VERDIER QUOTIENTS OF HOMOTOPY CATEGORIES

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ABSTRACT. We study Verdier quotients of diverse homotopy categories of a full additive subcategory $\mathcal{E}$ of an abelian category. In particular, we consider the categories $K^{x,y}(\mathcal{E})$ for $x \in \{\infty, +, -, b\}$, and $y \in \{\emptyset, b, +, -, \infty\}$ the homotopy categories of left, right, unbounded complexes with homology being 0, bounded, left or right bounded, or unbounded. Inclusion of these categories give a partially ordered set, and we study localisation sequences or recollement diagrams between the Verdier quotients, and prove that many pairs of quotient categories identify.

INTRODUCTION

Let A be a left Noetherian ring. Denote by $A - mod$ the category of finitely generated A-modules, and let $D^b(A - mod)$ be the bounded derived category of finitely generated left A-modules. The full subcategory of perfect objects of $D^b(A - mod)$ is the homotopy category of bounded complexes of finitely generated projective modules $K^b(A - proj)$. Buchweitz [6], and independently Orlov [17] defined and studied the Verdier quotient

$$D^b_{sg}(A) := D^b(A - mod)/K^b(A - proj).$$

Orlov named this category the (bounded) singularity category of A. The singularity category has attracted a lot of interest in recent years (cf e.g.[9, 13, 25, 26, 21, 22, 23]).

If A is self-injective, then Rickard [18] and Keller-Vossieck [12] showed that $D^b_{sg}(A) \simeq A - \underline{mod}$ is the stable category of finitely generated A-modules modulo projective objects. If A has finite global dimension, then the singularity category vanishes. In general, however, if A is not self-injective, then the singularity category can be very complicated.

We note that the definition of the singularity category is very much linked to the case of finite dimensional algebras. Unbounded derived categories are more natural in many cases (cf e.g. [11, 19]). We note that there are many more possible alternative Verdier quotients of homotopy categories, and then the question is legitimate, which of them may lead to new quotients, and which can be identified. In this note we consider a full additive subcategory $\mathcal{E}$ of an abelian category $\mathcal{A}$ and consider the homotopy categories $K^{x,y}(\mathcal{E})$ of complexes of objects in $\mathcal{E}$, and of type x, and homology being in $\mathcal{A}$ of type y. Here x can be unbounded, right bounded, left bounded or bounded (written as $\infty, -, +, b$ respectively), and also the homology can take these types, or if we consider exact complexes, y is put to $\emptyset$. This gives a rather complicated Hasse diagram of subcategories (†) displayed in Section 1.3 below.

We consider Verdier quotients of inclusions in this Hasse diagram, and consider equivalences between them first abstractly, and then we consider as a special case when $\mathcal{E} = \mathcal{A}$ is abelian. We do not show the equivalences of the categories directly, but rather prove the existence of stable t-structures of the quotients as introduced by [9] which imply then isomorphisms of quotients. Simplifying the work, a result of Jørgensen and Kato [10] reduces the verification of stable t-structures $(\mathcal{X}/(\mathcal{X} \cap \mathcal{Y}), \mathcal{Y}/(\mathcal{X} \cap \mathcal{Y}))$ in $\mathcal{T}/(\mathcal{X} \cap \mathcal{Y})$ to a verification of the ambient category $\mathcal{T}$ being the full subcategory of $\mathcal{T}$ of middle terms of distinguished triangles $\mathcal{X} * \mathcal{Y}$ with precisely the end terms in $\mathcal{X}$ respectively $\mathcal{Y}$. Quotients involving exact complexes are more subtle than those given by other boundedness conditions. The appropriate additional hypothesis to be able to deal with this situation is that of an abelian category, instead of just an additive subcategory of an abelian category. In the case of additive subcategories of abelian categories we use the distinguished triangle given by brutal truncation, and in case of an abelian category we use a distinguished triangle given by intelligent

truncation. In case we can use both triangles we show that the parallelograms in the Hasse diagrams all split in the following sense: If $\mathcal{X}$ contains two triangulated subcategories $\mathcal{Y}$ and $\mathcal{Z}$ occurring in the Hasse diagram, and $\mathcal{X}$ is generated by $\mathcal{Y}$ and $\mathcal{Z}$, let $\mathcal{D} = \mathcal{X} \cap \mathcal{Y}$. Then $\mathcal{X}/\mathcal{D} \simeq \mathcal{Y}/\mathcal{D} \times \mathcal{Z}/\mathcal{D}$.

We consider more in detail the case of projective objects $\mathcal{E} = \mathcal{P}$, and of injective objects $\mathcal{E} = \mathcal{I}$ in $\mathcal{A}$, where we get more detailed statements from a result due to Krause [13]. We are able to obtain as a corollary previous cases which appeared in a paper of Iyama-Kato-Miyachi [9]. The case of Gorenstein algebras is particularly simple, as we will show in a special treatment.

We further prove that the most curious case of the homotopy category of right (resp. left) bounded complexes of an abelian category modulo those with bounded homology allows a computation of the morphisms between objects as limits of morphisms of intelligently truncated complexes in the derived category.

We finally give a set of examples showing that the various quotients actually differ in specific cases.

The paper is organised as follows. In Section 1 we recall some known facts concerning homotopy and derived categories of complexes, as well as some methods available for triangulated categories which can be used to prove the presence of (co)localisation sequences, recollement or ladder diagrams, namely Miyachi's concept of stable t-structures [15]. In Section 2 we prove equivalences between Verdier quotients of homotopy categories of additive subcategories of abelian categories, and the more precise statements for homotopy categories of abelian categories. In Section 3 we study more specifically the category of projective (resp. injective) objects of a module category, and in particular in the case of Iwanaga-Gorenstein rings. We can give more precise statements there, using work of Krause [13] and of Iyama, Kato and Miyachi [9]. In Section 4 we study our most curious case of the homotopy categories of right bounded complexes modulo those with bounded homology. Here we show that homomorphism spaces can be computed as limits of truncated complexes analogous to the case of classical singularity categories, shown by Beligiannis [2]. Section 5 then gives examples showing that the Verdier quotients for the cases which were not shown to be equivalent actually differ in general.

1. Review on homotopy and derived categories

1.1. **Truncations.** We begin by recalling two possible truncation operators for complexes. Let

$$X = (\cdots \overset{d^{n-2}}{\to} X^{n-1} \overset{d^{n-1}}{\to} X^n \overset{d^n}{\to} X^{n+1} \overset{d^{n+1}}{\to} \cdots)$$

be a complex over an abelian category. For each $n \in \mathbb{Z}$, denote

$$\tau_{\geq n}X = (\cdots \to 0 \to 0 \to X^n \overset{d^n}{\to} X^{n+1} \overset{d^{n+1}}{\to} \cdots),$$

$$\tau_{\leq n}X = (\cdots \overset{d^{n-2}}{\to} X^{n-1} \overset{d^{n-1}}{\to} X^n \to 0 \to 0 \to \cdots),$$

the brutal or stupid truncation and let

$$\sigma_{\geq n}X = (\cdots \to 0 \to \mathrm{Im}(d^{n-1}) \to X^n \overset{d^n}{\to} X^{n+1} \overset{d^{n+1}}{\to} \cdots),$$

$$\sigma_{\leq n}X = (\cdots \overset{d^{n-1}}{\to} X^n \overset{d^n}{\to} \mathrm{Im}(d^n) \to 0 \to 0 \to \cdots)$$

be the intelligent or subtle truncation. Obviously we have a short exact sequence of complexes for each $n \in \mathbb{Z}$:

$$0 \to \tau_{\geq n}X \to X \to \tau_{\leq n-1}X \to 0$$

which is degree-wise split, and hence gives a distinguished triangle

(1) $$\tau_{\geq n}X \to X \to \tau_{\leq n-1}X \to (\tau_{\geq n}X)[1]$$

in the homotopy category; we shall call it the *stupid triangle* in the sequel. Note that because of our definition of intelligent truncations, we have no short exact sequence of complexes of the form

$$0 \to \sigma_{\leq n}X \to X \to \sigma_{\geq n+1}X \to 0.$$

However, we will show that this becomes a distinguished triangle in the homotopy category, which will be used repeatedly in the sequel.

Lemma 1.1. *There is a distinguished triangle*

$$\sigma_{\leq n}X \to X \to \sigma_{\geq n+1}X \to (\sigma_{\leq n}X)[1] \tag{2}$$

whenever all complexes are in the same homotopy category. We shall call it the intelligent triangle in the sequel.

Proof. We have the natural inclusion of complexes $\sigma_{\leq n}X \to X$ and let C be its cone. We claim that $C \simeq \sigma_{\geq n+1}X$ in the homotopy category. Indeed, C is the complex

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & X^{n-1} & \xrightarrow{-d^{n-1}} & X^n & \xrightarrow{-d^n} & \operatorname{im}(d^n) & \xrightarrow{0} & 0 & \longrightarrow \cdots \\
& \oplus & {\scriptstyle 1} & \oplus & {\scriptstyle 1} & \oplus & & \oplus & \\
\cdots \longrightarrow & X^{n-2} & \xrightarrow{d^{n-2}} & X^{n-1} & \xrightarrow{d^{n-1}} & X^n & \xrightarrow{d^n} & X^{n+1} & \xrightarrow{d^{n+1}} \cdots
\end{array}$$

But, this is homotopy equivalent to the complex $\sigma_{\geq n+1}X$ given above. Indeed, $X^i \xrightarrow{1} X^i$ is a direct factor for all $i \leq n$. We illustrate the most difficult case $i = n$ and this follows from the commutative diagram below.

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & 0 & \longrightarrow & X^n & \xrightarrow{1} & X^n & \longrightarrow & 0 & \longrightarrow \cdots \\
& \uparrow & & \uparrow {\scriptstyle (1\ d^{n-1})} & & \uparrow {\scriptstyle (0\ 1)} & & \uparrow & \\
\cdots \xrightarrow{-d^{n-2}} & X^{n-1} & \xrightarrow{-d^{n-1}} & X^n & \xrightarrow{-d^n} & \operatorname{im}(d^n) & \xrightarrow{0} & 0 & \longrightarrow \cdots \\
& \oplus {\scriptstyle 1} & & \oplus {\scriptstyle 1} & & \oplus {\scriptstyle 1} & & \oplus & \\
\cdots \xrightarrow{d^{n-3}} & X^{n-2} & \xrightarrow{d^{n-2}} & X^{n-1} & \xrightarrow{d^{n-1}} & X^n & \xrightarrow{d^n} & X^{n+1} & \xrightarrow{d^{n+1}} \cdots \\
& \uparrow & & \uparrow {\scriptstyle \binom{1}{0}} & & \uparrow {\scriptstyle \binom{-d^n}{1}} & & \uparrow & \\
\cdots \longrightarrow & 0 & \longrightarrow & X^n & \xrightarrow{1} & X^n & \longrightarrow & 0 & \longrightarrow \cdots
\end{array}$$

The left bottom and the right upper squares are trivially commutative. Since

$$(0\ 1)\cdot \begin{pmatrix} -d^n & 0 \\ 1 & d^{n-1} \end{pmatrix} = (1\ d^{n-1}) = (1\ d^{n-1})\cdot 1$$

and

$$\begin{pmatrix} -d^n & 0 \\ 1 & d^{n-1} \end{pmatrix}\cdot\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -d^n \\ 1 \end{pmatrix} = \begin{pmatrix} -d^n \\ 1 \end{pmatrix}\cdot 1,$$

the middle squares are commutative. The lower right square and the upper left square are both commutative as well, as it might be checked by the diligent reader. This shows the statement. ∎

1.2. **Stable t-structures, (co)localisation sequences, recollements and ladders of triangulated categories.** Let $\mathcal{T}$ be a triangulated category. A triangulated subcategory $\mathcal{T}'$ of $\mathcal{T}$ is called a *thick subcategory* if it is closed under taking direct summands. Then we can define the Verdier quotient $\mathcal{T}/\mathcal{T}'$; see [20]. It is still a triangulated category, and the quotient functor $j^* : \mathcal{T} \to \mathcal{T}/\mathcal{T}'$ is a triangle functor. In this case, we will say that $\mathcal{T}' \xrightarrow{i_*} \mathcal{T} \xrightarrow{j^*} \mathcal{T}''$ is a *short exact sequence of triangulated categories*, where i_* is the inclusion functor and $\mathcal{T}'' = \mathcal{T}/\mathcal{T}'$.

Given a short exact sequence of triangulated categories $\mathcal{T}' \xrightarrow{i_*} \mathcal{T} \xrightarrow{j^*} \mathcal{T}/\mathcal{T}'$, if i_* and/or j^* have a left adjoint, denoted by i^* and $j_!$ respectively, then the diagram

$$\mathcal{T}' \underset{i_*}{\overset{i^*}{\rightleftarrows}} \mathcal{T} \underset{j^*}{\overset{j_!}{\rightleftarrows}} \mathcal{T}''$$

is called a *colocalisation sequence* or a *left recollement*. Dually, given a short exact sequence of triangulated categories $\mathcal{T}' \xrightarrow{i_*} \mathcal{T} \xrightarrow{j^*} \mathcal{T}/\mathcal{T}'$, if i_* and/or j^* have a right adjoint, denoted by i^* and $j_!$ respectively, then the diagram

$$\mathcal{T}' \underset{i^!}{\overset{i_*}{\rightleftarrows}} \mathcal{T} \underset{j_*}{\overset{j^*}{\rightleftarrows}} \mathcal{T}''$$

is called a *localisation sequence* or a *right recollement*. Interchanging $\mathcal{T}'$ and $\mathcal{T}''$, we see that the notions of colocalisation sequences and of localisation sequences are equivalent. When i_* and/or j^* have a left adjoint and a right adjoint, the diagram

$$\mathcal{T}' \overset{i^*}{\underset{i^!}{\rightleftarrows}} \mathcal{T} \overset{j_!}{\underset{j_*}{\rightleftarrows}} \mathcal{T}''$$

is called a *recollement*.

