrow{\mathbb{Z}}\Delta)^\tau$.*

Here, $\tau = A^*[-1] \otimes^{\mathbb{L}}_A -$ and $\overrightarrow{\mathbb{Z}}\Delta$ is a quiver obtained by repetition of $\Delta$ (Ried-80). By this, Miyachi and Yekutieli get explicitly the case of $\Delta$ being a Dynkin diagram. No braid group arises, though the groups coming up are all of a nice algebraic nature.

**(4.6)** Helmut Lenzing and Hagen Meltzer study all kinds of properties for a rather special class of algebras, so-called canonical algebras. These are algebras of finite global dimension and are defined by quivers and relations. Moreover, these algebras can serve as a model for a 'weighted projective line'. For details the reader may consult Geigle and Lenzing (GL-87). Recently these algebras came into play by the attempt of Happel and Reiten to classify all abelian categories which are derived equivalent to a hereditary one. Canonical algebras play an important rôle in the list of categories they expect to get. Here is one of the main results on the group of self-equivalences of the derived category of these algebras.

THEOREM 8 (Meltzer and Lenzing (LM-00)). *Let $X$ be a weighted projective line of tubular type. Then the group of isomorphism classes of self-equivalences of $D^b(X)$ is isomorphic to the group $Braid(A_2) \rtimes (Pic_0(X) \rtimes AutX)$.*

For details on the definitions we refer to (GL-87). It is interesting to see that also in this case the generators for the braid group part are analogous to the case studied in (RZ-96) for Brauer tree algebras with two simple modules.

**(4.7)** It is surprising to get the braid group also in this non abelian situation. It seems that this happens since the assumption of the weighted projective line to be tubular has some regularizing effect, and so the situation becomes "almost abelian". In fact for general types the braid groups do not arise and actually the derived category in this case has only relatively few symmetries.

## 5. Action on the cohomology ring

Let $G$ be a finite group and let $R$ and be a complete discrete valuation ring of characteristic $0$ with residue field of characteristic $p$. In fact in order to establish the theory that follows the assumptions are much too strict on $R$ but the hypotheses on $R$ as above simplify a lot the presentation. Let $S$ be any commutative noetherian ring.

**(5.1)** In order to study the group $TrPic_R(B_0(RG))$ we shall look for natural modules. Since for any two $SG$-modules $M$ and $N$ we have

$$Ext^n_{SG}(N,M) \simeq Hom_{D^b(SG)}(N,M[n])$$

it is natural to consider $M = N$ and to look at the subgroup of $TrPic_S(SG)$ that fixes the module $M$. Then, morally at least, this group should act on $Ext^*_{SG}(M,M)$ in a multiplicative way. Since $H^*(G,S) = Ext^*(S,S)$ is the cohomology ring of the group $G$ with values in $S$, we are particularly interested in the case $M = S$. Denote

$$HD_S(G) := \{[X] \in TrPic_S(SG) \mid X \otimes_{SG} S \simeq S\}$$

It is reasonable to restrict in case of $S = R$ to the derived category of the principal block. We shall do this without mentioning and hope that this will not produce any confusion.
To get an action of $HD_S(G)$ on $H^*(G,S)$ there are many obstacles on the way. E.g. there are many isomorphisms we choose on the way, the functors are equivalences and not isomorphisms of categories to cite only a few. But, we get the following first result.

THEOREM 9 ((Z-99b)). *Keep the assumptions of this section. Then, the cohomology ring $H^*(G,S)$ is an $S$ $HD_S(G)$-module. This action is compatible with the ring structure of $H^*(G,S)$.*
*If $S_1 \longrightarrow S_2$ is a ring homomorphism, then this ring homomorphism induces a group homomorphism $HD_{S_1}(G) \longrightarrow HD_{S_2}(G)$ and this way the ring homomorphism $H^*(G,S_1) \longrightarrow H^*(G,S_2)$ is $S_1$ $HD_{S_1}(G)$-linear.*

The proof is not only a checking of being well defined, but an actual construction of a bigger group acting properly, and then seeing that the extension that had to be considered actually enlarges the group by something in the kernel of the action.

**(5.2)** What do we know with respect to functoriality of this structure in the first variable? Here we have to face the problem that derived equivalences behave very badly with respect to subrings, quotient rings, extension rings or similar constructions. There is one exception, the so-called splendid tilting complexes introduced by Jeremy Rickard (Ric-96).

