DERIVED INVARIANCE OF THE SUBSPACES OF p-SINGULAR CLASS SUMS IN BLOCKS

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Abstract. The centres of two derived equivalent algebras A and B are isomorphic, as was shown by Rickard. If A and B are blocks of group rings over a perfect field of characteristic $p > 0$, it is natural to ask if the subspaces of the centre generated by p-singular class sums also map to each other. A tool is given by Külshammer's work on p-power maps on the cocentre. We prove that the spaces generated by p-singular class sums are mapped to each other up to multiplication by a central unit.

Introduction

Let G be a group and k be a field of characteristic $p > 0$. The ring structure of the centre $Z(kG)$ of kG is of high interest, in particular for questions concerning stable and derived equivalences between blocks of group rings. Indeed, by a result due to Rickard the centres of two derived equivalent k-algebras A and B are isomorphic as algebras. Stable equivalences of Morita type between A and B still preserve a certain quotient, the so-called stable centre. Hence, the detailed structure of the centre of algebras proved to be highly useful to distinguish derived equivalence classes of algebras as well as algebras up to stable equivalence of Morita type (cf [2] or [26, Chapter 5] for more details).

Also, the centre of a group algebra or its blocks encode many interesting invariants of the group, as was developed by Külshammer in his sequence of papers [8, 9]. Külshammer studied in particular the p-power map on the cocentre $A/[A, A] = H_0(A)$, i.e. the degree 0 Hochschild homology of a finite dimensional k-algebra A. If the algebra A is symmetric, then the Hochschild homology is closely linked by a non degenerate bilinear pairing with the Hochschild cohomology of the same degree. In particular, the centre and the cocentre are isomorphic as vector spaces, but the p-power map on the cocentre has an adjoint on the centre, and its image is called the Külshammer ideals $T_n(A)^{\perp}$. Versions of these constructions were shown to be derived invariants [23, 7, 24] (cf e.g. [25], [26, Chapter 2.9] for an overview) and proved to be a very useful and efficient in distinguishing stable and derived equivalence classes of pairs of algebras which differ only by a few parameters in the relations [6, 5, 22, 3, 4, 27, 1, 20].

Another subspace of the centre of a group algebra was studied more recently. It is well-known that the conjugacy class sums form a $\mathbb{Z}$-basis of the centre of $\mathbb{Z}G$, and hence also a k-basis of the centre $Z(kG)$ of kG. The conjugacy classes C_g of elements $g \in G$ such that the order of g is prime to p are called the p-regular classes, the remaining classes are called p-singular. If k is sufficiently big its number coincides with the number isomorphism classes of simple modules by a result due to Brauer. The conjugacy class sums of p-regular elements span a k-vector space called $Z_{p'}(kG)$, or by multiplying with the corresponding block idempotent b, $Z_{p'}(B)$ for a block $B = kGb$. In [13, 14] Meyer found that $Z_{p'}(kG)$ with G having a Sylow p-subgroup being either commutative, dihedral, or semidihedral is a subring of the centre of kG. Meyer proved in [14] that this holds as well for $p = 2$ and G being the Suzuki group $Sz(2^r)$ and p any prime number. Murray found in [15] that for $p = 2$, the square of the centre of a group algebra equals to the 2-regular subspace in case of G being a symmetric or an alternating group, and hence $Z_{p'}(kG)$ is a ring. Suppose that B_1 is a block of kG_1 and B_2 is a block of kG_2. Fan and Külshammer [10] proved

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by purely character theoretic methods that if B_1 and B_2 have abelian defect group, and suppose that B_1 and B_2 are perfectly isometric, then $B_1 \cap Z_{p'}(kG_1) =: Z_{p'}(B_1)$ is a ring if and only if $B_2 \cap Z_{p'}(kG_2) =: Z_{p'}(B_2)$ is a ring. We should note that Meyer gave in [13] numerous examples for groups G such that $Z_{p'}(kG)$ is not a ring. This holds for example when $G = SL_2(q)$ for q a power of an odd prime, and k of characteristic 2.

In this paper we study yet another kind of substructures of the centre, again linked to p-power maps. More precisely, we study p-singular conjugacy classes C_g, that is conjugacy classes of elements $g \in G$ such that the order of g is divisible by p. We further form the conjugacy class sums and the corresponding subspace $Z_p(kG)$ is the k-vector space generated by the p-singular classes, respectively $Z_p(B)$ for a block B of KG. We shall also give intermediate spaces $Z_p^{(m)}(kG)$ respectively its block versions $Z_p^{(m)}(B)$ for all integers $m \geq 1$ corresponding to the question if g has p^m-th roots in G or not. We shall show that this gives an increasing sequence of subspaces of the centre $Z(kG)$, respectively $Z(B)$ for a block B of kG. We shall prove in this paper that for two finite groups G_1 and G_2 and a block B_1 of kG_1 and a block B_2 of kG_2 such that B_1 is derived equivalent to B_2, then the algebra isomorphism $Z(B_1) \simeq Z(B_2)$ induced by the derived equivalence maps $Z_p(B_1)$ to $u \cdot Z_p(B_2)$ for some unit $u \in Z(B_2)$. More generally, we show that in this case $Z_p^{(m)}(B_1)$ to $u \cdot Z_p^{(m)}(B_2)$ for all integers m, with the same unit u. The main tool is the p-power map on the cocentre $B_1/[B_1, B_1]$ (respectively $B_2/[B_2, B_2]$) and the non degenerate pairing between the centre and the cocentre of B_1, respectively B_2. The non degenerate pairing for B_1 is shown to be mapped to a pairing for B_2, and the difference with the standard pairing is given by a central unit of B_2. We then use work of König, Liu, Zhou and the third author [7, 24] to prove that this characterisation is a derived invariant up to a multiplicative with this central unit.

The paper is organised as follows. In Section 1 we review general properties on symmetric algebras and the p-power maps on the centre and the cocentre, as far as it is needed in the sequel. In Section 2 we consider the special case of a group algebra and define p-regular and p-singular subspaces of the centre. We also give the crucial characterisations of these spaces in terms of p-power maps. A blockwise definition of the spaces defined in Section 2 is then given in Section 3. In Section 4 we study the behaviour of these data under derived equivalences, and recall the results needed from previous work. Section 5 then contains our main result, namely the derived invariance of p-singular subspaces. Finally we compute an example to illustrate this concept and to show that the unit u is indeed necessary in the theorem.