If in a recollement diagram, the functors i^* and/or $j_!$ have still left adjoints, we then have a ladder of height two

Dually we can also have a ladder of height two

if the functors $i^!$ and/or j_* have still right adjoints. Here the height refers to the number of recollements contained in the ladder. Of course one can extend upwards or downwards to obtain ladders of larger height. More generally a *ladder* ([4, 1.2]) is a finite or an infinite diagram of triangle functors:

such that any two consecutive rows form a left or right recollement (or equivalently, any three consecutive rows form a recollement or an opposed recollement) of $\mathcal{C}$ relative to $\mathcal{C}'$ and $\mathcal{C}''$. Its *height* is the number of rows minus 2. Ladders of height 0 (resp. 1) are exactly left or right recollements (resp. recollements).

When considering localisation sequences or recollement diagrams, the notion of stable t-structures is rather useful.

Definition 1.2. [15, Page 467] *Let $\mathcal{T}$ be a triangulated category with suspension functor* [1]. *A pair $(\mathcal{U}, \mathcal{V})$ of full subcategories of $\mathcal{T}$ is called a stable t-structure in $\mathcal{T}$ if*

(1) $\mathcal{U} = \mathcal{U}[1]$ *and* $\mathcal{V} = \mathcal{V}[1]$
(2) $Hom_{\mathcal{T}}(\mathcal{U}, \mathcal{V}) = 0$
(3) *for every object X of $\mathcal{T}$ there is a distinguished triangle $U \to X \to V \to U[1]$ for an object U of $\mathcal{U}$, and an object V of $\mathcal{V}$.*

Stable t-structures, localisations and recollements are closely related. Miyachi showed in the remarks preceding [15, Proposition 2.6] that if $(\mathcal{U}, \mathcal{V})$ is a stable t-structure of $\mathcal{T}$, then the natural inclusion $R : \mathcal{V} \to \mathcal{T}$ is right adjoint to a functor $Q : \mathcal{T} \to \mathcal{V}$ and the natural inclusion $R : \mathcal{U} \to \mathcal{T}$ is left adjoint to a functor $Q' : \mathcal{T} \to \mathcal{U}$ such that Q induces an equivalence $\mathcal{V} \simeq \mathcal{T}/\mathcal{U}$ and Q' induces an equivalence $\mathcal{U} \simeq \mathcal{T}/\mathcal{V}$. [15, Proposition 2.6] shows that stable t-structures is precisely the same as localisation sequences.

One of our main tools is the following result due to Jørgensen and Kato [10]. For a triangulated category $\mathcal{T}$ and two subcategories $\mathcal{X}$ and $\mathcal{Y}$ of $\mathcal{T}$ let $\mathcal{X} * \mathcal{Y}$ be the full subcategory of $\mathcal{T}$ having as objects those objects E of $\mathcal{T}$ such that there is a distinguished triangle $X \longrightarrow E \longrightarrow Y \longrightarrow X[1]$ with $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$. We want to stress however that it is not hard to avoid the use of Theorem 1.3 and to give independent elementary and short proofs of all statements where Theorem 1.3 is used in our paper.

Theorem 1.3. [10] *Let $\mathcal{T}$ be a triangulated category. Let $\mathcal{X}, \mathcal{Y}$ be two thick subcategories of $\mathcal{T}$, and denote $\mathcal{U} := \mathcal{X} \cap \mathcal{Y}$. Then $(\mathcal{X}/\mathcal{U}, \mathcal{Y}/\mathcal{U})$ is a stable t-structure in $\mathcal{T}/\mathcal{U}$ provided that $\mathcal{X} * \mathcal{Y} = \mathcal{T}$.*

If we have both $\mathcal{X} * \mathcal{Y} = \mathcal{T} = \mathcal{Y} * \mathcal{X}$, then there exists a splitting equivalence

$$\mathcal{T}/\mathcal{U} \simeq \mathcal{X}/\mathcal{U} \times \mathcal{Y}/\mathcal{U}.$$

Proof. The first statement is the first row of [10, Theorem B]. The second statement can be shown using [24, Lemma 3.4(2)] and [10, Theorem B] for $\mathbf{Z} = \mathbf{X}$. For the convenience of the reader, we give a simple independent proof using the first statement. Denote by $Q : \mathcal{T} \to \mathcal{T}/\mathcal{U}$ the natural functor. Since $(Q\mathcal{X}, Q\mathcal{Y})$ is a stable t-structure, any object QT in $\mathcal{T}/\mathcal{U}$ is middle term of a distinguished triangle $QX \to QT \to QY \to QX[1]$ with $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$. Since by the first statement $(Q\mathcal{Y}, Q\mathcal{X})$ is a stable t-structure, there is no non zero morphism from objects in $Q\mathcal{Y}$ to objects in $Q\mathcal{X}$, this triangle splits. ∎

The following statement will simplify our work in the sequel.

Lemma 1.4. *Let $\mathcal{T}_1$ be a triangulated category, and let $\mathcal{T}_2, \mathcal{T}_3, \mathcal{T}_4, \mathcal{T}_5, \mathcal{T}_6$ be triangulated subcategories such that $\mathcal{T}_4 = \mathcal{T}_3 \cap \mathcal{T}_2$ and $\mathcal{T}_6 = \mathcal{T}_4 \cap \mathcal{T}_5$.*

If $(\mathcal{T}_2/\mathcal{T}_4, \mathcal{T}_3/\mathcal{T}_4)$ is a stable t-structure in $\mathcal{T}_1/\mathcal{T}_4$, and if $(\mathcal{T}_4/\mathcal{T}_6, \mathcal{T}_5/\mathcal{T}_6)$ is a stable t-structure in $\mathcal{T}_3/\mathcal{T}_6$, then

$$(\mathcal{T}_2/\mathcal{T}_6, \mathcal{T}_5/\mathcal{T}_6)$$

is a stable t-structure in $\mathcal{T}_1/\mathcal{T}_6$.

Proof. Without loss of generality, we can assume that $\mathcal{T}_6 = 0$.

By Beilinson, Bernstein, Deligne [3, 1.3.10] the operation $*$ on subcategories is associative. Hence, by Theorem 1.3 we get $\mathcal{T}_1 = \mathcal{T}_2 * \mathcal{T}_3$ and $\mathcal{T}_3 = \mathcal{T}_4 * \mathcal{T}_5$. Then

$$\mathcal{T}_2 * \mathcal{T}_5 = (\mathcal{T}_2 * \mathcal{T}_4) * \mathcal{T}_5 = \mathcal{T}_2 * (\mathcal{T}_4 * \mathcal{T}_5) = \mathcal{T}_2 * \mathcal{T}_3 = \mathcal{T}_1.$$

Theorem 1.3 then shows the statement. ∎

Recall from [9, Definition 0.3] that a triple $(\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3)$ of full subcategories of a triangulated category $\mathcal{T}$ forms a *triangle of recollements* if $(\mathcal{U}_1, \mathcal{U}_2)$, $(\mathcal{U}_2, \mathcal{U}_3)$ and $(\mathcal{U}_3, \mathcal{U}_1)$ are stable t-structures in $\mathcal{T}$. If $(\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3)$ is a triangle of recollements, by [9, Proposition 1.10] we get then that there is a recollement diagram

$$\mathcal{U}_k \overset{\longleftarrow}{\underset{\longleftarrow}{\longrightarrow}} \mathcal{T} \overset{\longleftarrow}{\underset{\longleftarrow}{\longrightarrow}} \mathcal{T}/\mathcal{U}_k$$

for all $k \in \{1, 2, 3\}$.

1.3. **Various homotopy and derived categories.** From now on, let $\mathcal{E}$ be a full additive subcategory of an abelian category $\mathcal{A}$. Define the following full triangulated categories of the homotopy category $K(\mathcal{E})$:

- $K^{\infty,+}(\mathcal{E})$ consists of all unbounded complexes with left bounded cohomology groups;
- $K^{\infty,-}(\mathcal{E})$ consists of all unbounded complexes with right bounded cohomology groups;
- $K^{\infty,b}(\mathcal{E})$ consists of all unbounded complexes with bounded cohomology groups;
- $K^+(\mathcal{E})$ consists of all complexes homotopy equivalent to left bounded complexes;
- $K^-(\mathcal{E})$ consists of all complexes homotopy equivalent to right bounded complexes;

- $K^{+,b}(\mathcal{E})$ consists of all complexes homotopy equivalent to left bounded complexes with bounded cohomology groups;
- $K^{-,b}(\mathcal{E})$ consists of all complexes homotopy equivalent to right bounded complexes with bounded cohomology groups;
- $K^{\infty,\emptyset}(\mathcal{E})$ consists of all unbounded exact complexes;
- $K^{+,\emptyset}(\mathcal{E})$ consists of all complexes homotopy equivalent to left bounded exact complexes;
- $K^{-,\emptyset}(\mathcal{E})$ consists of all complexes homotopy equivalent to right bounded exact complexes;
- $K^{b}(\mathcal{E})$ consists of all complexes homotopy equivalent to bounded complexes;
- $K^{b,\emptyset}(\mathcal{E})$ consists of all complexes homotopy equivalent to bounded exact complexes.

When $\mathcal{E} = \mathcal{A}$, we also have the following derived categories which are triangulated subcategories of the unbounded derived category $D(\mathcal{A})$ (when they exists).

- $D^{+}(\mathcal{A})$ consists of all unbounded complexes with left bounded cohomology groups;
- $D^{-}(\mathcal{A})$ consists of all unbounded complexes with right bounded cohomology groups;
- $D^{b}(\mathcal{A})$ consists of all unbounded complexes with bounded cohomology groups.

We have the following diagram of inclusion functors of triangulated categories. Note that we use the convention that lower categories are subcategories of upper ones.

These homotopy categories form a rather complicated Hasse diagram (†):
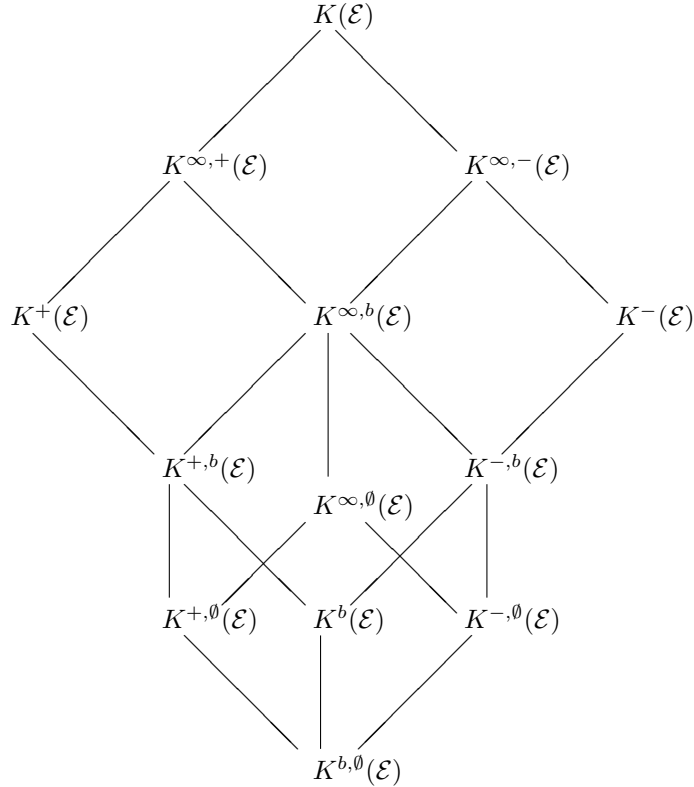
These homotopy categories and their various Verdier quotients play an important role in representation theory. By [16] the inclusions form thick subcategories in case $\mathcal{E}$ is Karoubian, i.e. every idempotent splits. Hence, the Verdier quotients of these homotopy categories are defined.