DEFINITION 5.1. **(Jeremy Rickard (Ric-96))** Let $G$ and $H$ be finite groups with a common $p$-Sylow subgroup $P$ and let $R$ be a complete discrete valuation domain of characteristic $0$ with residue field of characteristic $p$ or a field of characteristic $p$. Denote by $\Delta P$ a fixed diagonal embedding of $P$ in $G \times H$. Let $X$ be a bounded complex of $B_0(RG) \otimes_R B_0(RH)^{op}$ modules so that

- Each homogeneous component of $X$ is projective as $RG$-module and projective as $RH$-module,
- Each homogeneous component of $X$ is a $\Delta P$-projective $p$-permutation module (that is a direct factor of a permutation module),

– $Hom_{RG}(X,X)$ is *homotopic* to $B_0(RH)$ as complex of bimodules and $Hom_{RH}(X,X)$ is *homotopic* to $B_0(RG)$ as complex of bimodules.

**(5.3)** I do not know any twosided tilting complex between two principal blocks of group rings which cannot be modified by a Morita equivalence so that the modified complex is splendid.

**(5.4)** For splendid tilting complexes Rickard proves that the Brauer construction induces equivalences on the local levels. We shall explain this statement below.
Let $G$ be a finite group and let $k$ be a field of characteristic $p$. Then, for any $kG$-module $M$ and a $p$-subgroup $U$ of $G$ define the $kN_G(U)$-module $M(U) := M^U / \sum_{V<U} Tr_V^U(M^V)$. This construction, the Brauer construction, is functorial and carries over to the homotopy category. It is the right environment to restrict splendid derived equivalences to centralizers of subgroups.

THEOREM 10 (Jeremy Rickard (Ric-96)). *Let $X$ be a splendid tilting complex in $D^b(B_0(kG) \otimes_k B_0(kH))$. Then, for any $p$-subgroup $Q$ of $G$ the complex $X(\Delta Q)$ is a splendid tilting complex in $D^b(B_0(kC_G(Q)) \otimes_k B_0(kC_H(Q)))$.*

Moreover, Rickard proves that for any splendid tilting complex $\overline{X}$ in $D^b(B_0(kG) \otimes_k B_0(kH))$ there is an, up to isomorphism, unique splendid tilting complex $X$ in $D^b(B_0(RG) \otimes_R B_0(RH))$ with $X \otimes_R k \simeq \overline{X}$.

**(5.5)** Denote by $[X]$ the isomorphism class of a complex $X$ *in the derived category* and denote by $(X)$ the isomorphism class of a complex $X$ *in the homotopy category*. Note that being homotopy equivalent is stronger than being quasi-isomorphic.

**(5.6)** It is natural to define (Z-01)

$$SplenPic_R(G) := \{(X) \mid X \text{ is a splendid tilting complex in } D^b(B_0(RG) \otimes_R B_0(RG))\}$$

and similarly for $k$ as coefficient domain. It is readily verified that this in fact is a group under tensor product. It is clear that we get a group homomorphism $SplenPic_R(G) \longrightarrow TrPic_R(B_0(RG))$ by setting $\nu(\ (X)\ ) := [X]$. The same notation is used for $k$ as coefficient domain.
In view of the action on the cohomology ring we define

$$HSplen_R(RG) := \nu^{-1}(HD_R(G)) \cap SplenPic_R(RG)$$

and similarly for $k$ instead of $R$.

**(5.7)** The action of $HSplen_R(G)$ on $H^*(G,R)$ is via $\varphi$ though. So, working up to homotopy equivalence only or working up to quasi-isomorphism does not make any difference for the action on the cohomology ring of the group.

**(5.8)** Why do we like to work up to homotopy equivalence only instead of up to quasi-isomorphism? The reason is the following result.

PROPOSITION 5.1 ((Z-01)). *Let $k$ be a field of characteristic $p$, let $G$ be a finite group and let $Q$ be a $p$-subgroup of $G$. Then, the Brauer construction provides a group homomorphism $SplenPic_k(G) \longrightarrow SplenPic_k(C_G(Q))$*

If one would work over isomorphism classes in the derived category the Brauer construction is not necessarily well defined. Actually there is no obvious reason why $[B_0(kG)] = [X]$ implies $[B_0(kC_G(Q)))] = [X(\Delta Q)]$.