1. Review of some facts on symmetric algebras

1.1. Symmetric algebras, symmetrising forms and isomorphisms to the dual. A k-algebra A is symmetric if $Hom_k(A, k) \simeq A$ as $A-A$-bimodules. Then each isomorphism $\varphi : A \to Hom_k(A, k)$ defines a non degenerate associative and symmetric k-bilinear form

$$\langle\ ,\ \rangle : A \times A \longrightarrow k$$

by $\langle x, y\rangle := \varphi(x)(y)$ for all $x, y \in A$, and conversely any non degenerate associative and symmetric bilinear form $\langle\ ,\ \rangle$ defines an isomorphism $\varphi : A \to Hom_k(A, k)$ by the above formula $\langle x, y\rangle =: \varphi(x)(y)$. We call such a bilinear form symmetrising. A symmetrising form is not unique and two different forms correspond to two different choices of isomorphisms φ. Külshammer [8, Part IV] (see also [26, Lemma 2.9.10]) showed that $Z(A)^\perp = [A, A]$ where the orthogonal is taken with respect to the symmetrising form $\langle\ ,\ \rangle$. Hence the restriction of $\langle\ ,\ \rangle$ to $Z(A)$ in the left entry induces a non degenerate k-bilinear form

$$\langle\ ,\ \rangle : Z(A) \times A/[A, A] \longrightarrow k$$

by putting

$$\langle z, a + [A, A]\rangle := \langle z, a\rangle$$

for all $a \in A$ and $z \in Z(A)$.

1.2. **Modifying the symmetrising form.** The following lemma is well-known (cf e.g. [26, Lemma 1.10.25]), but we include the short proof for completeness.

Lemma 1.1. *Let k be a field and let A be a symmetric k-algebra with symmetrising form $\langle\ ,\ \rangle$. Let $u \in Z(A)$ be an invertible element. We define $\langle x, y\rangle_u := \langle u \cdot x, y\rangle$ for all $x, y \in A$. Then $\langle\ ,\ \rangle_u$ is a symmetrising form as well, and for every symmetrising form $\widetilde{\langle\ ,\ \rangle}$ on A there is a $u \in (Z(A))^\times$ so that $\widetilde{\langle\ ,\ \rangle} = \langle\ ,\rangle_u$.*

Proof. Since $\langle\ ,\ \rangle$ is non degenerate,

$$\bigcap_{x\in A} ker(\langle u \cdot x,\ \rangle) = \bigcap_{u\cdot x\in A} ker(\langle u \cdot x,\ \rangle) = \bigcap_{x'\in A} ker(x',\ \rangle) = 0$$

and so $\langle\ ,\ \rangle_u$ is non degenerate.

We compute for all $x, y \in A$

$$\langle x, y\rangle_u = \langle u \cdot x, y\rangle = \langle y, u \cdot x\rangle = \langle y \cdot u, x\rangle = \langle u \cdot y, x\rangle = \langle y, x\rangle_u$$

and hence $\langle\ ,\ \rangle_u$ is symmetric.

Moreover, for all $x, y, z \in A$ we get

$$\langle x, y \cdot z\rangle_u = \langle u \cdot x, y \cdot z\rangle = \langle u \cdot x \cdot y, z\rangle = \langle x \cdot y, z\rangle_u$$

and hence $\langle\ ,\ \rangle_u$ is associative.

Conversely, we will show that any symmetrising form is of the form $\langle\ ,\rangle_u$ for some $u \in Z(A)$. Indeed, $\langle\ ,\ \rangle$ induces an isomorphism of $A \otimes_k A^{op}$-modules $A \xrightarrow{\simeq} Hom_k(A, k)$ by $a \mapsto \langle a,\ \rangle$ (cf e.g. [26, Proposition 1.10.23]). Hence any symmetrising form will give such an isomorphism, whence two different forms will produce an automorphism $A \longrightarrow A$ of $A \otimes_k A^{op}$-modules. Since $End_{A\otimes_k A^{op}}(A) = Z(A)$ we see that any symmetrising form of A is of the form $\langle\ ,\ \rangle_u$ for some $u \in A$. ■

1.3. **Recall Külshammer's p-power constructions.** Let k be a perfect field of characteristic $p > 0$ and let A be a k-algebra. Then denote by $[A, A]$ the k-submodule of A generated by commutators $ab - ba$ for $a, b \in A$. Recall (cf [8, Part IV Section 1] or [26, Lemma 2.9.3]) that for every k-algebra A we may form the degree 0 Hochschild homology $HH_0(A) = A/[A, A]$ and we get a well-defined mapping

$$\begin{array}{ccc} A/[A, A] & \xrightarrow{\mu_p} & A/[A, A] \\ a + [A, A] & \mapsto & a^p + [A, A] \end{array}$$

We recall from [8, Part IV] a map κ (resp. ζ) on the cocentre (resp. the centre) which are associated to the p-power maps on the centre (resp. the cocentre) of an associative algebra.

Let A be a symmetric algebra over a perfect field k, and let $\langle\ ,\ \rangle$ be a symmetrising form on A. Then (cf [8, Part IV] or [26, Lemma 2.9.10]) $[A, A]^\perp = Z(A)$, where orthogonal spaces are taken with respect to the symmetrising form. Hence $\langle\ ,\ \rangle$ induces a non degenerate pairing

$$\langle\ ,\ \rangle : Z(A) \times A/[A, A] \to k.$$

We shall need to consider orthogonal spaces with respect to two different symmetrising forms.