2. Quotients for general homotopy categories

We first consider the general case when $\mathcal{E}$ is an arbitrary full additive subcategory of an abelian category. In this case, we need to consider the second diagram (‡) which is a subdiagram of (†):

$$K(\mathcal{E})$$

$$K^{\infty,+}(\mathcal{E}) \qquad K^{\infty,-}(\mathcal{E})$$

$$K^{+}(\mathcal{E}) \qquad K^{\infty,b}(\mathcal{E}) \qquad K^{-}(\mathcal{E})$$

$$K^{+,b}(\mathcal{E}) \qquad K^{-,b}(\mathcal{E})$$

$$K^{b}(\mathcal{E})$$
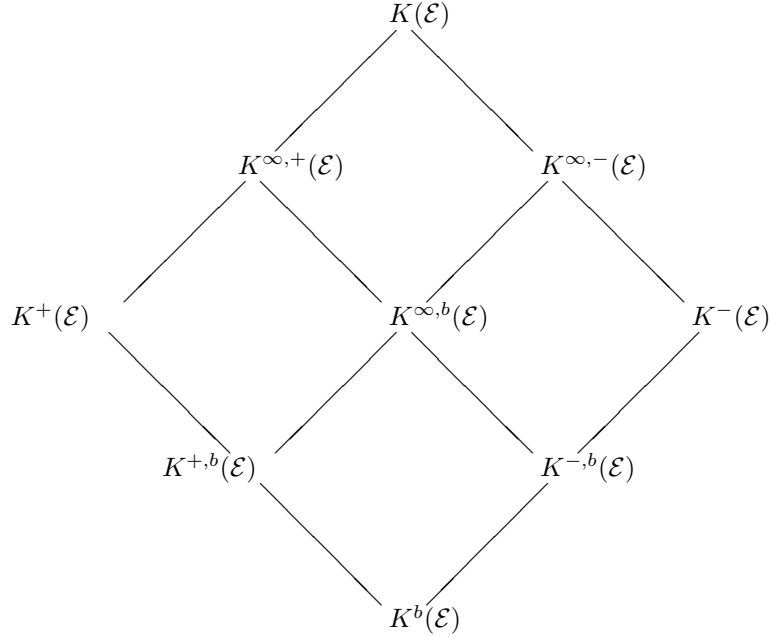
Theorem 2.1. *Let $\mathcal{E}$ be a full additive Karoubian subcategory of an abelian category $\mathcal{A}$. Then each parallelogram in the diagram ($\ddagger$) gives a stable t-structure, that is, there exist nine stable t-structures:*

(1) $(K^{+}(\mathcal{E})/K^{+,b}(\mathcal{E}), K^{\infty,b}(\mathcal{E})/K^{+,b}(\mathcal{E}))$ *in* $K^{\infty,+}(\mathcal{E})/K^{+,b}(\mathcal{E})$,

(2) $(K^{\infty,b}(\mathcal{E})/K^{-,b}(\mathcal{E}), K^{-}(\mathcal{E})/K^{-,b}(\mathcal{E}))$ *in* $K^{\infty,-}(\mathcal{E})/K^{-,b}(\mathcal{E})$,

(3) [9, Proposition 2.1] $(K^{+,b}(\mathcal{E})/K^{b}(\mathcal{E}), K^{-,b}(\mathcal{E})/K^{b}(\mathcal{E}))$ *in* $K^{\infty,b}(\mathcal{E})/K^{b}(\mathcal{E})$,

(4) $(K^{\infty,+}(\mathcal{E})/K^{\infty,b}(\mathcal{E}), K^{\infty,-}(\mathcal{E})/K^{\infty,b}(\mathcal{E}))$ *in* $K(\mathcal{E})/K^{\infty,b}(\mathcal{E})$,

(5) $(K^{+}(\mathcal{E})/K^{b}(\mathcal{E}), K^{-,b}(\mathcal{E})/K^{b}(\mathcal{E}))$ *in* $K^{\infty,+}(\mathcal{E})/K^{b}(\mathcal{E})$,

(6) $(K^{\infty,+}(\mathcal{E})/K^{-,b}(\mathcal{E}), K^{-}(\mathcal{E})/K^{-,b}(\mathcal{E}))$ *in* $K(\mathcal{E})/K^{-,b}(\mathcal{E})$,

(7) $(K^{+,b}(\mathcal{E})/K^{b}(\mathcal{E}), K^{-}(\mathcal{E})/K^{b}(\mathcal{E}))$ *in* $K^{\infty,-}(\mathcal{E})/K^{b}(\mathcal{E})$,

(8) $(K^{+}(\mathcal{E})/K^{+,b}(\mathcal{E}), K^{\infty,-}(\mathcal{E})/K^{+,b}(\mathcal{E}))$ *in* $K(\mathcal{E})/K^{+,b}(\mathcal{E})$,

(9) $(K^{+}(\mathcal{E})/K^{b}(\mathcal{E}), K^{-}(\mathcal{E})/K^{b}(\mathcal{E}))$ *in* $K(\mathcal{E})/K^{b}(\mathcal{E})$.

So we have nine localisation sequences (or right recollements)

(1) $\quad K^{+}(\mathcal{E})/K^{+,b}(\mathcal{E}) \rightleftarrows K^{\infty,+}(\mathcal{E})/K^{+,b}(\mathcal{E}) \rightleftarrows K^{\infty,b}(\mathcal{E})/K^{+,b}(\mathcal{E})$,

(2) $\quad K^{\infty,b}(\mathcal{E})/K^{-,b}(\mathcal{E}) \rightleftarrows K^{\infty,-}(\mathcal{E})/K^{-,b}(\mathcal{E}) \rightleftarrows K^{-}(\mathcal{E})/K^{-,b}(\mathcal{E})$,

(3) $\quad K^{+,b}(\mathcal{E})/K^{b}(\mathcal{E}) \rightleftarrows K^{\infty,b}(\mathcal{E})/K^{b}(\mathcal{E}) \rightleftarrows K^{-,b}(\mathcal{E})/K^{b}(\mathcal{E})$,

(4) $\quad K^{\infty,+}(\mathcal{E})/K^{\infty,b}(\mathcal{E}) \rightleftarrows K(\mathcal{E})/K^{\infty,b}(\mathcal{E}) \rightleftarrows K^{\infty,-}(\mathcal{E})/K^{\infty,b}(\mathcal{E})$,

(5) $\quad K^{+}(\mathcal{E})/K^{b}(\mathcal{E}) \rightleftarrows K^{\infty,+}(\mathcal{E})/K^{b}(\mathcal{E}) \rightleftarrows K^{-,b}(\mathcal{E})/K^{b}(\mathcal{E})$,

(6) $\quad K^{\infty,+}(\mathcal{E})/K^{-,b}(\mathcal{E}) \rightleftarrows K(\mathcal{E})/K^{-,b}(\mathcal{E}) \rightleftarrows K^{-}(\mathcal{E})/K^{-,b}(\mathcal{E})$,

(7) $\quad K^{+,b}(\mathcal{E})/K^{b}(\mathcal{E}) \rightleftarrows K^{\infty,-}(\mathcal{E})/K^{b}(\mathcal{E}) \rightleftarrows K^{-}(\mathcal{E})/K^{b}(\mathcal{E})$,

(8) $\quad K^{+}(\mathcal{E})/K^{+,b}(\mathcal{E}) \rightleftarrows K(\mathcal{E})/K^{+,b}(\mathcal{E}) \rightleftarrows K^{\infty,-}(\mathcal{E})/K^{+,b}(\mathcal{E})$,

(9) $\quad K^{+}(\mathcal{E})/K^{b}(\mathcal{E}) \rightleftarrows K(\mathcal{E})/K^{b}(\mathcal{E}) \rightleftarrows K^{-}(\mathcal{E})/K^{b}(\mathcal{E})$.

Proof. The localisation sequences follow from the stable t-structures, so we only need to show the existence of the nine stable t-structures. For last five ones, they can be deduced from the first four using Lemma 1.4 and its dual, so we only need to consider the first four stable t-structures. The third stable *t*-structure was proved in [9, Proposition 2.1]. We will see that this statement can be reproved by our method as well.

For a complex X, consider the stupid triangle

$$\tau_{\geq 1}X \to X \to \tau_{\leq 0}X \to \tau_{\geq 1}X[1]$$

in $K(\mathcal{E})$.
- When $X \in K^{\infty,+}(\mathcal{E})$, $\tau_{\geq 1}X \in K^{+}(\mathcal{E})$ and $\tau_{\leq 0}X \in K^{\infty,b}(\mathcal{E})$, whence the first stable t-structure, using Theorem 1.3;
- when $X \in K^{\infty,-}(\mathcal{E})$, $\tau_{\geq 1}X \in K^{\infty,b}(\mathcal{E})$ and $\tau_{\leq 0}X \in K^{-}(\mathcal{E})$, whence the second stable t-structure, using Theorem 1.3;
- when $X \in K^{\infty,b}(\mathcal{E})$, $\tau_{\geq 1}X \in K^{+,b}(\mathcal{E})$ and $\tau_{\leq 0}X \in K^{-,b}(\mathcal{E})$, whence the third stable t-structure, using Theorem 1.3;
- when $X \in K(\mathcal{E})$, $\tau_{\geq 1}X \in K^{+}(\mathcal{E}) \subseteq K^{\infty,+}(\mathcal{E})$ and $\tau_{\leq 0}X \in K^{-}(\mathcal{E}) \subseteq K^{\infty,-}(\mathcal{E})$, whence the fourth stable t-structure, using Theorem 1.3.

We finished the proof. ∎

When $\mathcal{E} = \mathcal{A}$, we can actually show much stronger results and even for the larger diagram (†).

Theorem 2.2. *If $\mathcal{E} = \mathcal{A}$ is an abelian category, then each parallelogram in the diagram (‡) gives a splitting equivalence and each parallelogram in the rest of the diagram (†) gives a stable t-structure. More precisely, we have the following nine triangle equivalences:*

(1) $K^{\infty,+}(\mathcal{A})/K^{+,b}(\mathcal{A}) \simeq K^{+}(\mathcal{A})/K^{+,b}(\mathcal{A}) \times K^{\infty,b}(\mathcal{A})/K^{+,b}(\mathcal{A})$,
(2) $K^{\infty,-}(\mathcal{A})/K^{-,b}(\mathcal{A}) \simeq K^{\infty,b}(\mathcal{A})/K^{-,b}(\mathcal{A}) \times K^{-}(\mathcal{A})/K^{-,b}(\mathcal{A})$,
(3) $K^{\infty,b}(\mathcal{A})/K^{b}(\mathcal{A}) \simeq K^{+,b}(\mathcal{A})/K^{b}(\mathcal{A}) \times K^{-,b}(\mathcal{A})/K^{b}(\mathcal{A})$,
(4) $K(\mathcal{A})/K^{\infty,b}(\mathcal{A}) \simeq K^{\infty,+}(\mathcal{A})/K^{\infty,b}(\mathcal{A}) \times K^{\infty,-}(\mathcal{A})/K^{\infty,b}(\mathcal{A})$,
(5) $K^{\infty,+}(\mathcal{A})/K^{b}(\mathcal{A}) \simeq K^{+}(\mathcal{A})/K^{b}(\mathcal{A}) \times K^{-,b}(\mathcal{A})/K^{b}(\mathcal{A})$,
(6) $K^{\infty,-}(\mathcal{A})/K^{b}(\mathcal{A}) \simeq K^{+,b}(\mathcal{A})/K^{b}(\mathcal{A}) \times K^{-}(\mathcal{A})/K^{b}(\mathcal{A})$,
(7) $K(\mathcal{A})/K^{-,b}(\mathcal{A}) \simeq K^{\infty,+}(\mathcal{A})/K^{-,b}(\mathcal{A}) \times K^{-}(\mathcal{A})/K^{-,b}(\mathcal{A})$,
(8) $K(\mathcal{A})/K^{+,b}(\mathcal{A}) \simeq K^{+}(\mathcal{A})/K^{+,b}(\mathcal{A}) \times K^{\infty,-}(\mathcal{A})/K^{+,b}(\mathcal{A})$,
(9) $K(\mathcal{A})/K^{b}(\mathcal{A}) \simeq K^{+}(\mathcal{A})/K^{b}(\mathcal{A}) \times K^{-}(\mathcal{A})/K^{b}(\mathcal{A})$.

and there are five stable t-structures:

(a) $(K^{-,\emptyset}(\mathcal{A})/K^{b,\emptyset}(\mathcal{A}), K^{+,\emptyset}(\mathcal{A})/K^{b,\emptyset}(\mathcal{A}))$ *in* $K^{\infty,\emptyset}(\mathcal{A})/K^{b,\emptyset}(\mathcal{A})$,
(b) $(K^{\infty,\emptyset}(\mathcal{A})/K^{+,\emptyset}(\mathcal{A}), K^{+,b}(\mathcal{A})/K^{+,\emptyset}(\mathcal{A}))$ *in* $K^{\infty,b}(\mathcal{A})/K^{+,\emptyset}(\mathcal{A})$,
(c) $(K^{-,b}(\mathcal{A})/K^{-,\emptyset}(\mathcal{A}), K^{\infty,\emptyset}(\mathcal{A})/K^{-,\emptyset}(\mathcal{A}))$ *in* $K^{\infty,b}(\mathcal{A})/K^{-,\emptyset}(\mathcal{A})$,
(d) $(K^{b}(\mathcal{A})/K^{b,\emptyset}(\mathcal{A}), K^{+,\emptyset}(\mathcal{A})/K^{b,\emptyset}(\mathcal{A}))$ *in* $K^{+,b}(\mathcal{A})/K^{b,\emptyset}(\mathcal{A})$,
(e) $(K^{-,\emptyset}(\mathcal{A})/K^{b,\emptyset}(\mathcal{A}), K^{b}(\mathcal{A})/K^{b,\emptyset}(\mathcal{A}))$ *in* $K^{-,b}(\mathcal{A})/K^{b,\emptyset}(\mathcal{A})$,

Proof. In order to show the nine triangle equivalences (1)-(9), by Theorem 1.3 and Lemma 1.4 we only need to prove the first four equivalences.
- Consider (1). Given X in $K^{\infty,+}(\mathcal{A})$, we have that $\sigma_{\leq 0}X \in K^{-,\emptyset}(\mathcal{A})$ and $\sigma_{\geq 1}X \in K^{+}(\mathcal{A})$. Hence, the intelligent triangle from Lemma 1.1 and (1) from Theorem 2.1 show that the hypothesis of shows that the hypothesis of Theorem 1.3 are verified.
- The point (2) follows from (1) considering the opposite category.
- Consider (3) (resp. (4)). Given X in $K^{\infty,b}(\mathcal{A})$ (resp. $K(\mathcal{A})$), we have that $\sigma_{\leq 0}X \in K^{-,b}(\mathcal{A})$ (resp. $K^{-}(\mathcal{A})$) and $\sigma_{\geq 1}X \in K^{+,b}(\mathcal{A})$ (resp. $K^{+}(\mathcal{A})$). Again, the intelligent triangle from Lemma 1.1 and point (3) (resp. (4)) from Theorem 2.1 shows that the hypothesis of Theorem 1.3 are verified.