**(5.9)** Let us study the obstruction. The equation $[B_0(kG)] = [X]$ implies that there is a quasi-isomorphism $B_0(kG) \stackrel{s}{\longrightarrow} X$ (and not necessarily in the other direction) since the homology of $X$ is concentrated in degree 0 and is isomorphic to $B_0(kG)$. The mapping cone of $s$ is therefore acyclic (i.e. with homology 0). Form the mapping cone by the standard construction

$$C(s) := (B_0(kG)[1] \oplus X, \begin{pmatrix} d_{B_0(kG)} & s \\ 0 & d_X \end{pmatrix}) .$$

Then, it is easily verified that $C(s(\Delta Q)) = (C(s))(\Delta Q)$ . In order to define the mapping in Proposition 5.1 in the derived category we have to prove that the Brauer functor applied to an acyclic complex is acyclic. Actually this is slightly stronger than what is needed since not every acyclic complex is the mapping cone of a quasi-isomorphism as above. Nevertheless, we shall give an example showing that it is not correct that the Brauer construction applied to an acyclic complex of $p$-permutation modules is acyclic.

**(5.10)** Let $k$ be the field with 2 elements. The sequence $0 \longrightarrow k \longrightarrow kC_2 \longrightarrow k \longrightarrow 0$ coming from the augmentation of the cyclic group of order 2 is exact. Moreover, this exact sequence can be seen as acyclic complex of $kC_2$-permutation modules with three non zero terms. Applying the Brauer functor $-(C_2)$ to this complex gives $(\ldots \longrightarrow 0 \longrightarrow k \longrightarrow 0 \longrightarrow k \longrightarrow 0 \longrightarrow \ldots)$ since $kC_2(C_2) = 0$ and $k(C_2) = k$. Obviously, this complex is not acyclic.

THEOREM 11 ((Z-01; Z-00b)). *Let $k$ be a field of characteristic $p$, let $G$ be a finite group, $P$ a $p$-Sylow subgroup and let $Q$ be a $p$-subgroup of $G$. Then, for any $[X] \in HSplen_k(G)$ so that $X(\Delta Q) \in HSplen_k(C_G(Q))$, we get $HSplen_R(G) \simeq HSplen_k(G)$ and the diagrams*

$$\begin{array}{ccc} H^*(G,k) & \stackrel{X\otimes -}{\longrightarrow} & H^*(G,k) \\ res^G_{C_G(Q)} \downarrow & & res^G_{C_G(Q)} \downarrow \\ H^*(C_G(Q),k) & \stackrel{X(\Delta Q)\otimes -}{\longrightarrow} & H^*(C_G(Q),k) \end{array} \quad \text{and} \quad \begin{array}{ccc} H^*(G,k) & \stackrel{X\otimes -}{\longrightarrow} & H^*(G,k) \\ \uparrow trans^G_{C_G(Q)} & & \uparrow trans^G_{C_G(Q)} \\ H^*(C_G(Q),k) & \stackrel{X(\Delta Q)\otimes -}{\longrightarrow} & H^*(C_G(Q),k) \end{array}$$

*are commutative. In case $P$ is abelian, an $X \in HSplen_k(G) \cap (-(\Delta P))^{-1}(HSplen_k(C_G(P)))$ acts on $H^*(G,k)$ via automorphisms of $P$.*

**(5.11)** It should be noted that the first point is not a trivial consequence of Rickard's result that $SplenPic_k(G) \simeq SplenPic_R(G)$. It has to be shown that the unique lift fixes the trivial module as well.
The second relation is essentially proved by first showing that $X(\Delta Q) \otimes_{C_G(Q)} k \simeq X^G(Q)$.
The third item uses the second and then as essential ingredient the theorem of Roggenkamp and Scott on the isomorphism problem for integral group rings of $p$-groups (RS-87; Rog-92).

