HOCHSCHILD HOMOLOGY INVARIANTS OF KÜLSHAMMER TYPE OF DERIVED CATEGORIES

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Abstract. For a perfect field k of characteristic $p > 0$ and for a finite dimensional symmetric k-algebra A Külshammer studied a sequence of ideals of the centre of A using the p-power map on degree 0 Hochschild homology. In joint work with Bessenrodt and Holm we removed the condition to be symmetric by passing through the trivial extension algebra. If A is symmetric then the dual to the Külshammer ideal structure was generalised to higher Hochschild homology in earlier work [23]. In the present paper we follow this program and propose an analogue of the dual to the Külshammer ideal structure on the degree m Hochschild homology theory also to not necessarily symmetric algebras.

Introduction

Let k be a perfect field of characteristic $p > 0$ and let A be a finite dimensional k-algebra. For a symmetric algebra A Külshammer introduced in [14] a descending sequence of ideals $T_n(A)^\perp$ of the centre of A satisfying various interesting properties. In [6] Héthelyi, Horváth, Külshammer and Murray continued to study this sequence of ideals, proved their invariance under Morita equivalence, and showed further properties linked amongst others to the Higman ideal of the algebra. In order to show these properties Külshammer introduced a sequence of mappings $\zeta_n : Z(A) \longrightarrow Z(A)$ with image being $T_n(A)^\perp$. In a completely dual procedure he introduced mappings $\kappa_n : A/KA \longrightarrow A/KA$ again in case A is symmetric. These mappings encode many representation theoretic informations, in particular in case A is a group algebra. The reader may refer to Külshammer's original articles [14] and [15] for more details. Moreover, in a recent development it could be proved in [22] that these mappings actually are an invariant of the derived category of A. This fact could be used to distinguish derived categories in very subtle situations, such as some parameter questions for blocks of dihedral or semidihedral type in joint work with Holm [10], such as very delicate questions to distinguish two families of symmetric algebras of tame domestic representation type by Holm and Skowroński [9], or in joint work with Holm [11] to distinguish derived equivalence classes of deformed preprojective algebras of generalised Dynkin type, as defined by Białkowski, Erdmann and Skowroński [3] (modified slightly for type E; cf [5, page 238]) with respect to different parameters. Most recently in joint work [17] with Yuming Liu and Guodong Zhou we gave a version which is invariant under stable equivalences of Morita type. This approach gives a link to Auslander's conjecture on the invariance of the number of simple modules under stable equivalences of Morita type. König, Liu and Zhou [13] continued the work with focus being higher Hochschild and cyclic homology invariants.

In joint work with Bessenrodt and Holm [2] we could get rid of the assumption that A needs to be symmetric for the definition of the images of ζ_n. We used the trivial extension algebra $\mathbb{T}A$ of A, computed the image of ζ_n for $\mathbb{T}A$ and interpreted the result purely in terms of A. The derived invariant Külshammer's ideal structure of the centre of A becomes available for any finite dimensional algebra over perfect fields of positive characteristic.

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Since the centre of the algebra is the degree 0 Hochschild cohomology and A/KA is the 0-degree Hochschild homology, it is natural to try to understand the Külshammer construction for higher degree Hochschild (co-)homology. Denote by $HH^m(A)$ the m-degree Hochschild cohomology of A and by $HH_m(A)$ the m-degree Hochschild homology of A. We defined in [23] for a symmetric algebra A certain mappings $\kappa_n^{(m);A} : HH_{p^n m}(A) \longrightarrow HH_m(A)$ in completely analogous manner as Külshammer's κ_n. Moreover we showed that $\kappa_n^{(0);A} = \kappa_n$, and proved its derived invariance as well (cf Theorem 1 below).

In Definition 1 of Section 2 we propose a mapping $\hat{\kappa}_n^{(m);A}$, denoted also $\hat{\kappa}_n^{(m)}$ if no confusion may occur, in analogy of $\kappa_n^{(m);A}$ for algebras which are not necessarily symmetric. Further, we prove that $\hat{\kappa}_n^{(m)}$ is invariant under a derived equivalence: If $F : D^b(A) \simeq D^b(B)$ is a standard equivalence then it induces coherent isomorphisms of Hochschild homology groups $HH_m(F) : HH_m(A) \longrightarrow HH_m(B)$ and we get $HH_m(F) \circ \hat{\kappa}_n^{(m);A} = \hat{\kappa}_n^{(m);B} \circ HH_{p^n m}(F)$ for all positive integers m and n.