Consider the five stable t-structures (a)-(e). We will again apply Theorem 1.3 and the intelligent triangle from Lemma 1.1.
- If X is in $K^{\infty,\emptyset}(\mathcal{A})$, then $\sigma_{\leq 0}X \in K^{-,\emptyset}(\mathcal{A})$ and $\sigma_{\geq 1}X \in K^{+,\emptyset}(\mathcal{A})$. This shows (a).
- If X is in $K^{\infty,b}(\mathcal{A})$, then its homology is bounded and for small enough n we get $\sigma_{\leq n}X \in K^{-,\emptyset}(\mathcal{A})$ and $\sigma_{\geq n+1}X \in K^{+,b}(\mathcal{A})$. This shows (b).
- If X is in $K^{+,b}(\mathcal{A})$, then its homology is bounded and for large enough n we get $\sigma_{\leq n}X \in K^{b}(\mathcal{A})$ and $\sigma_{\geq n+1}X \in K^{+,\emptyset}(\mathcal{A})$. This shows (d).
- Finally, point (c) follows from (b) and (e) follows from (d) considering the opposite category.

∎

Corollary 2.3. *Let $\mathcal{E}$ be a full additive Karoubian subcategory of an abelian category $\mathcal{A}$. There are triangle equivalences*

(i) $K^{\infty,+}(\mathcal{E})/K^{+}(\mathcal{E}) \simeq K^{\infty,b}(\mathcal{E})/K^{+,b}(\mathcal{E}) \simeq K^{-,b}(\mathcal{E})/K^{b}(\mathcal{E})$,
(ii) $K^{\infty,-}(\mathcal{E})/K^{-}(\mathcal{E}) \simeq K^{\infty,b}(\mathcal{E})/K^{-,b}(\mathcal{E}) \simeq K^{+,b}(\mathcal{E})/K^{b}(\mathcal{E})$,
(iii) $K(\mathcal{E})/K^{\infty,-}(\mathcal{E}) \simeq K^{\infty,+}(\mathcal{E})/K^{\infty,b}(\mathcal{E}) \simeq K^{+}(\mathcal{E})/K^{+,b}(\mathcal{E})$,
(iv) $K(\mathcal{E})/K^{\infty,+}(\mathcal{E}) \simeq K^{\infty,-}(\mathcal{E})/K^{\infty,b}(\mathcal{E}) \simeq K^{-}(\mathcal{E})/K^{-,b}(\mathcal{E})$,
(v) $K(\mathcal{E})/K^{-}(\mathcal{E}) \simeq K^{\infty,+}(\mathcal{E})/K^{-,b}(\mathcal{E}) \simeq K^{+}(\mathcal{E})/K^{b}(\mathcal{E})$,
(vi) $K(\mathcal{E})/K^{+}(\mathcal{E}) \simeq K^{\infty,-}(\mathcal{E})/K^{+,b}(\mathcal{E}) \simeq K^{-}(\mathcal{E})/K^{b}(\mathcal{E})$.

If $\mathcal{E} = \mathcal{A}$, then

(1) $K^{+,b}(\mathcal{E})/K^{+,\emptyset}(\mathcal{E}) \simeq K^{b}(\mathcal{E})/K^{b,\emptyset}(\mathcal{E}) \simeq K^{-,b}(\mathcal{E})/K^{-,\emptyset}(\mathcal{E}) \simeq K^{\infty,b}(\mathcal{E})/K^{\infty,\emptyset}(\mathcal{E})$,
(2) $K^{\infty,b}(\mathcal{E})/K^{+,b}(\mathcal{E}) \simeq K^{\infty,\emptyset}(\mathcal{E})/K^{+,\emptyset}(\mathcal{E}) \simeq K^{-,\emptyset}(\mathcal{E})/K^{b,\emptyset}(\mathcal{E})$,
(3) $K^{\infty,b}(\mathcal{E})/K^{-,b}(\mathcal{E}) \simeq K^{\infty,\emptyset}(\mathcal{E})/K^{-,\emptyset}(\mathcal{E}) \simeq K^{+,\emptyset}(\mathcal{E})/K^{b,\emptyset}(\mathcal{E})$.

Indeed, this follows from the fact that stable t-structures yield equivalences of the corresponding quotient categories.

The third diagram ($\natural$) consists of derived categories:

$$D(\mathcal{A})$$

$$D^{+}(\mathcal{A}) \qquad\qquad D^{-}(\mathcal{A})$$

$$D^{b}(\mathcal{A})$$

For this diagram ($\natural$), we can also show a splitting equivalence.

Proposition 2.4. *For an abelian category $\mathcal{A}$, there is a triangle equivalence*

$$D(\mathcal{A})/D^{b}(\mathcal{A}) \simeq D^{+}(\mathcal{A})/D^{b}(\mathcal{A}) \times D^{-}(\mathcal{A})/D^{b}(\mathcal{A}),$$

whenever these categories exist.

Proof. By Theorem 1.3, the result follows from the two distinguished triangles (1) and (2) from Lemma 1.1 and the remark preceding it. ∎

The results of this section suggest that we need to consider the quotient categories in the following definition.

Definition 2.5. *Let $\mathcal{E}$ be an additive subcategory of an abelian category $\mathcal{A}$. Then*

- – $K^{-}(\mathcal{E})/K^{b}(\mathcal{E}) =: D^{-}_{sg}(\mathcal{E})$ *is the* right bounded singularity category.
  – $K^{+}(\mathcal{E})/K^{b}(\mathcal{E}) =: D^{+}_{sg}(\mathcal{E})$ *is the* left bounded singularity category.
- – $K^{-,b}(\mathcal{E})/K^{b}(\mathcal{E}) =: D^{-,b}_{sg}(\mathcal{E})$ *is the* right bounded and homologically bounded singularity category.
  – $K^{+,b}(\mathcal{E})/K^{b}(\mathcal{E}) =: D^{+,b}_{sg}(\mathcal{E})$ *is the* left bounded and homologically bounded singularity category.
- – $K^{\infty,b}(\mathcal{E})/K^{b}(\mathcal{E}) =: D^{\infty,b}_{sg}(\mathcal{E})$ *is the* homologically bounded singularity category.
  – $K^{\infty,-}(\mathcal{E})/K^{b}(\mathcal{E}) =: D^{\infty,-}_{sg}(\mathcal{E})$ *is the* homologically right bounded singularity category.
  – $K^{\infty,+}(\mathcal{E})/K^{b}(\mathcal{E}) =: D^{\infty,+}_{sg}(\mathcal{E})$ *is the* homologically left bounded singularity category.
- $K(\mathcal{E})/K^{b}(\mathcal{E}) =: D^{\infty}_{sg}(\mathcal{E})$ *is the* unbounded singularity category.
- – $K^{-}(\mathcal{E})/K^{-,b}(\mathcal{E}) =: D^{-}_{\infty}(\mathcal{E})$ *is the* right bounded and homologically infinite category.
  – $K^{+}(\mathcal{E})/K^{+,b}(\mathcal{E}) =: D^{+}_{\infty}(\mathcal{E})$ *is the* left bounded and homologically infinite category.

Remark 2.6. When $\mathcal{E}$ is the category of finitely generated projective A-modules for a Noetherian algebra A, then the right bounded and cohomologically bounded singularity category of $\mathcal{E}$

$$D^{-,b}_{sg}(\mathcal{E}) = K^{-,b}(\mathcal{E})/K^{b}(\mathcal{E}) = D^{b}_{sg}(\mathcal{A})$$

is the usual singularity category of $\mathcal{A}$ as defined by Buchweitz and Orlov.

The most curious categories seem to be $D^{-}_{\infty}(\mathcal{E})$ and $D^{+}_{\infty}(\mathcal{E})$.

3. THE SPECIAL CASE OF THE CATEGORY OF INJECTIVES, PROJECTIVES, AND THE GORENSTEIN SITUATION

3.1. **Quotients for homotopy categories of injectives.** Let $\mathcal{A}$ be an abelian category with enough injectives and $\mathcal{E} = \mathcal{I}$ be the full subcategory of injective objects. Since we are dealing with the subcategory of injectives, left bounded exact complexes are homotopy equivalent to 0. Hence $K^{b,\emptyset}(\mathcal{I}) = K^{+,\emptyset}(\mathcal{I}) = 0$. In this case, we may reduce the diagram (†) to a smaller one:
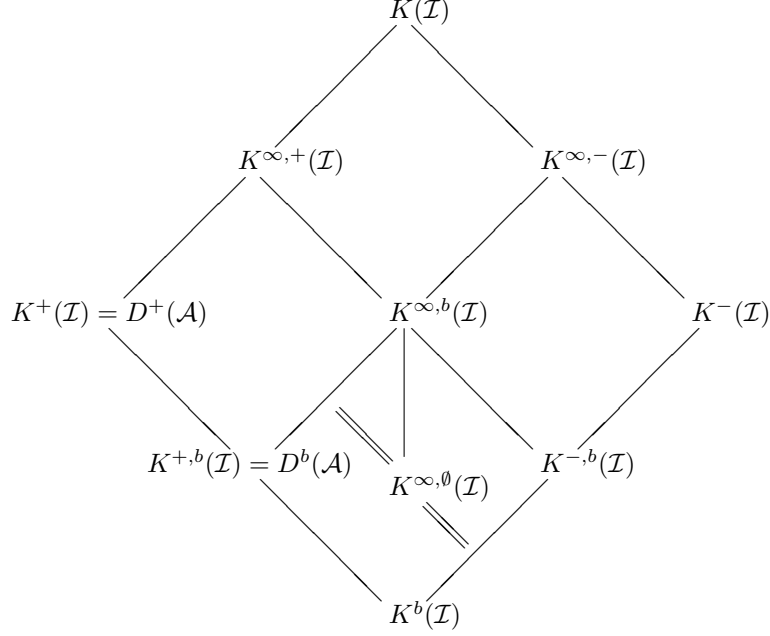
An interesting fact is that now $K^{\infty,\emptyset}(\mathcal{I})$ is equivalent to $K^{-,b}(\mathcal{I})/K^b(\mathcal{I})$ hence also to the quotients $K^{\infty,b}(\mathcal{I})/K^{+,b}(\mathcal{I})$ and $K^{\infty,+}(\mathcal{I})/K^+(\mathcal{I})$, as indicated in the diagram. More precisely, Iyama, Kato and Miyachi [9, Proposition 2.2] give a localisation sequence

$$K^{\infty,\emptyset}(\mathcal{I}) \rightleftarrows K^{\infty,b}(\mathcal{I}) \rightleftarrows K^{+,b}(\mathcal{I}) \ ,$$

and they also show in Theorem 2.4(i) of *loc. cit.* that there exists a recollement

$$K^{+,b}(\mathcal{I})/K^b(\mathcal{I}) \rightleftarrows K^{\infty,b}(\mathcal{I})/K^b(\mathcal{I}) \rightleftarrows K^{\infty,b}(\mathcal{I})/K^{+,b}(\mathcal{I})$$

Here $K^{\infty,b}(\mathcal{I})/K^{+,b}(\mathcal{I}) \simeq K^{\infty,\emptyset}(\mathcal{I}) \simeq K^{-,b}(\mathcal{I})/K^b(\mathcal{I})$.

Krause's result [13, Corollary 4.3] says that for $\mathcal{A}$ a locally Noetherian Grothendieck category, there exists a recollement

$$K^{\infty,\emptyset}(\mathcal{I}) \rightleftarrows K(\mathcal{I}) \rightleftarrows D(\mathcal{A}) \ ,$$

provided $D(\mathcal{A})$ is compactly generated, in particular when $\mathcal{A} = R - Mod$ for a left Noetherian ring.

We shall use [13, Corollary 4.3] to strengthen and generalise Iyama, Kato and Miyachi's results [9, Proposition 2.2 and Theorem 2.4(1)].

We need a small observation similar to [9, Proposition 2.2(1)].