## 6. The alternating group of degree 4

We shall give the group ring $R\mathfrak{A}_4$ for $R$ being a complete discrete valuation ring of characteristic 0 and residue field of characteristic 2, so that $R$ has a third root of unity. For this one may as well use the description in (N-98) of $R\ SL_2(4)$ by Gabriele Nebe using that $SL_2(4) \simeq \mathfrak{A}_5$ and the explicit tilting complex inducing a splendid equivalence between the principal block of $R\ SL_2(4)$ and $R\mathfrak{A}_4$ given by Jeremy Rickard in (Ric-96). Nevertheless, a more direct approach is possible as is shown below.
Let $k$ be a field of characteristic 2 containing a third root of unity and let $R$ be a complete discrete valuation ring of characteristic 0 containing a third root of unity and with residue field $k$ of characteristic 2.
Since $\hat{Z}_2[\zeta_3]$ is a splitting field for $\mathfrak{A}_4$ and all of its subgroups, it is enough to compute $R\mathfrak{A}_4$ for $R$ being the (unique) unramified extension of $\hat{\mathbb{Z}}_2$ of degree 2. Let $K$ be the field of fractions of $R$.
Since $\mathfrak{A}_4 \simeq (C_2 \times C_2) \rtimes C_3$ $K$ is a splitting field for $\mathfrak{A}_4$. We use a representation $A_4 \simeq < a,b,c : a^2, b^2, c^3, (a,b), a^c = ab, b^c = a >$. Now, $\mathbb{Q}\mathfrak{A}_4$ is a direct product of a copy of $\mathbb{Q}$, a copy of $\mathbb{Q}[\zeta_3]$ and a $3 \times 3$ matrix ring over $\mathbb{Q}$. Moreover, $K \otimes_{\mathbb{Q}} \mathbb{Q}[\zeta_3] \simeq K \times K$.
We are interested, how $R\mathfrak{A}_4$ is embedded into $K\mathfrak{A}_4$. Over $R$, however, we get 3 indecomposable projective modules, over $\hat{\mathbb{Z}}_2$ we get 2 projective indecomposable modules. Corresponding to the irreducible characters $\chi_i$, $i = 1,2,3$ of $C_3$, we get the idempotents $e_i := \frac{1}{3}\sum_{j=1}^{3} \chi_i(c^j)c^j$ Over $\hat{\mathbb{Z}}_2$ we get the two indecomposable idempotents $\chi_1$ and $\chi_2 + \chi_3$. We know, that for $i \neq j$ $e_i R\mathfrak{A}_4 e_j$ is a non zero fractional $R$–ideal in $K$ and for $i = j$ we know that $e_i R A_4 e_i$ is an $R$–order in $K \times K$.

The central idempotent $\delta := \frac{1}{4}(1 + a + ab + b)$ in $K\mathfrak{A}_4$ corresponds to the natural projection onto $KC_3$. This induces a pullback diagram

$$\begin{array}{ccc} R\mathfrak{A}_4 & \longrightarrow & RC_3 \\ \downarrow & & \downarrow \\ RA_4 \cdot (1-\delta) & \longrightarrow & (R/4R)C_3 \end{array} .$$

$\Lambda := R\mathfrak{A}_4 \cdot (1 - \delta)$ is an $R$-order in $\begin{pmatrix} K & K & K \\ K & K & K \\ K & K & K \end{pmatrix}$. We use the induced representation from a one dimensional character of the Klein Four group

$$a \longrightarrow \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, b \longrightarrow \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } c \longrightarrow \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Then, we conjugate by $\begin{pmatrix} 1 & 1 & 1 \\ 1 & \zeta_3^2 & -\zeta_3 \\ 1 & -\zeta_3 & \zeta_3^2 \end{pmatrix}$ to get $\Lambda = \begin{pmatrix} R & 2R & 2R \\ 2R & R & 2R \\ 2R & 2R & R \end{pmatrix}$. After some computation we get $R\mathfrak{A}_4 \simeq \{(a_1, a_2, a_3, (b_{i,j})_{1 \le i,j \le 3}) \in R \times R \times R \times \Lambda \mid b_{i,i} - a_i \in 4R \; \forall 1 \le i \le 3\}$

## 7. The braid group generators

We keep the hypotheses of the beginning of section 6. The group ring $R\mathfrak{A}_4$ has three projective indecomposable modules $P_1$, $P_2$, $P_3$ up to isomorphism. Denote by $P_1$ the projective cover of the trivial module. Note that if $R$ does not have a third root of unity, then there are only two projective indecomposable modules. The discussion in that case differs considerably of what we shall develop here.

Denote in order not to overload the notation by $A$ the group ring $R\mathfrak{A}_4$. We denote by $S_i$ the complex

$$\ldots \longrightarrow 0 \longrightarrow P_i \otimes P_i^* \longrightarrow A \longrightarrow 0 \longrightarrow \ldots$$

and the functors $F_i := S_i \otimes_A -$ for any $i \in \{1, 2, 3\}$. The following result was known to Joe Chuang [1] for $k$ as coefficient domain as action on the simple modules. We shall include a proof for $R$ and this implies at once the result for $k$.