Lemma 1.2. *Let A be a symmetric k-algebra, let $u \in Z(A)^\times$, and let $\langle\ ,\ \rangle$ and $\langle\ ,\ \rangle_u$ (according to the notation in Lemma 1.1). For each $M \subset A/[A, A]$ let $M^{\perp_{\langle\ ,\ \rangle}} := \{c \in Z(A) \mid \langle c, y\rangle = 0\ \forall y \in M\}$, and let $M^{\perp_{\langle\ ,\ \rangle_u}} := \{c \in Z(A) \mid \langle c, y\rangle_u = 0\ \forall y \in M\}$. Then*

$$u \cdot M^{\perp_{\langle\ ,\ \rangle_u}} = M^{\perp_{\langle\ ,\ \rangle}}.$$

Proof.

$$v \in M^{\perp_{\langle\ ,\ \rangle_u}} \Leftrightarrow \langle v, y\rangle_u = \langle uv, y\rangle = 0\ \forall y \in M \Leftrightarrow uv \in M^{\perp_{\langle\ ,\ \rangle}}.$$

Whence the statement. ■

2. Centre and the cocentre of a group ring with respect to p-power maps

2.1. p-power filtration of the centre of a group ring. Let G be a finite group and let k be a field of characteristic $p > 0$. Let

$$C_h := \{g \in G | \; \exists x \in G : g = xhx^{-1}\}$$

be the conjugacy class of $h \in G$. It is well-known that the centre $Z(kG)$ of the group ring kG of G over k has a k-basis given by the conjugacy class sums, i.e. set of elements $\underline{C_h} := \sum_{g \in C_h} g$ for $h \in G$.

Definition 2.1. Let $p > 0$ be a prime, G be a finite group with $p| \; |G|$, and k be a perfect field of characteristic p.

- The p-regular subspace of the centre is $Z_{p'}(kG) := \langle \underline{C_h} \mid p \not| \; |h| \rangle_k$
- The p^m-regular subspace of the centre is $Z_{p'}(kG) := \langle \underline{C_h} \mid p^m \not| \; |h| \rangle_k$
- The p-singular subspace of the centre is $Z_p(kG) := \langle \underline{C_h} \mid p| \; |h| \rangle_k$
- The p^m-singular subspace of the centre is $Z_p^{(m)}(kG) := \left\langle \underline{C_h} \mid \neg \left(\exists g \in G : \; g^{p^m} = h \right) \right\rangle_k$.

For any integer n we denote by n_p the unique integer p^m such that $p^m | n$ but $p^{m+1} \not| n$. For any $g \in G$ there are uniquely determined elements $g_p, g_{p'} \in G$ be commuting elements such that $g = g_p \cdot g_{p'}$ and g_p has p-power order, and the order of $g_{p'}$ is prime to p.

Lemma 2.2. *Let G be a finite group. Then the set of elements of order prime to p coincides with the set of elements of G which admit a p^s-th root for some $s > \log_p(exp(G)_p)$.*

Proof. If p does not divide the order of h, then by Bézout's lemma there are integers u_s and v_s such that $1 = u_s p^s + v_s |h|$. Therefore,

$$h = h^{u_s p^s + v_s |h|} = (h^{u_s})^{p^s} \left(h^{|h|} \right)^{v_s} = h_s^{p^s}$$

for $h_s := h^{u_s}$.

Conversely, suppose that h has arbitrary p^s-th roots. Hence, for all $s \in \mathbb{N}$ there is $h_s \in G$ such that $h_s^{p^s} = h$. Now, let $h = h_p h_{p'}$ for commuting elements h_p and $h_{p'}$ such that h_p is of p-power order p^t, and $h_{p'}$ has order q prime to p. Then

$$(h_s^q)^{p^s} = \left(h_s^{p^s} \right)^q = h^q = h_p^q.$$

By Bézout's lemma there are $u, v \in \mathbb{Z}$ such that $1 = up^t + vq$ and hence

$$h_p = \left(h_p^q \right)^v = (h_s^{vq})^{p^s}.$$

This implies that h_p has arbitrary p^s-th roots g_s in G for all $s \in \mathbb{N}$. But then the order of h_s is p^{t+s}, and hence p^{t+s} divides the order of G for all $s \in \mathbb{N}$. This provides a contradiction to the finiteness of G. ∎

Corollary 2.3. *The p^s-singular space for $s > exp(G)_p$ is just the p-singular space.*

Proof. Indeed, if g does not has a p^s-th root, then the order of g has to be divisible by p, by Lemma 2.2. This shows $Z_p(kG) \supseteq Z_p^{(s)}(kG)$ for any s. Conversely, suppose that the order of g is divisible by p and write $g = g_p g_{p'}$ for commuting elements g_p and $g_{p'}$ such that the order of g_p is a non trivial prime power p^t and the order of $g_{p'}$ is not divisible by p. Suppose there is $h \in G$ such that $h^{exp(G)_p} = g_p$. Then h is of order $p^t \cdot exp(G)_p$, which is a contradiction to the definition of the exponent of a group. This shows the Corollary. ∎

Remark 2.4. Let $e := \log_p(exp(G)_p)$. If g has a p^n-th root h, then g has a p^{n-1}-th root h^p. Therefore we have an ascending sequence of vector spaces

$$Z_p^{(1)}(kG) \subsetneq Z_p^{(2)}(kG) \subsetneq \cdots \subsetneq Z_p^{(e)}(kG) \subsetneq Z_p^{(e+1)}(kG) = Z_p(kG).$$

The sequence is stationary from $Z_p^{(e+1)}(kG)$ onwards (i.e. $Z_p^{(e+1)}(kG) = Z_p^{(e+s)}(kG) \;\; \forall s \geq 1$), since we can write any element $h = h_p h_{p'}$ for commuting elements h_p and $h_{p'}$ such that

h_p is a p-power order, and such that the order of $h_{p'}$ is not divisible by p. By definition of an exponent the p-part h_p of h cannot have order bigger than p^e. For the p'-part $h_{p'}$ we can use Lemma 2.2.

2.2. **Group rings as symmetric algebras.** Recall that for any field k and any finite group G the group ring kG is a symmetric algebra by putting

$$\langle\sum_{g\in G} x_g g, \sum_{h\in G} y_h h\rangle := \sum_{g\in G} x_g y_{g^{-1}}.$$

Lemma 2.5. *Let k be a field and let*

$$G = C_{h_1} \dot\cup C_{h_2} \dot\cup \cdots \dot\cup C_{h_s}$$

be a decomposition of G into a disjoint union of conjugacy classes. Then $kG/[kG,kG]$ has a basis given by

$$\{h_i + [kG,kG] \mid i \in \{1,2,\dots,s\}\}.$$

In particular for $g,h \in kG$, then $g + [kG,kG] = h + [kG,kG]$ if and only if g is conjugate to h in G.