Lemma 3.1. *Let $\mathcal{A}$ be an abelian category with enough injectives and $\mathcal{I}$ be the full subcategory of injective objects. Then*

- *$(K^{\infty,\emptyset}(\mathcal{I}), K^+(\mathcal{I}))$ is a stable t-structure in $K^{\infty,+}(\mathcal{I})$. So*
- *$(K^{\infty,\emptyset}(\mathcal{I}), K^+(\mathcal{I})/K^b(\mathcal{I}))$ is a stable t-structure in $K^{\infty,+}(\mathcal{I})/K^b(\mathcal{I})$ and*
- *$(K^{\infty,\emptyset}(\mathcal{I}), K^+(\mathcal{I})/K^{+,b}(\mathcal{I}))$ is a stable t-structure in $K^{\infty,+}(\mathcal{I})/K^{+,b}(\mathcal{I})$.*

Proof. For the first statement, given an $X \in K^{\infty,+}(\mathcal{I})$, for $n \ll 0$ the intelligent triangle (2)

$$\sigma_{\leq n}X \to X \to \sigma_{\geq n+1}X \to \sigma_{\leq n}X[1]$$

shows that $K^{\infty,\emptyset}(\mathcal{I}) * K^+(\mathcal{I}) = K^{\infty,+}(\mathcal{I})$. Since $K^{\infty,\emptyset}(\mathcal{I}) \cap K^+(\mathcal{I}) = K^{+,\emptyset}(\mathcal{I}) = 0$, by Theorem 1.3, $(K^{\infty,\emptyset}(\mathcal{I}), K^+(\mathcal{I}))$ is a stable t-structure in $K^{\infty,+}(\mathcal{I})$.

The second and the third statements follow from the first and [9, Proposition 1.5]. ∎

Proposition 3.2. *Let $\mathcal{A}$ be an abelian category satisfying Ab4* with enough injectives and $\mathcal{I}$ be the full subcategory of injective objects. Then we have localisation sequences*

(1) [9, Proposition 2.2] $K^{\infty,\emptyset}(\mathcal{I}) \rightleftarrows K^{\infty,b}(\mathcal{I}) \rightleftarrows D^b(\mathcal{A})$,

(2) $K^{\infty,\emptyset}(\mathcal{I}) \rightleftarrows K^{\infty,+}(\mathcal{I}) \rightleftarrows D^+(\mathcal{A})$,

(3) $K^{\infty,\emptyset}(\mathcal{I}) \rightleftarrows K^{\infty,-}(\mathcal{I}) \rightleftarrows D^-(\mathcal{A})$,

(4) $K^{\infty,\emptyset}(\mathcal{I}) \rightleftarrows K(\mathcal{I}) \rightleftarrows D(\mathcal{A})$,

where the localisation sequences (1), (2) and (3) are restrictions of the localisation sequence (4).

If moreover $\mathcal{A}$ is $R-Mod$ for some left Noetherian ring R, then the above localisation sequences become recollements

(1') $K^{\infty,\emptyset}(\mathcal{I}) \rightleftarrows K^{\infty,b}(\mathcal{I}) \rightleftarrows D^b(\mathcal{A})$,

(2') $K^{\infty,\emptyset}(\mathcal{I}) \rightleftarrows K^{\infty,+}(\mathcal{I}) \rightleftarrows D^+(\mathcal{A})$,

(3') $K^{\infty,\emptyset}(\mathcal{I}) \rightleftarrows K^{\infty,-}(\mathcal{I}) \rightleftarrows D^-(\mathcal{A})$,

(4') [13, Corollary 4.3] $K^{\infty,\emptyset}(\mathcal{I}) \rightleftarrows K(\mathcal{I}) \rightleftarrows D(\mathcal{A})$,

where the recollements (1'), (2') and (3') are restrictions of the recollements (4').

Proof. The localisation sequence (1) is [9, Proposition 2.2] and (2) follows from the first statement of Lemma 3.1. However, we will give an alternative proof.

The localisation sequence (4)

$$K^{\infty,\emptyset}(\mathcal{I}) \underset{\overset{i^!}{\longleftarrow}}{\overset{i_*}{\longrightarrow}} K(\mathcal{I}) \underset{\overset{j_*}{\longleftarrow}}{\overset{j^*}{\longrightarrow}} D(\mathcal{A})$$

is well known, where the functor $i_* : K^{\infty,\emptyset}(\mathcal{I}) \to K(\mathcal{I})$ is the inclusion functor, the functor $j^* : K(\mathcal{I}) \to D(\mathcal{A})$ is the composition $K(\mathcal{I}) \to K(\mathcal{A}) \to D(\mathcal{A})$ and the functor $j_* : D(\mathcal{A}) \to K(\mathcal{I})$ is taking DG-injective resolution. Obviously $(j^*)^{-1}(D^\pm(\mathcal{A})) = K^{\infty,\pm}(\mathcal{I})$ and $(j^*)^{-1}(D^b(\mathcal{A})) = K^{\infty,b}(\mathcal{I})$. Moreover, the functor j_* sends $D^\pm(\mathcal{A})$ (resp. $D^b(\mathcal{A})$) to $K^{\infty,\pm}(\mathcal{I})$ (resp. $K^{\infty,b}(\mathcal{I})$). Hence, we have the localisation sequences (1)(2)(3).

The category $\mathcal{A} = R - Mod$ is a locally Noetherian Grothendieck category, and hence we can apply Krause's recollement (4')

$$K^{\infty,\emptyset}(\mathcal{I}) \underset{\overset{i^!}{\longleftarrow}}{\overset{\overset{i^*}{\longleftarrow}}{\overset{i_*}{\longrightarrow}}} K(\mathcal{I}) \underset{\overset{j_*}{\longleftarrow}}{\overset{\overset{j_!}{\longleftarrow}}{\overset{j^*}{\longrightarrow}}} D(\mathcal{A}) \ ,$$

which extends the localisation sequence (4), and the fully faithful functor $j_! : D(\mathcal{A}) \to K(\mathcal{I})$ identifies $D(\mathcal{A})$ with the localising subcategory of $K(\mathcal{I})$ generated by $\mathbf{i}A, A \in D(\mathcal{A})^c = K^b(R - proj)$, where $\mathbf{i}A$ is an injective resolution of A and $D(\mathcal{A})^c$ is the full subcategory of compact objects of $D(\mathcal{A})$ which is exactly $K^b(R - proj)$. We claim that $j_!$ sends $D^\pm(\mathcal{A})$ (resp. $D^b(\mathcal{A})$) to $K^{\infty,\pm}(\mathcal{I})$ (resp. $K^{\infty,b}(\mathcal{I})$). Indeed,

$$\begin{aligned}
H^n(X) &\simeq Hom_{D(R-Mod)}(R, X[n]) \\
&\simeq Hom_{K(\mathcal{I})}(j_!R, j_!X[n]) \\
&\simeq Hom_{K(\mathcal{I})}(\mathbf{i}R, j_!X[n]) \\
&\simeq Hom_{K(R-Mod)}(R, j_!X[n]) \\
&\simeq H^n(j_!X),
\end{aligned}$$

where the second isomorphism holds because $j_!$ is fully faithful and the fourth follows from [13, Lemma 2.1]. This shows that the homology behaviour is preserved under $j_!$. So we have the three recollements (1')(2')(3'). ∎

Proposition 3.3. *Let $\mathcal{A}$ be an abelian category with enough injectives and $\mathcal{I}$ the full subcategory of injective objects. Then there are recollements*

(i) [9, Theorem 2.4(1)] $K^{+,b}(\mathcal{I})/K^b(\mathcal{I}) \rightleftarrows K^{\infty,b}(\mathcal{I})/K^b(\mathcal{I}) \rightleftarrows K^{\infty,b}(\mathcal{I})/K^{+,b}(\mathcal{I})$,

(ii) $K^+(\mathcal{I})/K^b(\mathcal{I}) \rightleftarrows K^{\infty,+}(\mathcal{I})/K^b(\mathcal{I}) \rightleftarrows K^{\infty,+}(\mathcal{I})/K^+(\mathcal{I})$,

(iii) $K^+(\mathcal{I})/K^{+,b}(\mathcal{I}) \rightleftarrows K^{\infty,+}(\mathcal{I})/K^{+,b}(\mathcal{I}) \rightleftarrows K^{\infty,+}(\mathcal{I})/K^+(\mathcal{I})$.

If moreover, $\mathcal{A} = R - Mod$ for R a left Noetherian algebra, then the above recollements can be extended one step upwards so that we have ladders of height two

(i')
$$K^{+,b}(\mathcal{I})/K^b(\mathcal{I}) \rightleftarrows K^{\infty,b}(\mathcal{I})/K^b(\mathcal{I}) \rightleftarrows K^{\infty,b}(\mathcal{I})/K^{+,b}(\mathcal{I}) \ ,$$

(ii')
$$K^+(\mathcal{I})/K^b(\mathcal{I}) \rightleftarrows K^{\infty,+}(\mathcal{I})/K^b(\mathcal{I}) \rightleftarrows K^{\infty,+}(\mathcal{I})/K^+(\mathcal{I}) \ ,$$

(iii')
$$K^+(\mathcal{I})/K^{+,b}(\mathcal{I}) \rightleftarrows K^{\infty,+}(\mathcal{I})/K^{+,b}(\mathcal{I}) \rightleftarrows K^{\infty,+}(\mathcal{I})/K^+(\mathcal{I}) \ .$$

Proof. The recollement (i) is proved in [9, Theorem 2.4(1)]. By the proof of [9, Theorem 2.4(1)], the left recollement contained in this recollement (i) is induced by the stable t-structure $(K^{\infty,\cdot}(\mathcal{I}), K^{+,b}(\mathcal{I})/K^b(\mathcal{I}))$ in $K^{\infty,b}(\mathcal{I})/K^b(\mathcal{I})$.

The recollement (1') of Proposition 3.2

$$K^{\infty,\emptyset}(\mathcal{I}) \xrightarrow[\substack{\xleftarrow{i^*} \\ \xrightarrow{i_*} \\ \xleftarrow{i^!}}]{} K^{\infty,b}(\mathcal{I}) \xrightarrow[\substack{\xleftarrow{j_!} \\ \xrightarrow{j^*} \\ \xleftarrow{j_*}}]{} K^{+,b}(\mathcal{I})$$

shows that $(j_!(K^{+,b}(\mathcal{I})), K^{\infty,\emptyset}(\mathcal{I}))$ is a stable t-structure in $K^{\infty,b}(\mathcal{I})$. We claim that $j_!(K^{+,b}(\mathcal{I}))$ contains $K^b(\mathcal{I})$. In fact, by [9, Proposition 1.5(2)], $(j_!(K^{+,b}(\mathcal{I})) \cap K^b(\mathcal{I}), K^{\infty,\emptyset}(\mathcal{I}) \cap K^b(\mathcal{I}))$ is a stable t-structure in $K^b(\mathcal{I})$. However, $K^{\infty,\emptyset}(\mathcal{I}) \cap K^b(\mathcal{I}) = K^{b,\emptyset}(\mathcal{I}) = 0$, so $j_!(K^{+,b}(\mathcal{I})) \cap K^b(\mathcal{I}) = K^b(\mathcal{I})$ and $K^b(\mathcal{I})$ is contained in $j_!(K^{+,b}(\mathcal{I}))$. Now by [9, Proposition 1.5(1)], $(j_!(K^{+,b}(\mathcal{I}))/K^b(\mathcal{I}), K^{\infty,\emptyset}(\mathcal{I}))$ is a stable t-structure in $K^{\infty,b}(\mathcal{I})/K^b(\mathcal{I})$.

Given this new stable t-structure and combined with the recollement, we see that the recollement (i) can be extended one step upwards so that we have the ladder (i') of height two.

The proofs of the recollements (ii), (iii) and of the ladders (ii') (iii') are similar. We only give the proof of (ii) and (ii').

By Theorem 2.1(9), $(K^+(\mathcal{I})/K^b(\mathcal{I}), K^-(\mathcal{I})/K^b(\mathcal{I}))$ is a stable t-structure in $K(\mathcal{I})/K^b(\mathcal{I})$. By Lemma 3.1 $(K^{\infty,\emptyset}(\mathcal{I}), K^+(\mathcal{I})/K^b(\mathcal{I}))$ is a stable t-structure in $K^{\infty,+}(\mathcal{I})/K^b(\mathcal{I})$. Combining these two stable t-structures, we get the recollement (ii). Then use the recollement (2') of Proposition 3.2 to show the existence of the ladder (ii'). ∎

3.2. **Quotients for homotopy categories of projectives.** We suppose that $\mathcal{E} = \mathcal{P}$ is the full subcategory of projective objects in an abelian category $\mathcal{A}$ with enough projectives. Then $K^{-,\emptyset}(\mathcal{P})$ and $K^{b,\emptyset}(\mathcal{P})$ vanish since then these complexes are actually zero homotopic.

Taking opposite categories we do have the analogous results for projective objects dualising the injective situation.

Lemma 3.4. *Let $\mathcal{A}$ be an abelian category with enough projectives and $\mathcal{P}$ be the full subcategory of injective objects. Then we have several stable t-structures*

(i) [9, Proposition 2.3(1)] $(K^{-,b}(\mathcal{P}), K^{\infty,\emptyset}(\mathcal{P}))$ in $K^{\infty,b}(\mathcal{P})$,
(ii) $(K^-(\mathcal{P}), K^{\infty,\emptyset}(\mathcal{P}))$ in $K^{\infty,-}(\mathcal{P})$,
(iii) $(K^{-,b}(\mathcal{P})/K^b(\mathcal{P}), K^{\infty,\emptyset}(\mathcal{P}))$ in $K^{\infty,b}(\mathcal{P})/K^b(\mathcal{P})$,
(iv) $(K^-(\mathcal{P})/K^b(\mathcal{P}), K^{\infty,\emptyset}(\mathcal{P}))$ in $K^{\infty,-}(\mathcal{P})/K^b(\mathcal{P})$,
(v) $(K^-(\mathcal{P})/K^{-,b}(\mathcal{P}), K^{\infty,\emptyset}(\mathcal{P}))$ in $K^{\infty,-}(\mathcal{P})/K^{-,b}(\mathcal{P})$.