PROPOSITION 7.1. *$S_i \otimes_A S_j \otimes_A S_i \simeq S_j \otimes_A S_i \otimes_A S_j$ for any pair $(i, j) \in \{1, 2, 3\} \times \{1, 2, 3\}$.*

Proof. It is obvious by the symmetry of $R\mathfrak{A}_4$ that it is enough to prove the statement for $i = 1$ and $j = 2$. We compute $S_1 \otimes P_1 \simeq P_1 \longrightarrow 0$, $S_1 \otimes P_2 \simeq P_1 \longrightarrow P_2$ and $S_1 \otimes P_3 \simeq P_1 \longrightarrow P_3$ where the right most position for the complexes on the right of the isomorphism is in degree 0. Analogous results hold for $S_2$ instead of $S_1$. Then, $S_2 \otimes_A S_1 \otimes_A P_1 \simeq S_2 \otimes_A P_1[1] \simeq P_2 \longrightarrow P_1 \longrightarrow 0$ again with the convention that the right hand side of the complex on the right is meant to be in degree 0.

$$S_2 \otimes_A S_1 \otimes_A P_2 \simeq F_2(cone(P_1 \longrightarrow P_2)) \simeq cone(F_2(P_1) \longrightarrow F_2(P_2)) \simeq P_1[1] .$$

Likewise $S_2 \otimes_A S_1 \otimes_A P_3 \simeq F_2(cone(P_1 \longrightarrow P_3)) \simeq P_2 \longrightarrow P_1 \oplus P_2 \longrightarrow P_3$ . Furthermore, $F_1(P_2 \longrightarrow P_1 \longrightarrow 0) \simeq P_2[2]$ and $F_1(P_1[1]) \simeq P_1[2]$. Compute the image of $P_3$:

$$\begin{array}{rcl} F_1(F_2(F_1(P_3))) & \simeq & F_1(cone((P_2 \to P_1) \to (P_2 \to P_3))) \\ & \simeq & cone(F_1(P_2 \to P_1) \to F_1(P_2 \to P_3)) \\ & \simeq & cone(P_2[1] \to (P_2 \longrightarrow P_1 \oplus P_2 \to P_3)) \\ & \simeq & \ldots \to 0 \begin{array}{c} \nearrow P_1 \longrightarrow P_2 \searrow \\ \searrow P_2 \longrightarrow P_1 \nearrow \end{array} P_3 \longrightarrow 0 \longrightarrow \ldots \end{array}$$

[1] In September 1999 in a private communication

where in the final complex the differential in degree 2 maps the projective $P_1$ in degree 2 to the projective $P_2$ in degree 1 and the projective $P_2$ in degree 2 to the projective $P_1$ in degree 1.
In any case the result is invariant under changing indices 1 and 2. By (RZ-96) this proves that $F_1F_2F_1$ differ from $F_2F_1F_2$ only by an automorphism of $R\mathfrak{A}_4$. As is easily seen from the structure of $R\mathfrak{A}_4$ in section 6, apart from the identity there is no $R$-linear automorphism which fixes each projective indecomposable up to isomorphism. We proved the statement. ■

COROLLARY 7.1. *Let $Br(\tilde{A}_2)$ be the braid group associated to the affine Dynkin diagram $\tilde{A}_2$. Then, there is a group homomorphism $Br(\tilde{A}_2) \longrightarrow SplenPic_R(\mathfrak{A}_4)$.*

It is clear that the group generated by $F_2$ and $F_3$ is the image under the above homomorphism of an Artin braid group on three strings. Moreover, these two self-equivalences stabilize the trivial module.

## 8. Studying the action on the group cohomology

**(8.1)** The outer automorphism group of $\mathfrak{A}_4$ is isomorphic to the cyclic group of order 2 since any automorphism of $\mathfrak{A}_4$ induces an automorphism of $\mathfrak{S}_4$ where each automorphism is inner. Moreover (A-M 94, Chapter III Theorem 1.3), $H^*(\mathfrak{A}_4, \mathbb{F}_2) = \mathbb{F}_2[A, B](C)/(C^3 + A^2 + B^2 + AB)$ where $A = f_1^3 + f_2^3$, $B = \zeta f_1^3 + \zeta^2 f_2^3$ and $C = f_1 f_2$. Here, $\zeta$ is a primitive third root of unity over $\mathbb{F}_2$. The elements $f_1$ and $f_2$ are the two canonical generators of $H^*(C_2^2, \mathbb{F}_2)$ in degree 1.
The group of automorphisms of $C_2^2$ is $GL_2(2) \simeq \mathfrak{S}_3$. The automorphisms of order 3 are inner in $\mathfrak{A}_4$ and induce hence the identity on the subring $H^*(\mathfrak{A}_4, \mathbb{F}_2)$ of $H^*(C_2^2, \mathbb{F}_2)$. It is an easy and straightforward calculation that the elements of $GL_2(2)$ of order 2 act trivially on degree 2 cohomology and act on degree 3 cohomology as regular representation ($A \longrightarrow A$ and $B \longrightarrow -A - B$).