Proof. Since by [8, Part IV] or [26, Lemma 2.9.10] we have $Z(kG)^\perp = [kG,kG]$, we obtain that $Z(kG) \simeq kG/[kG,kG]$ as vector spaces. Hence the two spaces have the same dimension. Since the dimension of $Z(kG)$ is the same as the number of conjugacy classes of G, this is also the dimension of $kG/[kG,kG]$. Since

$$h - xhx^{-1} = (hx^{-1})x - x(hx^{-1}) \in [kG,kG]$$

for all $h \in G$ and $x \in G$, and since G is a k-basis of kG, G generates $kG/[kG,kG]$ as k-vector space, and moreover, two conjugate elements in G belong to the same class modulo $[kG,kG]$. The dimension argument shows that $\{h_i + [kG,kG] \mid i \in \{1,\dots,s\}$ is actually a basis of $kG/[kG,kG]$. This shows as well that $g + [kG,kG] = h + [kG,kG]$ in $kG/[kG,kG]$ if and only if g is conjugate to h in G. ■

We can now determine the restriction of the above symmetrising form $\langle\ ,\ \rangle$ to the centre $Z(kG)$ of kG. We get

$$\langle \underline{C}_g, h\rangle = \begin{cases} 1 & \text{if } g \text{ is conjugate in } G \text{ to } h^{-1} \\ 0 & \text{else} \end{cases}.$$

Indeed, recall that C_g is a conjugacy class, and observe that for $x \in kG$ and $h \in G$ we get

$$h - xhx^{-1} = x^{-1}\cdot xh - xh\cdot x^{-1} = [x^{-1}, xh] \in [kG,kG]$$

so that the value of the restriction of the bilinear form above indeed depends only on the conjugacy class of h in G. Therefore the restriction to $Z(kG)$ of the given symmetrising form above on kG is the non degenerate form

$$\langle\ ,\rangle : Z(kG) \times kG/[kG,kG] \longrightarrow k$$

given by

$$\langle \underline{C}_g, h + [kG,kG]\rangle = \begin{cases} 1 & \text{if } g \text{ is conjugate in } G \text{ to } h^{-1} \\ 0 & \text{else} \end{cases}.$$

2.3. **p-power maps on the commutator quotient of kG.** We shall study the p-power map μ_p on $kG/[kG,kG]$. Throughout, let k be a perfect field of characteristic $p > 0$. Consider the set

$$M^t = im(\mu_p^t).$$

It is clear that

$$M^1 \supseteq M^2 \supseteq \cdots \supseteq \bigcap_{t\in\mathbb{N}} M^t.$$

Lemma 2.6. *Let k be a perfect field of characteristic $p > 0$, let G be a finite group and let $h \in G$. Then*

$$\left(h + [kG, kG] \in \bigcap_{t \leq m} M^t = M^m\right) \Leftrightarrow \left(\exists g \in G : g^{p^m} = h\right).$$

In particular,

$$\left(h + [kG, kG] \in \bigcap_{t \in \mathbb{N}} M^t\right) \Leftrightarrow (p \not| \, |h|).$$

Proof. We first prove the first equivalence. We know that

$$\dim_k(Z(kG)) = \dim_k(kG/[kG, kG])$$

and a k-basis of $Z(kG)$ is given by the conjugacy class sums of G. Hence a k-basis of $kG/[kG, kG]$ is given by representatives of conjugacy classes of G.

We claim that

$$h + [kG, kG] \in M^m \Leftrightarrow \exists g \in G : g^{p^m} = h \; .$$

Suppose $\exists g \in G : g^{p^m} = h$, then $h = \mu_p^m(g)$ trivially. This shows the implication "$\Leftarrow$".

Now suppose that $x + [kG, kG] \in M^m$. By Lemma 2.5 a basis for the commutator quotient is given by representatives of the conjugacy classes of G. Let

$$x = \sum_{g \in G/\text{conj}} k_g g + [kG, kG] \in kG/[kG, kG]$$

and compute

$$\mu_p^m(x + [kG, kG]) = \sum_{g \in G/\text{conj}} k_g^{p^m} g^{p^m} + [kG, kG].$$

This shows that M^m is k-linearly generated by those elements $g + [kG, kG]$ for which there is $h_m(g) \in G$ with $h_m(g)^m - g \in [kG, kG]$. By Lemma 2.5 this is precisely the set of elements $g + [kG, kG]$ for which there is $h_m(g) \in G$ with $h_m(g)^m$ is conjugate in G to g. Hence, this is precisely the set of elements $g + [kG, kG]$ for which there is $h_m(g) \in G$ with $h_m(g)^{p^m} = g$. This shows "$\Rightarrow$".

The second statement follows from the first equivalence and Lemma 2.2. ■

2.4. **Obtaining the p-regular subspace.** We digress with a construction which was used in [19], but which proves not to be an invariant under change of bilinear forms.

Given a symmetric k-algebra A with symmetrising form $\langle \; , \; \rangle$, then for any k-basis B of $Z(A)$ we get a k-basis B', the dual basis of B, of $A/[A, A]$ by the condition

$$\langle b, c' \rangle = \delta_{b,c}$$

for $\delta_{b,c}$ being the Kronecker symbol; i.e. the linear form $\langle b, - \rangle$ vanishes on all elements of B' except on b', on which it is 1. Indeed, there is a linear independent system of $|B|$ linear equations $\langle c, - \rangle = \delta_{b,c}$ for $b, c \in B$, and hence there is a unique solution B' to this set of linear equations. Hence we get an identification

$$\begin{array}{ccc} Z(A) & \xrightarrow{\delta} & A/[A, A] \\ \sum_{b \in B} \lambda_b b & \mapsto & \sum_{b' \in B'} \lambda_b b' \end{array}$$

This map depends on the choice of B and the choice of the symmetrising form $\langle \; , \; \rangle$.