Proposition 3.5. *Let $\mathcal{A}$ be an abelian category satisfying Ab4 with enough projectives and $\mathcal{P}$ be the full subcategory of projective objects. Then we have colocalisation sequences*

(1) [9, Proposition 2.3(2)] $K^{\infty,\emptyset}(\mathcal{P}) \rightleftarrows K^{\infty,b}(\mathcal{P}) \rightleftarrows D^b(\mathcal{A})$,

(2) $K^{\infty,\emptyset}(\mathcal{P}) \rightleftarrows K^{\infty,-}(\mathcal{P}) \rightleftarrows D^-(\mathcal{A})$,

(3) $K^{\infty,\emptyset}(\mathcal{P}) \xLeftarrow{\quad} K^{\infty,+}(\mathcal{P}) \xLeftarrow{\quad} D^+(\mathcal{A})$,

(4) $K^{\infty,\emptyset}(\mathcal{P}) \xLeftarrow{\quad} K(\mathcal{P}) \xLeftarrow{\quad} D(\mathcal{A})$,

where the colocalisation sequences (1), (2) and (3) are restrictions of the colocalisation sequence (4).

Proposition 3.6. *Let $\mathcal{A}$ be an abelian category with enough projectives and $\mathcal{P}$ the full subcategory of projective objects. Then there are recollements*

  (i) [9, Theorem 2.4(2)] $K^{-,b}(\mathcal{P})/K^b(\mathcal{P}) \xLeftrightarrow{\quad} K^{\infty,b}(\mathcal{P})/K^b(\mathcal{P}) \xLeftrightarrow{\quad} K^{\infty,b}(\mathcal{P})/K^{-,b}(\mathcal{P})$,

  (ii) $K^-(\mathcal{P})/K^b(\mathcal{P}) \xLeftrightarrow{\quad} K^{\infty,-}(\mathcal{P})/K^b(\mathcal{P}) \xLeftrightarrow{\quad} K^{\infty,-}(\mathcal{P})/K^-(\mathcal{P})$,

  (iii) $K^-(\mathcal{P})/K^{-,b}(\mathcal{P}) \xLeftrightarrow{\quad} K^{\infty,-}(\mathcal{P})/K^{-,b}(\mathcal{P}) \xLeftrightarrow{\quad} K^{\infty,-}(\mathcal{P})/K^-(\mathcal{P})$.

3.3. **Gorenstein situation.** Let $\mathcal{A}$ be an abelian category with enough projective objects and $\mathcal{E} = \mathcal{P}$ be the full subcategory of projective objects. Then an object X in $K(\mathcal{P})$ is *acyclic* if for every P in $\mathcal{P}$ the complex $Hom_{\mathcal{A}}(P, X)$ is an acyclic complex of abelian groups. An acyclic complex X is *totally acyclic* if for every P in $\mathcal{P}$ the complex $Hom_{\mathcal{P}}(X, P)$ is an acyclic complex of abelian groups. Let $K_{tac}(\mathcal{P})$ be the full subcategory of $K(\mathcal{P})$ formed by totally acyclic complexes. By [5, Theorem 3.1] stupid truncation at degree 0 gives a fully faithful triangle functor

$$\beta_{\mathcal{P}} : K_{tac}(\mathcal{P}) \to K^{-,b}(\mathcal{P})/K^b(\mathcal{P}) = D^b_{sg}(\mathcal{A}).$$

Obviously, every totally acyclic complex is acyclic. For an Iwanaga-Gorenstein ring, that is, a two-sided Noetherian ring which has finite injective dimension on itself for both sides, the converse is also true; see [6, Theorem 4.4.1]. In this case, we have equivalences

$$K^{\infty,\emptyset}(\mathcal{P}) \simeq K_{tac}(\mathcal{P}) \overset{\beta_{\mathcal{P}}}{\simeq} K^{-,b}(\mathcal{P})/K^b(\mathcal{P}) = D^b_{sg}(\mathcal{A}).$$

Proposition 3.7. *Let A be an Iwanaga-Gorenstein ring, denote $\mathcal{P} := A-proj$. Then $K^{+,\emptyset}(\mathcal{P}) = 0$.*
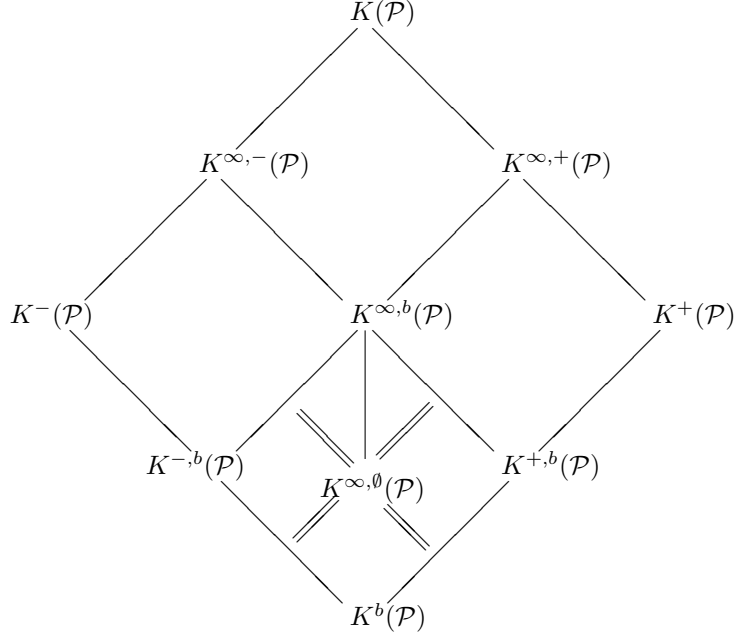
Proof. Let

$$X := \quad \ldots \to 0 \to P^n \overset{\partial^n}{\to} P^{n+1} \to P^{n+2} \to \ldots$$

be a left bounded acyclic complex of projectives, which is therefore totally acyclic. We shall show that X is zero homotopic. We show that ∂^n is a split monomorphism. In fact, we apply $Hom_{\mathcal{A}}(-, P^n)$ to this complex and obtain by total acyclicity that $Hom_{\mathcal{A}}(X, P^n)$ is exact. But then the identity id_{P^n} is in the image of $Hom_{\mathcal{A}}(\partial^n, P^n)$, and therefore ∂^n is a split monomorphism. We can then suppose that X begins with P^{n+1} and the same argument applies. This shows that $K^{+,\emptyset}(\mathcal{P}) = 0$.

Another proof is given by the observation that the A-dual $Hom_A(-, A)$ induces an equivalence between $K^{+,\emptyset}(A - proj)$ and $K^{-,\emptyset}(A^{op} - proj)$, while the latter is zero. ∎

We should mention that in the case of projective modules over an Iwanaga-Gorenstein algebra, by [9, Theorem 2.4], [6, Theorem 4.4.1] and Theorem 2.1, we have a very nice particular way of visualising things, that is, in the Gorenstein case, we may reduce the diagram (†) to a smaller one:

$$K(\mathcal{P})$$

$$K^{\infty,-}(\mathcal{P}) \qquad\qquad K^{\infty,+}(\mathcal{P})$$

$$K^{-}(\mathcal{P}) \qquad\qquad K^{\infty,b}(\mathcal{P}) \qquad\qquad K^{+}(\mathcal{P})$$

$$K^{-,b}(\mathcal{P}) \qquad K^{\infty,\emptyset}(\mathcal{P}) \qquad K^{+,b}(\mathcal{P})$$

$$K^{b}(\mathcal{P})$$

Note that there is a left-right symmetry in this diagram.

Moreover, Iyama, Kato, Miyachi [9, Theorem 2.7] shows that

$$(K^{+,b}(\mathcal{P})/K^{b}(\mathcal{P}), K^{-,b}(\mathcal{P})/K^{b}(\mathcal{P}), K^{\infty,\emptyset}(\mathcal{P}))$$

is a triangle of recollements. They [9, Proposition 3.6, Theorem 4.8] also show that $K^{\infty,b}(\mathcal{P})/K^{b}(\mathcal{P}) \simeq D_{sg}^{b}(T_2(A))$, where $T_2(A)$ is the upper triangular algebra $\left(\begin{smallmatrix} A & A \\ 0 & A \end{smallmatrix}\right)$.

For Iwanaga-Gorenstein rings, besides the bounded singularity category $D_{sg}^{b}(\mathcal{A}) = K^{-,b}(\mathcal{P})/K^{b}(\mathcal{P})$, it seems that the triangulated categories $D_{\infty}^{-}(\mathcal{P}) = K^{-}(\mathcal{P})/K^{-,b}(\mathcal{P})$ and $D_{\infty}^{+}(\mathcal{P}) = K^{+}(\mathcal{P})/K^{+,b}(\mathcal{P})$ are rather interesting.

Question 3.8. *Are $D_{\infty}^{-}(\mathcal{P}) = K^{-}(\mathcal{P})/K^{-,b}(\mathcal{P})$ and $D_{\infty}^{+}(\mathcal{P}) = K^{+}(\mathcal{P})/K^{+,b}(\mathcal{P})$ equivalent as triangulated categories for Iwanaga-Gorenstein rings?*

4. On the Hom-spaces of the categories D_{∞}^{-} and D_{∞}^{+}

For the category $D_{sg}^{b}(A - Proj)$ we know (cf [2] or e.g. [26, Proposition 6.9.18]) that if X and Y are two A-modules, considered as complexes in degree 0, then

$$Hom_{D_{sg}^{b}(A-Proj)}(X,Y) \simeq \varinjlim_{n} \underline{Hom}_A(\Omega^n(X), \Omega^n(Y))$$

where $\underline{Hom}_A$ denotes the morphism space in the stable category modulo projectives. We note that this property is the main tool in Wang's construction of the Gerstenhaber bracket on Tate-Hochschild cohomology [23].

We should ask if a variant of this holds more generally for the infinite singular category, and in particular in the left (or right) bounded infinite homology category.

Remark 4.1. We first recall that intelligent truncation is a well-defined additive functor on the homotopy categories. This follows from [8, Lemma 7.5.e), page 77]. For the convenience of the reader we shall give the easy argument in this special case within a few lines. If $\alpha : X \to Y$ is a morphism of complexes. Then, $\alpha_{n+1}d_n^X = d_n^Y\alpha_{n-1}$, and hence $\alpha_{n+1}|_{\text{im}(d_n^X)} : \text{im}(d_n^X) \to \text{im}(d_n^Y)$. We may hence define α on the truncations in the natural way by restriction to the subcomplexes, giving then a morphism of complexes $\sigma_{\leq n}\alpha : \sigma_{\leq n}X \to \sigma_{\leq n}Y$. If

$$\alpha = hd_X + d_Y h : X \to Y$$

is a zero homotopic morphism of complexes over an abelian category $\mathcal{A}$, then defining $h'_n := h_n|_{\text{im}(d_n^X)} : \text{im}(d_n^X) \to Y_n$, and $h'_m = h_m$ for all $m < n$, and $h'_m = 0$ for $m > n$ gives a zero

homotopic map $\sigma_{\leq n}\alpha : \sigma_{\leq n}X \longrightarrow \sigma_{\leq n}Y$. This shows that $\alpha \mapsto \sigma_{\leq n}\alpha$ is a well-defined map on the morphisms of homotopy categories.

Notice, however, that the intelligent truncation $\sigma_{\leq n}$ is *not* a triangle functor on homotopy categories. In fact, let

$$X \xrightarrow{f} Y \to \mathrm{cone}(f) \to X[1]$$

be a distinguished triangle in a homotopy category. Then we have the commutative diagram with exact rows

$$\sigma_{\leq n}X \xrightarrow{\sigma_{\leq n}f} \sigma_{\leq n}Y \longrightarrow \mathrm{cone}(\sigma_{\leq n}f) \longrightarrow \sigma_{\leq n}X[1]$$

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow \varphi \qquad\qquad \downarrow$$

$$X \xrightarrow{\quad f \quad} Y \longrightarrow \mathrm{cone}(f) \longrightarrow X[1]$$

The long exact sequences of cohomology groups show that φ induces a quasi-isomorphism $\mathrm{cone}(\sigma_{\leq n}f) \to \sigma_{\leq n}\mathrm{cone}(f)$, still denoted by φ. However, as complexes, $\mathrm{cone}(\sigma_{\leq n}f)$ has the form

$$\cdots \xrightarrow{-d_X^{n-1}} X^n \xrightarrow{-d_X^n} \mathrm{Im}d_X^n$$
$$f^{n-1} \searrow \quad \oplus \quad f^n \searrow \quad \oplus \quad f^{n+1}\searrow$$
$$\cdots \longrightarrow Y^{n-1} \xrightarrow{d_Y^{n-1}} Y^n \xrightarrow{d_Y^n} \mathrm{Im}d_Y^n \longrightarrow 0$$

and $\sigma_{\leq n}\mathrm{cone}(f)$ has a different form

$$\cdots \xrightarrow{-d_X^{n-1}} X^n \xrightarrow{-d_X^n} X^{n+1}$$
$$f^{n-1}\searrow \; \oplus \qquad f^n \searrow \; \oplus \longrightarrow \mathrm{Im}d_{\mathrm{cone}(f)}^n \longrightarrow 0$$
$$\cdots \longrightarrow Y^{n-1} \xrightarrow{d_Y^{n-1}} Y^n$$

So $\mathrm{cone}(\sigma_{\leq n}f)$ is a subcomplex of $\sigma_{\leq n}\mathrm{cone}(f)$, but they are not homotopy equivalent in general.