**(8.2)** Carlson and Rouquier proved (CR-00) that for a group $G = P \rtimes Q$ where $P$ is an abelian $p$-group and $Q$ is a cyclic $p'$-group, the group $StPic_{\mathbb{F}_p}(\mathbb{F}_pG)$ of self-equivalences of Morita type of the stable category is generated by the syzygy operator and the usual Picard group of $\mathbb{F}_pG$. Hence, the action of $HD_{\mathbb{F}_p}(G)$ on $H^*(G, \mathbb{F}_p)$ is via automorphisms of $\mathbb{F}_pG$. In case $G = \mathfrak{A}_4$ and $p = 2$, the action described above is the only possible non trivial action of $HSplen_{\mathbb{F}_2}(\mathfrak{A}_4)$ on $H^*(\mathfrak{A}_4, \mathbb{F}_2)$.

**(8.3)** Recall that $< F_2, F_3 >$, which is the homomorphic image of an Artin braid group, is in $HSplen_R(R\mathfrak{A}_4)$. By (Z-00b) we conclude in our case of abelian Sylow subgroups that $HSplen_R(\mathfrak{A}_4)$ acts by automorphisms of the Sylow subgroup on the cohomology. Since for a splendid tilting complex $X$ one has $X(\Delta Q) \otimes_{C_G(Q)} R \simeq X^G(Q)$ (where $G$ acts on the right and $Q$ on the left), one sees easily that any element of $< F_2(\Delta C_2^2), F_3(\Delta C_2^2) >$ fixes the trivial module as well. Moreover, it is easy to see that $< F_2, F_3 >$ acts trivially an $H^*(\mathfrak{A}_4, \mathbb{F}_2)$ since this action factors via the action of the group of stable self-equivalences (see (Z-99b)) and there $S_2$ and $S_3$ become just isomorphic to $R\mathfrak{A}_4$. So, we proved that $< F_2, F_3 >$ does not contain $Out(\mathfrak{A}_4)$.
Denoting by $\gamma$ the element (of order 2) in $Out(R\mathfrak{A}_4)$ given by conjugation by an element in $\mathfrak{S}_4 \setminus \mathfrak{A}_4$, we get $< F_2, F_3, \gamma > = < F_2, F_3 > \rtimes C_2 \leq HD_R(\mathfrak{A}_4)$ .

**(8.4)** We observe that $F_2(P_2) \simeq P_2[1]$ and $F_2(P_3) \simeq (P_2 \longrightarrow P_3)$ with homology in degree 0 and 1, and likewise for $F_3$ with the rôles of $P_2$ and $P_3$ interchanged. The very same proof as in (RZ-96, Theorem 4.6) shows that the morphism $Br(A_2) \longrightarrow < F_2, F_3 > \subseteq HSplen_{\mathbb{F}_2}(\mathfrak{A}_4)$ is injective.

**(8.5)** Since the symmetric group of degree 3 acts naturally on the three generators of $Br(\tilde{A}_2)$ and by the fact that the relations obey the same symmetry also on $Br(\tilde{A}_2)$, we get

$$\begin{array}{ccccc}
Br(\tilde{A}_2) \rtimes \mathfrak{S}_3 & \simeq & (ker\beta \rtimes Br(A_2)) \rtimes (C_3 \rtimes C_2) & \longrightarrow\!\!\!\!\!\rightarrow & Br(A_2) \rtimes C_2 \\
\downarrow & & & & \cap\!\downarrow \\
< F_1, F_2, F_3 > \rtimes \mathfrak{S}_3 & & & & < F_2, F_3 > \rtimes < \gamma > \\
\cap & & & & \cap \\
TrPic_R(R\mathfrak{A}_4) & & & & HD_R(\mathfrak{A}_4)
\end{array}$$

**(8.6)** May it be that derived categories "in an abelian situation" have generalized braid group symmetries? I believe that the above observations are not just an accident.

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