Corollary 2.7. *Let G be a finite group and let k be a perfect field of characteristic $p > 0$. Let $\langle \; , \; \rangle$ be the standard symmetrising form of kG from Section 2.2, and take the conjugacy class sums as basis of $Z(kG)$. Then the dual basis map δ for these data maps $Z_{p'}^{(m)}(kG)$ to M^m.*

Proof. We know that the dual basis element to $\underline{C}_g$ is $g^{-1} + [kG, kG]$. Hence the dual basis element of $\underline{C}_g \in Z_{p'}(kG)$ is exactly $g^{-1} + [kG, kG]$ for $p \not| \ |g|$, and the dual basis element of $g^{-1} + [kG, kG]$ for $p \not| \ |g|$ is precisely $\underline{C}_g \in Z_{p'}(kG)$. Hence $\delta(Z_{p'}^{(m)}(kG)) = M^m$ by Lemma 2.6. ■

Example 2.8. We consider $p = 2$ and let k be an algebraically closed field of characteristic 2. Let C_2 be the cyclic group of order 2 generated by c. Then $kC_2 \simeq k[X]/X^2$ as algebras. The isomorphism φ is given by

$$\begin{array}{ccc} kC_2 & \longrightarrow & k[X]/X^2 \\ c & \mapsto & X-1 \\ 1 & \mapsto & 1 \end{array}$$

The standard symmetrising form on kC_2 is given by

$$\langle 1, 1\rangle_{grp} = \langle c, c\rangle_{grp} = 1 \ , \ \langle 1, c\rangle_{grp} = \langle c, 1\rangle_{grp} = 0$$

which is mapped by φ to

$$\langle 1, 1\rangle_{grp} = \langle X, 1\rangle_{grp} = \langle 1, X\rangle_{grp} = 1 \ , \ \langle X, X\rangle_{grp} = 0.$$

The form on $k[X]/X^2$ from [5, Proposition 3.1] is however given by

$$\langle 1, X\rangle_{alg} = \langle X, 1\rangle_{alg} = 1 \ , \ \langle 1, 1\rangle_{alg} = \langle X, X\rangle_{alg} = 0$$

since $rad(k[X]/X^2) = soc(k[X]/X^2) = Xk[X]/X^2$ with basis $\{X\}$. The Gram matrices are therefore

$$G_{alg} := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ for } \langle\ ,\ \rangle_{alg} \text{ and basis } \{1, X\}$$

and

$$G_{grp} := \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \text{ for } \langle\ ,\ \rangle_{grp} \text{ and basis } \{1, X\}.$$

However, there is no invertible matrix T with $TG_{grp}T^{tr} = G_{alg}$. Hence these two symmetric forms are not equivalent. Nevertheless, accordingly with Lemma 1.1,

$$\langle a, b\rangle_{grp} = \langle a \cdot (1 + X), b\rangle_{alg}$$

for all $a, b \in k[X]/X^2$ where $1 + X$ is a unit, as is easily checked on the value of the basis elements. Note that $Z_{2'}(kC_2) = k\cdot 1$, $Z_2(kC_2) = k \cdot c$, and $\bigcap_{t\in\mathbb{N}} M^t = k\cdot 1$. Now, $\delta_{grp}^{-1}(1) = 1$, whereas $\delta_{alg}^{-1}(1) = X = c + 1$. We see that the two elements 1 and $c + 1$ belong to different orbits of the multiplication action of $Z(kC_2)^\times$ on $Z(kC_2)$.

3. Going blockwise

Throughout this section let k be a perfect field of characteristic $p > 0$. Let B be a block of kG and let $kG = \bigoplus_{i=0}^n B_i$ be the block decomposition of kG. Then $HH_0(kG)$ decomposes into

$$HH_0(kG) = \bigoplus_{i=0}^{n} HH_0(B_i)$$

and of course also

$$Z(kG) = HH^0(kG) = \bigoplus_{i=0}^{n} HH^0(B_i) = \bigoplus_{i=0}^{n} Z(B_i).$$

Moreover, the p-power map on $HH_0(kG)$ restricts to p-power maps $\mu_p(B_i)$ on $HH_0(B_i)$ for each $i \in \{0, \ldots, n\}$. As is well-known, each B_i is symmetric using the restriction of the symmetrising form $\langle\ ,\ \rangle$ to B_i. For the reader's convenience we recall the easy argument. Indeed, it is clear that the restriction of the form to the block is again symmetric and associative. We need to show that the restriction of the form is non degenerate. Let

$x = xb \in B_i$ for $b^2 = b$ being the block idempotent of B_i. Then since the form on kG is non degenerate, there is $y \in kG$ such that $\langle x, y\rangle \neq 0$. But now,

$$0 \neq \langle x, y\rangle = \langle xb, y\rangle = \langle xb^2, y\rangle = \langle xb, by\rangle = \langle xb, yb\rangle$$

and $yb \in B_i$. This proves that the restriction of the symmetrising form to a block is a symmetrising form on this block. By the general argument we get again that $[B, B]^\perp = Z(B)$ for the restriction of the symmetrising form to B. Hence, the symmetrising form induces again a non degenerate pairing

$$Z(B) \times B/[B, B] \to k.$$

Since $\mu_p : kG/[kG, kG] \to kG/[kG, kG]$ restricts to $\mu_p(B_i) : B_i/[B_i, B_i] \to B_i/[B_i, B_i]$, we get that $M^m = \bigoplus_{i=0}^n M^m(B_i)$, where $M^m(B_i) = M^m \cdot b_i$, where b_i is the block idempotent of B_i. In the same way we may decompose the centre and the p^m-regular and the p^m-singular subspace of the centre. If we define

$$Z_{p'}^{(m)}(kG)b_i =: Z_{p'}^{(m)}(B_i) \text{ and } Z_p^{(m)}(kG)b_i =: Z_p^{(m)}(B_i)$$

for the block idempotent b_i of B_i, then we get

$$Z_{p'}^{(m)}(B_i) = Z_{p'}^{(m)}(kG) \cap B_i \text{ and } Z_p^{(m)}(B_i) = Z_p^{(m)}(kG) \cap B_i$$

and observe that

$$Z_{p'}^{(m)}(kG) = \bigoplus_{i=0}^n Z_{p'}^{(m)}(B_i) \text{ and } Z_p^{(m)}(kG) = \bigoplus_{i=0}^n Z_p^{(m)}(B_i).$$