We shall show that intelligent truncation does not only pass to homotopy categories but also to the derived category. This fact should be well-known, and actually is sort of implicit in [3], but we could not find an explicit reference. In any case, since our definition of $\sigma_{\leq n}$ differs slightly from the usual one, it is appropriate to verify that the necessary constructions work also in our case.

Lemma 4.2. *Let $\mathcal{A}$ be an abelian category and let $\alpha : Z \longrightarrow \sigma_{\leq n}X$ be a morphism in $K^-(\mathcal{A})$. If the cone of α is an object in $K^{-,\emptyset}(\mathcal{A})$, then α induces an isomorphism $H^i(\alpha) : H^i(Z) \longrightarrow H^i(\sigma_{\leq n}X)$ for all $i \in \mathbb{Z}$. Moreover, in this case $\sigma_{\leq n}\alpha \in \mathrm{Hom}_{K^-(\mathcal{A})}(\sigma_{\leq n}Z, \sigma_{\leq n}X)$ is a quasi-isomorphism. More precisely, if the cone of α is in $K^{-,\emptyset}(\mathcal{A})$, then the cone of $\sigma_{\leq n}\alpha$ is in $K^{-,\emptyset}(\mathcal{A})$.*

Proof. We have a distinguished triangle

$$Z \xrightarrow{\alpha} \sigma_{\leq n}X \longrightarrow C \longrightarrow Z[1]$$

with a C in $K^{-,\emptyset}(\mathcal{A})$. The distinguished triangle induces a long exact sequence in homology

$$H^{i-1}(C) \longrightarrow H^i(Z) \longrightarrow H^i(\sigma_{\leq n}X) \longrightarrow H^i(C) \longrightarrow H^{i+1}(Z).$$

Since C is in $K^{-,\emptyset}(\mathcal{A})$, we get $H^j(C) = 0$ for all $j \in \mathbb{Z}$. Therefore $H^i(\alpha) : H^i(Z) \longrightarrow H^i(\sigma_{\leq n}X)$ is an isomorphism.

For the second statement, suppose that α is a quasi-isomorphism. Then $\sigma_{\leq n}\alpha$ is the composition of $\alpha : Z \longrightarrow \sigma_{\leq n}X$ with the canonical morphism $\sigma_{\leq n}Z \longrightarrow Z$. Since by the first step α induces an isomorphism between the homology of Z and the homology of $\sigma_{\leq n}X$, and since $\sigma_{\leq n}X$ is exact in degrees higher than n, also $\sigma_{\leq n}Z \longrightarrow Z$ is a quasi-isomorphism. Therefore, since the composition of quasi-isomorphisms is a quasi-isomorphism, also $\sigma_{\leq n}\alpha$ is a quasi-isomorphism. ∎

Lemma 4.3. *Let $\mathcal{A}$ be an abelian category and let $\alpha : Z \longrightarrow X$ be a morphism in $K^-(\mathcal{A})$ with cone in $K^{-,\emptyset}(\mathcal{A})$. Then α induces a morphism $\sigma_{\leq n}\alpha : \sigma_{\leq n}Z \longrightarrow \sigma_{\leq n}X$ with cone in $K^{-,\emptyset}(\mathcal{A})$.*

Proof. Suppose α is a quasi-isomorphism. Then $H^i(\alpha)$ is an isomorphism for all $i \in \mathbb{Z}$ and $\sigma_{\leq n}\alpha$ coincides with α in degrees smaller or equal to n. Since $H^i(Z) = H^i(\sigma_{\leq n}Z)$ for all $i \leq n$, and likewise $H^i(X) = H^i(\sigma_{\leq n}X)$ for all $i \leq n$, and since $H^i(\sigma_{\leq n}Z) = H^i(\sigma_{\leq n}X) = 0$ for $i \geq n+1$, we get that $\sigma_{\leq n}\alpha$ is a quasi-isomorphism as well. ∎

Lemma 4.4. *Let $\mathcal{A}$ be an abelian category. Then $\sigma_{\leq n}$ induces a functor $K^-(\mathcal{A})/K^{-,\emptyset}(\mathcal{A}) \longrightarrow K^-(\mathcal{A})/K^{-,\emptyset}(\mathcal{A})$.*

Proof. Suppose given a diagram

$$X \xleftarrow{\ \alpha\ } Z \xrightarrow{\ \beta\ } Y$$

which represents a morphism in $K^-(\mathcal{A})/K^{-,\emptyset}(\mathcal{A})$ and hence α is a quasi-isomorphism. By Remark 4.1 this induces a diagram

$$\sigma_{\leq n}X \xleftarrow{\ \sigma_{\leq n}\alpha\ } \sigma_{\leq n}Z \xrightarrow{\ \sigma_{\leq n}\beta\ } \sigma_{\leq n}Y.$$

By Lemma 4.3 we get that $\sigma_{\leq n}\alpha$ is a quasi-isomorphism again. Hence, this diagram presents a morphism in $K^-(\mathcal{A})/K^{-,\emptyset}(\mathcal{A})$. Now, if

is a commutative diagram such that α, α', α'', γ and γ' are quasi-isomorphisms. Then

is a diagram such that $\sigma_{\leq n}\alpha$, $\sigma_{\leq n}\alpha'$, $\sigma_{\leq n}\alpha''$, $\sigma_{\leq n}\gamma$ and $\sigma_{\leq n}\gamma'$ are quasi-isomorphisms, using again Lemma 4.3. Hence, truncations of two diagrams representing the same morphism in the derived category yield two morphisms representing the same morphism in the derived category.

Clearly, $\sigma_{\leq n}\mathrm{id}_X = \mathrm{id}_{\sigma_{\leq n}X}$. Moreover, composition is compatible as well, in the sense that

$$\sigma_{\leq n}(\alpha_1, \beta_1) \circ \sigma_{\leq n}(\alpha_2, \beta_2) = \sigma_{\leq n}((\alpha_1, \beta_1) \circ (\alpha_2, \beta_2))$$

in the obvious sense and notation. This follows from just applying truncation to the diagram representing the composition, and applying Lemma 4.2 and Lemma 4.3. ∎

Remark 4.5. Using Lemma 4.4 we can form an inductive system over these morphism spaces

$$
\begin{aligned}
Hom_{K^-(\mathcal{A})/K^{-,\emptyset}(\mathcal{A})}(\sigma_{\leq n}X, \sigma_{\leq n}Y) \quad &\to\quad Hom_{K^-(\mathcal{A})/K^{-,\emptyset}(\mathcal{A})}(\sigma_{\leq n-1}X, \sigma_{\leq n-1}Y) \\
&\to\quad Hom_{K^-(\mathcal{A})/K^{-,\emptyset}(\mathcal{A})}(\sigma_{\leq n-2}X, \sigma_{\leq n-2}Y) \\
&\to\quad \dots
\end{aligned}
$$

and get the inductive limit $\varinjlim_{(n \in \mathbb{Z};\geq)}(Hom_{K^-(\mathcal{A})/K^{-,\emptyset}(\mathcal{A})}(\sigma_{\leq n}X, \sigma_{\leq n}Y))$. Note that here $\mathbb{Z}$ is ordered by *decreasing* order! Further, we have the canonical distinguished triangle

$$\sigma_{\leq n}X \xrightarrow{\ i_n^X\ } X \longrightarrow \sigma_{\geq n+1}X \longrightarrow \sigma_{\leq n}X[1]$$

and if X is an object of $K^-(\mathcal{A})$, then $\sigma_{\geq n+1}X$ is an object of $K^{-,b}(\mathcal{A})$. Therefore, the natural functor

$$G : K^-(\mathcal{A})/K^{-,\emptyset}(\mathcal{A}) \longrightarrow K^-(\mathcal{A})/K^{-,b}(\mathcal{A})$$

identifies X with $\sigma_{\leq n}X$ for all n, along the natural morphisms

$$\ldots \longrightarrow \sigma_{\leq n-1}X \longrightarrow \sigma_{\leq n}X \longrightarrow \ldots \longrightarrow X.$$

Therefore, the functor G induces a series of compatible maps

$$Hom_{K^-(\mathcal{A})/K^{-,\emptyset}(\mathcal{A})}(\sigma_{\leq n}X, \sigma_{\leq n}Y) \longrightarrow Hom_{K^-(\mathcal{A})/K^{-,b}(\mathcal{A})}(GX, GY)$$

sending a morphism $f : \sigma_{\leq n}X \to \sigma_{\leq n}Y$ in $K^-(\mathcal{A})/K^{-,\emptyset}(\mathcal{A})$ to the composition

$$X \xrightarrow{(i_n^X)^{-1}} \sigma_{\leq n}X \xrightarrow{f} \sigma_{\leq n}Y \xrightarrow{i_n^Y} Y$$

in $K^-(\mathcal{A})/K^{-,b}(\mathcal{A})$, which hence in turn induce a map

$$\Upsilon : \varinjlim_{(n\in\mathbb{Z};\geq)} (Hom_{K^-(\mathcal{A})/K^{-,\emptyset}(\mathcal{A})}(\sigma_{\leq n}X, \sigma_{\leq n}Y)) \to Hom_{D_\infty^-(\mathcal{A})}(GX, GY).$$

Proposition 4.6. *Let $\mathcal{A}$ be an abelian category, let $D^-(\mathcal{A}) = K^-(\mathcal{A})/K^{-,\emptyset}(\mathcal{A})$ be the right bounded derived category of $\mathcal{A}$, and let $D_\infty^-(\mathcal{A}) = K^-(\mathcal{A})/K^{-,b}(\mathcal{A})$. Denote by $G : D^-(\mathcal{A}) \to D_\infty^-(\mathcal{A})$ the canonical quotient functor. Then, considering the inductive system ($n \in \mathbb{Z}; \geq$), with decreasing integers n,*

$$\varinjlim_{(n\in\mathbb{Z};\geq)} (Hom_{D^-(\mathcal{A})}(\sigma_{\leq n}X, \sigma_{\leq n}Y)) \to Hom_{D_\infty^-(\mathcal{A})}(GX, GY)$$

is an isomorphism.

Proof. Consider the map Υ from Remark 4.5. We claim that this is an isomorphism. Let

$$X \xleftarrow{\alpha} Z \xrightarrow{\beta} Y$$

be a diagram in $K^-(\mathcal{A})$, representing a morphism $\gamma \in Hom_{D_\infty^-(\mathcal{A})}(X, Y)$, i.e. such that $C := \text{cone}(\alpha)$ is in $K^{-,b}(\mathcal{A})$. Then there is an n_0 such that $\sigma_{\leq n}C$ is in $K^{-,\emptyset}(\mathcal{A})$ for $n \leq n_0$. Hence

$$\sigma_{\leq n}X \xleftarrow{\sigma_{\leq n}\alpha} \sigma_{\leq n}Z$$

is an isomorphism in $D^-(\mathcal{A})$, whenever $n \leq n_0$. This shows that γ is the image of $(\sigma_{\leq n}\beta) \circ (\sigma_{\leq n}\alpha)^{-1}$ under Υ. This implies that Υ is surjective.

We now prove that Υ is injective. Recall that the colimit is constructed explicitly in e.g. [26, Proposition 3.1.18]. Let $\gamma_n : \sigma_{\leq n}X \to \sigma_{\leq n}Y$ be a morphism in $D^-(\mathcal{A})$ representing $\gamma \in \varinjlim_{(n\in\mathbb{Z};\geq)}(Hom_{D^-(\mathcal{A})}(\sigma_{\leq n}X, \sigma_{\leq n}Y))$, with $\Upsilon(\gamma) = 0$. Denote by i_n^Y the natural morphism $\sigma_{\leq n}Y \to Y$. Now, γ_n is represented by the equivalence class of diagrams of morphisms of complexes

$$\sigma_{\leq n}X \xleftarrow{\alpha_n} Z_n \xrightarrow{\beta_n} \sigma_{\leq n}Y$$

such that the cone of α_n is acyclic for all n. If $\Upsilon(\gamma) = 0$, then there is a morphism $s_n : T_n \to Z_n$ in $K^-(\mathcal{A})$ with cone in $K^{-,b}(\mathcal{A})$ such that $i_n^Y \circ \beta_n \circ s_n = 0$. Since $\text{cone}(s_n)$ is in $K^{-,b}(\mathcal{A})$, its homology is bounded and $\sigma_{\leq m}\text{cone}(s_n) = \text{cone}(\sigma_{\leq m}s_n) = 0$, for small enough m. This implies that $\gamma_n = 0$ in $\varinjlim_n Hom_{D^-(\mathcal{A})}(\sigma_{\leq n}X, \sigma_{\leq n}Y)$, identifying γ_n with $\sigma_{\leq m}\gamma_n$. ∎

Dually we get

Proposition 4.7. *Let $\mathcal{A}$ be an abelian category. let $D^+(\mathcal{A}) = K^+(\mathcal{A})/K^{+,\emptyset}(\mathcal{A})$ be the left bounded derived category of $\mathcal{A}$, and let $D_\infty^+(\mathcal{A}) := K^+(\mathcal{A})/K^{+,b}(\mathcal{A})$. Then, considering the inductive system ($n \in \mathbb{Z}; \leq$), with increasing integers n,*

$$\varinjlim_{(n\in\mathbb{Z};\leq)} (Hom_{D^+(\mathcal{A})}(\sigma_{\geq n}X, \sigma_{\geq n}Y)) \to Hom_{D_\infty^+(\mathcal{A})}(GX, GY)$$

is an isomorphism for $G : D^+(\mathcal{A}) \to D_\infty^+(\mathcal{A})$ being the canonical quotient functor.