A symmetrising form $\langle\ ,\rangle$ on B induces by restriction to the centre in the first component a non degenerate pairing

$$\langle\ ,\ \rangle : Z(B) \times B/[B, B] \to k.$$

4. p-power structure on the cocentre under derived equivalences of standard type

Fix a commutative ring k, and let A and B be symmetric k-algebras. In particular A and B are projective as k-modules. Suppose $D^b(A) \simeq D^b(B)$ and let ${}_AX_B$ be a twosided tilting complex, which we may assume with homogeneous components projective on either side (cf [26, Corollary 6.5.7]). Therefore we may replace the left derived tensor product by ordinary tensor products. Then, by a result of Rickard ([17] in connection with [18]) $Hom_k(X, k) =: Y$ is a twosided tilting complex inverse to X. Observe that for any k-algebra A we get a natural isomorphism

$$A/[A, A] \simeq A \otimes_{A\otimes A^{op}} A$$

given by multiplication of the factors. Then, $A \simeq X \otimes_B Y$ in the derived category of $A - A$ bimodules, and similarly $B \simeq Y \otimes_A X$ in the derived category of $B - B$ bimodules. Hence

$$A/[A, A] \simeq X \otimes_B Y \otimes_{A\otimes A^{op}} X \otimes_B Y$$

and

$$B/[B, B] \simeq Y \otimes_A X \otimes_{B\otimes B^{op}} Y \otimes_A X.$$

Therefore we may define a homomorphism of complexes (cf [24])

$$\begin{array}{ccc} (Y \otimes_A X) \otimes_{B\otimes B^{op}} (Y \otimes_A X) & \longrightarrow & (X \otimes_B Y) \otimes_{A\otimes A^{op}} (X \otimes_B Y) \\ (y_1 \otimes_A x_1) \otimes_{B\otimes B^{op}} (y_2 \otimes_A x_2) & \mapsto & (x_2 \otimes_B y_1) \otimes_{A\otimes A^{op}} (x_1 \otimes_B y_2) \end{array}$$

which becomes an isomorphism $A/[A, A] \xrightarrow{\varphi_X} B/[B, B]$ when regarded in the derived category of bimodules.

Remark 4.1. We did not verify if $Hom_k(\varphi_X, k) : Hom_k(B/[B, B], k) \to Hom_k(A/[A, A], k)$ maps to an algebra isomorphism $Z(B) \to Z(A)$ when we apply the isomorphism $Z(A) \to Hom_k(A/[A, A], k)$ and respectively $Z(B) \to Hom_k(B/[B, B], k)$ given by the symmetrising form on A and its image under the derived equivalence.

Suppose now that k is a perfect field of characteristic $p > 0$. Let B_G be a block of kG and let B_H be a block of kH for two finite groups G and H. Suppose that there is an equivalence $D^b(B_G) \simeq D^b(B_H)$ of triangulated categories and suppose that this equivalence is given by $X \otimes^{\mathbb{L}}_{B_G} -$ for some twosided tilting complex $X \in D^b(B_H \otimes B_G^{op})$ with inverse $Y \in D^b(B_G \otimes B_H^{op})$. By the above we may suppose that X and Y are projective as modules over B_G and over B_H and we can replace the left derived tensor product by the ordinary tensor product. We have

$$B_G \simeq Y \otimes_{B_H} X$$

as $B_G - B_G$-bimodules, and

$$B_H \simeq X \otimes_{B_G} Y$$

as $B_H - B_H$-bimodules, which holds in the respective derived categories. We showed in [23, 24] that the p-power map commutes with the above isomorphism on the degree 0 Hochschild homology given by the equivalence $X \otimes_{B_G} -$ of standard type.

Another isomorphism of Hochschild homology is given by a trace (or transfer) map generalising Hattori-Stallings traces, and described by Bouc. The approach is described in [26, Definition 5.9.16 and Definition 5.8.6]. We denote the corresponding map by

$$HH_0(A) \xrightarrow{tr_X} HH_0(B)$$

for a derived equivalence $X \otimes^{\mathbb{L}}_A - : D^b(A) \to D^b(B)$ of standard type. Linckelmann defined a transfer (trace) map on the Hochschild cohomology of symmetric algebras, and showed in [11, Remark 2.13] that this actually gives an isomorphism of the Hochschild cohomology algebras, including in degree 0. König, Liu and Zhou gave in [7] another definition of the transfer map on Hochschild cohomology, and showed that their transfer on the cohomology is basically the same as Linckelmann's. They further show in [7, Theorem 2.10] compatibility of the transfer map on Hochschild cohomology and Bouc's trace map on Hochschild homology, including in degree 0.

The p-power map on the degree 0 Hochschild homology is an invariant under derived equivalences of standard type as well, and this has the following consequence.

Proposition 4.2. *Let k be a perfect field of characteristic $p > 0$, and let G and H be two finite groups. Let B_G be a block of kG and let B_H be a block of kH. Suppose that $D^b(B_G) \simeq D^b(B_H)$ and let X be a two-sided tilting complex in $D^b(B_H \otimes B_G^{op})$, where we may suppose that X and its inverse Y are projective as B_G-modules, and as B_H-modules. Then the isomorphism*

$$B_G/[B_G, B_G] \xrightarrow{\varphi_X} B_H/[B_H, B_H]$$

and also Bouc's trace tr_X as map

$$B_G/[B_G, B_G] \xrightarrow{tr_X} B_H/[B_H, B_H]$$

map $M^m(B_G)$ bijectively to $M^m(B_H)$ for all $m \in \mathbb{N}$.

Proof. Denote by μ_p the p-power map. By the proof of [23, Theorem] and [7, Proposition 6.4] for the isomorphism tr_X related to the Hattori-Stallings trace map between the cocentres we see that

$$\begin{array}{ccc} HH_0(B_G) & \xrightarrow{\varphi_X} & HH_0(B_H) \\ {\scriptstyle \mu_p(B_G)}\downarrow & & \downarrow{\scriptstyle \mu_p(B_H)} \\ HH_0(B_G) & \xrightarrow[\varphi_X]{} & HH_0(B_H) \end{array} \qquad \begin{array}{ccc} HH_0(B_G) & \xrightarrow{tr_X} & HH_0(B_H) \\ {\scriptstyle \mu_p(B_G)}\downarrow & & \downarrow{\scriptstyle \mu_p(B_H)} \\ HH_0(B_G) & \xrightarrow[tr_X]{} & HH_0(B_H) \end{array}$$

are commutative diagrams. This shows the statement. ■

5. p-singular subspaces and derived equivalences

Let k be a perfect field of characteristic $p > 0$. We consider the p^m-singular subspace. Recall from Definition 2.1 that this space, denoted $Z_p^{(m)}(kG)$, is the subspace of kG generated by the conjugacy class sums $\underline{C_g}$ such that g does not have a p^m-th root. According to Section 3 we define the block version of it as $Z_p(B_i) = Z_p(kG) \cap B_i = Z_p(kG)b_i$, where B_i is a block of kG with block idempotent b_i.

Lemma 5.1. *Let k be a field of characteristic $p > 0$ and let G be a finite group. Let $\langle\ ,\ \rangle_G$ be the standard symmetrising form of kG, and let $\langle\ ,\ \rangle_i$ be its restriction to the block B_i. Then, taking orthogonal spaces with respect to the non degenerate pairing $Z(kG) \times kG/[kG, kG] \to k$, respectively $Z(B_i) \times B_i/[B_i, B_i] \to k$ with respect to these forms $\langle\ ,\ \rangle_G$, respectively $\langle\ ,\ \rangle_i$, we get $(M^m)^\perp = Z_p^{(m)}(kG)$, respectively $(M^m b_i)^\perp = Z_p^{(m)}(B_i)$.*

Proof. By Definition 2.1 and Lemma 2.6 we have

$$M^m =< g + [kG, kG] \mid \exists h \in G : h^{p^m} = g >_k .$$

In particular, using Lemma 2.6, we get $\bigcap_{t\in\mathbb{N}} M^t =< g + [kG, kG] \mid p \not| \ |g| >_k$. Now, $\langle h, g\rangle_G = 0$ for all g with $p \not| \ |g|$ if and only if $p|\ |h^{-1}|$. Since $|h| = |h^{-1}|$ this proves that, taking orthogonal spaces with respect to $\langle\ ,\ \rangle_G$, respectively $\langle\ ,\ \rangle_i$, we obtain $\left(\bigcap_{t\in\mathbb{N}} M^t\right)^\perp = Z_p(kG)$, and multiplying the equation by a block idempotent b_i shows that $\left(\bigcap_{t\in\mathbb{N}} M^t b_i\right)^\perp = Z_p(B_i)$. In the same way $(M^m)^\perp = Z_p^{(m)}(kG)$ and multiplying with the block idempotent, $(M^m b_i)^\perp = Z_p^{(m)}(B_i)$ ■

We shall now prove the main result of the paper.

Theorem 1. *Let k be a field of characteristic $p > 0$ and let G and H be finite groups. Let B_G be a block of kG and let B_H be a block of kH and suppose that $D^b(B_G) \simeq D^b(B_H)$. Then there is a unit $v \in Z(kH)$ such that the isomorphism tr^X induced by $\mathfrak{F}_X$ maps $Z_p^{(m)}(B_G)$ to $v \cdot Z_p^{(m)}(B_H)$ for all $m \in \mathbb{N}$. In particular, $tr^X(Z_p(B_G)) = v \cdot Z_p(B_H)$.*

Proof. We may suppose that there is a derived equivalence of standard type $X \otimes_{B_G} -$ with a two-sided tilting complex X, which is projective on either side. Let Y be its inverse, and we may assume again that also Y is projective on either side. By Proposition 4.2 we get that isomorphism $tr_X : B_G/[B_G, B_G] \to B_H/[B_H, B_H]$ satisfies

$$tr_X(M^m(B_G)) = M^m(B_H).$$

It was shown by Rickard [17] and generalised in [21] that $\mathfrak{F}_X^{bi} := X \otimes_{B_G} - \otimes_{B_G} Y$ maps the $B_G - B_G$-bimodule $Hom_k(B_G, k)$ to the $B_H - B_H$-bimodule $Hom_k(B_H, k)$. Moreover, this functor maps an isomorphism $B_G \overset{\alpha}{\to} Hom_k(B_G, k)$ of $B_G - B_G$-bimodules to an isomorphism $B_H \overset{\mathfrak{F}_X^{bi}(\alpha)}{\longrightarrow} Hom_k(B_H, k)$ of $B_H - B_H$-bimodules. If α is the isomorphism coming from the standard symmetrising form, then its image $\mathfrak{F}_X^{bi}(\alpha)$ induces another symmetrising form $\widetilde{\langle\ ,\ \rangle}_H$ on B_H. By Lemma 1.1 we obtain for all $x, y \in B_H$ that there is a unit $u \in Z(B_H)$ such that

$$\widetilde{\langle x, y\rangle}_H = \langle ux, y\rangle_H$$

for the standard symmetrising form $\langle\ ,\ \rangle_H$ on B_H. Again these forms induce non degenerate pairings between the centres and the cocentres of the blocks. If y ranges over $M^m(B_G)$, we consider the orthogonal spaces, by Lemma 5.1 we obtain $Z_p^{(m)}(B_G)$, and applying tr_X the standard form $\langle\ ,\ \rangle_G$ on B_G maps to $\widetilde{\langle\ ,\ \rangle}_H$ on B_H. The image of $M^m(B_G)$ under tr_X is $M^m(B_H)$, and hence $Z_p^{(m)}(B_G)$ is mapped to the orthogonal of $M^m(B_H)$ with respect to

$$\widetilde{\langle\ ,\ \rangle}_H = \langle u\cdot\ ,\ \rangle_H.$$

By Lemma 5.1 the orthogonal of $M^m(B_H)$ with respect to $\langle\ ,\ \rangle_H$ is $Z_p^{(m)}(B_H)$, whence, using Lemma 1.2, the orthogonal of $M^m(B_H)$ with respect to $\langle u\cdot\ ,\ \rangle_H$ is $u^{-1}\cdot Z_p^{(m)}(B_H)$. König, Liu and Zhou (cf [7, Theorem 2.10, Corollary 2.11]) show that the dual of tr_X with respect to the symmetrising forms is tr^X. The last statement follows by Remark 2.4. We proved the statement. ∎

Remark 5.2. The central unit v which appears in Theorem 1 comes from the construction of the image of a symmetrising form. If F is a derived equivalence of standard type, given by a twosided tilting complex X, between two blocks B and B', then an isomorphism $\lambda : B \to Hom_k(B,k)$ is mapped to $F\lambda : B' \to Hom_k(B',k)$ and if we consider the standard isomorphism λ and λ' for group rings, then $F\lambda$ differs from λ' by this central unit v. It would be most interesting to find out a method to determine v in terms of the twosided tilting complex X.