Proof. The arguments are precisely dual to the case of $D_\infty^-(\mathcal{A})$. ∎

Remark 4.8. Generally speaking, Proposition 4.6 and Proposition 4.7 give an interpretation of $D_\infty^+(K - mod)$ and $D_\infty^-(K - mod)$ as the categories that detect large degree behaviour of (co-)homology complexes.

Indeed, let A be a K-algebra for a commutative ring K, and denote $A^e := A \otimes_K A^{op}$. Then, for any $A - A$-bimodule M the Hochschild cohomology complex $\mathbb{R}Hom_{A^e - Mod}(A, M)$ is non zero in $D_\infty^-(K - mod)$ if and only if M has bounded Hochschild cohomology.

Similarly, following [23, Definition 3.2], Wang's Tate-Hochschild homology complex $C_{sg}^*(A, M)$ is a complex in $D(K - Mod)$ and the image of $C_{sg}^*(A, M)$ in $D(K - Mod)/D^b(K - Mod)$ is in $D_\infty^+(K - Mod)$ if and only if the Tate-Hochschild homology $Hom_{D_{sg}(A)}(A, A[n])$ has only finitely many negative degrees, and is in $D_\infty^-(K - Mod)$ if and only if the Tate-Hochschild homology $Hom_{D_{sg}(A)}(A, A[n])$ has only finitely many positive degrees.

5. Examples

We will give examples of categories showing that certain of the categories which we introduced in Definition 2.5 differ from each other.

5.1. **The case of semisimple $\mathcal{E}$.** We compute in the case of a semisimple abelian category $\mathcal{E}$ each of the categories of Definition 2.5.

5.1.1. *Right bounded singularity category.* Recall the definition of the right bounded singularity category $D_{sg}^-(\mathcal{E}) = K^-(\mathcal{E})/K^b(\mathcal{E})$. It can be identified with end pieces of sequences with values in $\mathcal{E}$. Indeed since $\mathcal{E}$ is semisimple, $K^-(\mathcal{E})$ can be identified with the left infinitely sequences of objects in $K^-(\mathcal{E})$ with zero differential. Likewise $K^b(\mathcal{E})$ consists of bounded sequences of objects of $K^b(\mathcal{E})$. Hence, $D_{sg}^-(\mathcal{E})$ consists of equivalence classes of the left unbounded sequences of objects of $\mathcal{E}$ where we identify two such sequences if they become equal after a finite number of steps. In case $\mathcal{E} = K - mod$, we see that we can interpret this situation as quotient $\prod_{n \in \mathbb{N}} \mathcal{E}/\coprod_{n \in \mathbb{N}} \mathcal{E}$. We hence abbreviate the result as $\prod \mathcal{E}/\coprod \mathcal{E}$.

5.1.2. *Left bounded singularity category.* Likewise $D_{sg}^+(\mathcal{E})$ consists of equivalence classes of the right unbounded sequences of objects of $\mathcal{E}$ where we identify two such sequences if they become equal after a finite number of steps.

5.1.3. *Bounded singularity category.* Since for semisimple categories $\mathcal{E}$ we get $K^{-,b}(\mathcal{E}) = K^b(\mathcal{E})$, we get $D_{sg}^b(\mathcal{E}) = 0$.

5.1.4. *Cohomologically bounded singularity category.* By the discussion above we get for the homology bounded singular category $D_{sg}^{\infty,b}(\mathcal{E}) = K^{\infty,b}(\mathcal{E})/K^b(\mathcal{E}) = 0$.

5.1.5. *Left and right bounded infinite homology category.* Since for semisimple $\mathcal{E}$ we get $K^{-,b}(\mathcal{E}) = K^b(\mathcal{E})$, we obtain that in this case $D_{sg}^-(\mathcal{E}) = D_\infty^-(\mathcal{E})$ and $D_{sg}^+(\mathcal{E}) = D_\infty^+(\mathcal{E})$. Moreover, as abstract categories we get that $D_\infty^+(\mathcal{E}) \simeq D_\infty^-(\mathcal{E})$. It suffices to identify the degree n component of $D_\infty^+(\mathcal{E})$ with the degree $-n$ components of $D_\infty^-(\mathcal{E})$.

5.1.6. *Cohomologically left/right bounded singularity category.* The cohomologically right bounded singularity category $D_{sg}^{\infty,-}(\mathcal{E}) = K^{\infty,-}(\mathcal{E})/K^b(\mathcal{E})$ consists of equivalence classes of all right unbounded sequences of objects of $\mathcal{E}$ where we identify two such sequences if they become equal after a finite number of steps.

The cohomologically left bounded singularity category $D_{sg}^{\infty,+}(\mathcal{E}) = K^{\infty,+}(\mathcal{E})/K^b(\mathcal{E})$ consists of equivalence classes of all left unbounded sequences of objects of $\mathcal{E}$ where we identify two such sequences if they become equal after a finite number of steps.

5.1.7. *Unbounded singularity category.* The unbounded singularity category $D_{sg}^\infty(\mathcal{E}) = K(\mathcal{E})/K^b(\mathcal{E})$ consists of equivalence classes of all two-sided unbounded sequences of objects of $\mathcal{E}$ where we identify two such sequences if they become equal after a finite number of steps in two directions.

5.2. **The case of $\mathcal{E}$ being the dual numbers.** Let K be a field and $A = K[X]/(X^2)$. Let $\mathcal{E} = \mathcal{P} = A - proj$ be the category of finitely generated projective modules over the dual numbers. We shall study the different singular categories in this case. The indecomposable complexes over $\mathcal{P}$ are classified in [1, Theorem B] (see also [14, Lemma 3.1], which can be used as well). Let P is the minimal projective resolution of the only non projective A-module K and I be its injective resolution and PI its complete resolution, that is,

$$P = (\cdots \xrightarrow{x} A \xrightarrow{x} A \xrightarrow{x} A \xrightarrow{x} A \to 0 \to \cdots),$$
$$I = (\cdots \to 0 \to A \xrightarrow{x} A \xrightarrow{x} A \xrightarrow{x} A \xrightarrow{x} \cdots),$$
$$PI = (\cdots \xrightarrow{x} A \xrightarrow{x} A \xrightarrow{x} A \xrightarrow{x} A \xrightarrow{x} \cdots).$$

It reveals that indecomposable objects of $K(A - Proj)$, $K(\mathcal{P})$, $K^{\infty,+}(\mathcal{P})$, $K^{\infty,-}(\mathcal{P})$, $K^{\infty,b}(\mathcal{P})$ coincide which are exactly shifts of P, I and PI and their stupid truncations; indecomposable objects of $K^+(\mathcal{P})$ and of $K^{+,b}(\mathcal{P})$ coincide which are exactly shifts of I and their stupid truncations; indecomposable objects of $K^-(\mathcal{P})$ and of $K^{-,b}(\mathcal{P})$ coincide which are exactly shifts of I and their stupid truncations; indecomposable objects of $K^b(\mathcal{P})$ are exactly shifts of $\tau_{\leq n}I$ with $n \geq 0$.

5.2.1. *Bounded singularity category.* First we see that $A := K[X]/(X^2)$ is a symmetric algebra. Hence $D_{sg}^b(A) = K^{-,b}(\mathcal{P})/K^b(\mathcal{P}) \simeq A - \underline{mod}$, the stable module category, as triangulated categories. Under this equivalence, P is sent to the unique indecomposable non-projective A-module, namely $K[X]/(X) = K$, where X acts as 0. Since the endomorphisms of this module are scalar multiples of the identity, we see that $A - \underline{mod} \simeq K - mod$, the semisimple category of finite dimensional vector spaces with suspension functor being the identity on vector spaces and linear maps.

5.2.2. *Right bounded singularity category.* The bounded singularity category $A - \underline{mod} = D_{sg}^b(A)$ is a full triangulated subcategory of $D_{sg}^-(\mathcal{P})$. In particular, $K = K[0]$ is a non zero objects with endomorphism ring K. All indecomposable complexes are in $D_{sg}^b(A)$. Hence, in order to find objects in $D_{sg}^-(\mathcal{P})$ which are not in $D_{sg}^b(\mathcal{P})$ we need to consider infinite sums of indecomposable objects.

5.2.3. *Left bounded singularity category.* The case of the left bounded singularity category is dealt with analogously. Taking K-duals gives an equivalence as abstract K-linear categories $D_{sg}^-(\mathcal{P}) \simeq D_{sg}^+(\mathcal{P})^{op}$.

5.2.4. *Cohomologically bounded singularity category.* Iyama, Kato, Miyachi [9, Theorem 2.7] shows that $(K^{+,b}(\mathcal{P})/K^b(\mathcal{P}), K^{-,b}(\mathcal{P})/K^b(\mathcal{P}), K^{\infty,\emptyset}(\mathcal{P}))$ is a triangle of recollements. So in our case, there is a triangle of recollements $(K - mod, K - mod, K - mod)$ in $D_{sg}^{\infty,b}(\mathcal{P})$, but it is not equivalent to $K - mod \times K - mod$.

5.2.5. *Left/right bounded and cohomologically infinite category.* We need to study the categories $D_\infty^-(\mathcal{P}) = K^-(\mathcal{P})/K^{-,b}(\mathcal{P})$, respectively $D_\infty^+(\mathcal{P}) = K^+(\mathcal{P})/K^{+,b}(\mathcal{P})$. Since A is symmetric, taking K-duals gives an equivalence as K-linear categories between $D_\infty^-(\mathcal{P})$ and $D_\infty^+(\mathcal{P})^{op}$.

As seen before, all indecomposable objects of $D_\infty^-(\mathcal{P})$ come from indecomposable complexes in $K^-(\mathcal{P})$, so they are in $K^{-,b}(\mathcal{P})$. If an object M in $D_\infty^-(\mathcal{P})$ would have endomorphism ring K, it is indecomposable, and hence is in $K^{-,b}(\mathcal{P})$, which implies that $M = 0$. Hence $D_\infty^-(\mathcal{P})$ does not contain any object with endomorphism ring K.

5.3. **A non Gorenstein algebra of infinite global dimension.** We consider the algebra

$$A = K[x,y]/(x^2, y^2, xy).$$

This algebra is a string algebra. Let $\mathcal{P} = A - proj$.

All A-modules M of Loewy length 2 without a projective direct factor are not submodules of projective modules, since each projective indecomposable is free, and has have Loewy length 2. A monomorphism $M \hookrightarrow A^n$, with minimal n splits on each of the components in which the top is in the image. Minimality of n shows that M has a projective direct factor, a contradiction. Therefore only semisimple modules may be submodules of projectives, which shows that only semisimple modules can be syzygies of a complex in $K^{\infty,\emptyset}(\mathcal{P})$. Hence, let M be a semisimple module and consider an exact complex X which is M in degree 0, and which has projective homogeneous components in degrees bigger than 0. Suppose that X has no zero homotopic direct factors. Then M is semisimple

of dimension n, say. Now, the cokernel M^1 of $M \to X^1$ is again a submodule of X^2, and hence M identifies with the socle of X_1. Therefore $dim(M^1) = dim(M)/2$, and $dim(M^1)$ is even. Since $dim(M)$ is finite, there is n such that $M^n = 0$, and we reach a contradiction.

Hence an exact complex of A-modules is actually right bounded, and therefore $K^{\infty,\emptyset}(\mathcal{P}) = K^{-,\emptyset}(\mathcal{P})$. However, $K^{-,\emptyset}(\mathcal{P}) = 0$, since such complexes are 0-homotopic.

Therefore

$$
\begin{array}{rcl}
K^{+,b}(\mathcal{P}) & = & K^b(\mathcal{P}) \\
K^{\infty,b}(\mathcal{P}) & = & K^{-,b}(\mathcal{P}) \\
K^{\infty,\emptyset}(\mathcal{P}) & = & 0 \\
K^{+,\emptyset}(\mathcal{P}) & = & 0
\end{array}
$$

which shows that

$$
K^b(\mathcal{P}) = (K^{+,b}/K^{+,\emptyset})(\mathcal{P}) \hookrightarrow (K^{\infty,b}/K^{\infty,\emptyset})(\mathcal{P}) = K^{-,b}(\mathcal{P})
$$

which shows that $D_{sg}^{+,b}(\mathcal{P}) = K^{+,b}(\mathcal{P})/K^b(\mathcal{P})$ is zero, whereas $D_{sg}^{-,b}(\mathcal{P}) = K^{-,b}(\mathcal{P})/K^b(\mathcal{P}) \neq 0$, as is studied in much broader generality by Xiao-Wu Chen [7] are really different in this case.

5.4. **Summary.** We see that we obtained examples for coefficient categories $\mathcal{E}$ having different singularity category quotients. We summarise the above results in the following scheme.

	D_{sg}^b	$D_{sg}^{\infty,b}$	D_{sg}^-	D_∞^-
semisimple	0	0	$\prod \mathcal{E} / \coprod \mathcal{E}$	$\prod \mathcal{E} / \coprod \mathcal{E}$
$K[X]/(X^2)$	$K-\mathrm{mod}$	non semisimple	there is an object with endomorphism ring K	no object has endomorphism ring K

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