Remark 5.3. Let k be a field of characteristic $p > 0$ and let A be a symmetric k-algebra. Then we can define the $p^{(m)}$-singular subspace $Z_p^{(m)}(A)$ of the centre of A as the orthogonal of the image of the p^m-th power map on the cocentre of A. We then obtain analogous results as in Theorem 1.

Example 5.4. Let k be a field of characteristic $p > 0$ and let $A = N_n^{nm+1}$ be a symmetric Nakayama k-algebra given by a cyclic quiver Q with n vertices v_i, labelled by $i \in \mathbb{Z}/n\mathbb{Z}$, arrows $\alpha_i : v_i \longrightarrow v_{i+1}$ and relations R_m. Let $C_i := \alpha_i \dots \alpha_{i-1}$ for all $i \in \mathbb{Z}/n\mathbb{Z}$ be the full cycle of length m. Then R_m is generated by $\{C_i^m\alpha_i \mid i \in \mathbb{Z}/n\mathbb{Z}\}$. This is equivalent to saying that paths of length $mn + 1$ are 0. By e.g. [26, Proposition 5.10.36] blocks of group rings over algebraically closed base fields k with normal cyclic defect groups are algebras of this type. Let e_i be the lazy paths corresponding to the vertices, and let put $C := \sum_{i\in\mathbb{Z}/n\mathbb{Z}} C_i$. Then, following [19], $Z(A)$ has a k-basis given by C^s for $s \in \{0,\dots,m-1\}$ and C_i^m for $i \in \mathbb{Z}/n\mathbb{Z}$. A k-basis of $A/[A,A]$ is given by the classes of e_i for $i \in \mathbb{Z}/n\mathbb{Z}$ and C_1^s for $s \in \{1,\dots,m\}$. Following [5, 27], a non degenerate pairing $\langle\ ,\ \rangle : Z(A)\times A/[A,A] \to k$ given by a symmetrising form is then given by putting $\langle C^s, C_1^j\ \rangle = \delta_{s+j,m}$ and $\langle C_i^m, e_j\ \rangle = \delta_{s,j}$, using the Kronecker symbol δ. Hence,

$$M^t = \left\langle e_i; C_1^{sp^t} \mid i \in \mathbb{Z}/n\mathbb{Z}, 1 \le s \le \left\lfloor \frac{m}{p^t} \right\rfloor \right\rangle_k$$

and therefore

$$\left(M^t\right)^\perp = \bigcap_{i\in\mathbb{Z}/n\mathbb{Z}} e_i^\perp \cap \bigcap_{s=1}^{\left\lfloor \frac{m}{p^t} \right\rfloor} \left(C_1^{sp^t}\right)^\perp.$$

Clearly, $\bigcap_{i\in\mathbb{Z}/n\mathbb{Z}} e_i^\perp$ has a k-basis given by C^s for $0 \le s < m$. Moreover, inside this subspace, $\bigcap_{s=1}^{\left\lfloor \frac{m}{p^t} \right\rfloor} \left(C_1^{sp^t}\right)^\perp$ has k-basis given by those C^ℓ such that $p^t \not| (m-\ell)$. Therefore

$$Z_p^{(t)}(N_n^{mn+1}) = \langle C^\ell \mid p^t \not| (m-\ell)\rangle_k$$

and

$$Z_p(N_n^{mn+1}) = \langle C^\ell \mid 0 \le \ell \le m-1\rangle_k.$$

In particular, $Z_p^{(m)}$ is not an ideal of $Z(N_m^{nm+1})$ and hence we cannot get rid of the central unit which occurs in Theorem 1.

Example 5.5. We recall Example 2.8. We have $\bigcap M^t = k\cdot 1$ and $1^{\perp_{\langle\ ,\ \rangle_{alg}}} = k\cdot 1$ whereas $1^{\perp_{\langle\ ,\ \rangle_{grp}}} = k\cdot(1+X)$. Note that the forms

$$\langle\ ,\ \rangle_{grp} = \langle u\cdot\ ,\ \rangle_{alg}$$

differ by the central unit $u = (1+X)$, and conformably to the statement of Theorem 1 the orthogonal spaces do as well.

Remark 5.6. We note that if A is given by a quiver with (admissible) relations, and if the symmetrising form is given by [5, 27], then $rad(A)$ is nilpotent and hence $\bigcap_{t\in\mathbb{N}} M^t$ has a basis given by the primitive idempotents. $Z_p(A)$ can then be computed using the basis given in [5, Proposition 3.1]. This does not automatically imply that a basis of $Z_p(A)$ is given by all elements in $\mathcal{B}\setminus\mathcal{B}_s$, where $\mathcal{B}$ is the basis of A containing a basis $\mathcal{B}_s$ of the socle mentioned in [5, Proposition 3.1], since $\mathcal{B}$ is not an orthogonal basis in general. We remind the reader that [5, Proposition 3.1] only gives a Frobenius form, and only a necessary criterion for a symmetric form is given in [27]. The basis used for a symmetric form using the construction [5, Proposition 3.1] may involve linear combinations of paths rather than paths only.

6. Concluding remarks

A dual proof will lead to dual statements for a connection of p-singular elements in the cocentre. We do not elaborate on this in detail since we do not know any direct application. However, there is a Külshammer structure on the higher cocentres, i.e. the Hochschild homology (cf [24]). The natural generalisation of the present situation to Hochschild homology does not seem to make sense there, since the p-power map on the higher degree Hochschild homology does not preserve the degree. Hence we cannot expect immediately stabilisation results.

A different subject is nevertheless the Gerstenhaber bracket on Hochschild cohomology. Again in degrees strictly higher than 1 the p-power map of the restricted Lie algebra structure will not preserve degrees (cf [24]). However, the degree 1 Hochschild homology is a restricted Lie algebra, and it will be interesting to see the group theoretical applications and the outcome of an analogous study of the above arguments. There are quite a few technical difficulties, which we believe to be solvable. We intend to pursue this topic in subsequent work.

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