Finally we give an elementary example in order to illustrate that these invariants are computable and to show that they are not trivial.

The paper is organised as follows. In Section 1.1 we recall some known facts about Hochschild (co-)homology. In Section 1.2 the known constructions concerning Külshammer ideals in degree 0 for symmetric and non symmetric algebras, using trivial extension algebras are recalled, as well as the m-degree Hochschild cohomology generalisation for symmetric algebras. Section 1.3 recalls that Hochschild homology is functorial, a statement which is used at a prominent point in our construction. In Section 2 finally we provide a Külshammer mapping for not necessarily symmetric algebras and higher Hochschild homology. We show in Theorem 2 that the new mapping $\hat{\kappa}_n^{(m)}$ is an invariant of the derived category. Section 3 is devoted to a detailed computation of an example.

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1. Setup of the basic tools

1.1. **Hochschild homology and cohomology.** Recall some well-known facts from Hochschild (co-)homology. We refer to Loday [18] for a complete presentation of this theory. Let A be a finitely generated k-algebra for a commutative ring k, and suppose A is projective as k-module. Then for every $A \otimes_k A^{op}$-module M we define the Hochschild homology of A with values in M as

$$HH_n(A, M) := Tor_n^{A \otimes_k A^{op}}(A, M)$$

and Hochschild cohomology of A with values in M as

$$HH^n(A, M) := Ext^n_{A \otimes_k A^{op}}(A, M).$$

Often and frequently throughout the paper, for a k-algebra R we abbreviate $R^e := R \otimes_k R^{op}$. There is a standard way to compute Hochschild (co-)homology by the bar resolution. We recall its definition in order to set up the notations.

The bar complex $\mathbb{B}A$ is given by

$$(\mathbb{B}A)_n := A^{(\otimes_k)^{n+1}} := \underbrace{A \otimes_k A \otimes_k \cdots \otimes_k A}_{n+1 \text{ factors}}$$

and differential

$$d(a_0 \otimes \dots a_n) = \sum_{j=0}^{n-1} (-1)^j a_0 \otimes \cdots \otimes a_{j-1} \otimes a_j a_{j+1} \otimes a_{j+2} \otimes \cdots \otimes a_n.$$

One observes that $d^2 = 0$, that $(\mathbb{B}A)_n$ is a free $A \otimes_k A^{op}$-module for every $n > 0$ and that the complex $(\mathbb{B}A, d)$ is a free resolution of A as a $A \otimes_k A^{op}$-module, called the bar resolution. Hence,

$$HH_n(A, M) \simeq H_n(\mathbb{B}A \otimes_{A \otimes A^{op}} M) \text{ and } HH^n(A, M) \simeq H^n(Hom_{A \otimes A^{op}}(\mathbb{B}A, M)).$$

Moreover, the bar complex is functorial in the sense that whenever $f : A \longrightarrow B$ is a homomorphism of k-algebras, then

$$(\mathbb{B}A)_n \ni a_0 \otimes \cdots \otimes a_n \mapsto f(a_0) \otimes \cdots \otimes f(a_n) \in \mathbb{B}B$$

induces a morphism of complexes $\mathbb{B}A \longrightarrow \mathbb{B}B$.

1.2. **The Külshammer ideals; constructions and their Hochschild generalisation.** We recall the constructions introduced by Külshammer in [14] and [15] as well as their generalisation introduced in [23] to higher degree Hochschild cohomology and their generalisations to 0-degree Hochschild cohomology for not necessarily symmetric algebras.

1.2.1. *The Külshammer ideal theory for algebras which are symmetric.* Let k be a perfect field of characteristic $p > 0$ and let A be a k-algebra. Then

$$KA := < ab - ba \mid a, b \in A >_{k-\text{space}}$$

is a $Z(A)$-module and for all $n \in \mathbb{N}$

$$T_n(A) := \{x \in A \mid x^{p^n} \in KA\}$$

is a $Z(A)$ submodule of A. If A is symmetric with symmetrising form

$$< \, , \, >: A \otimes_k A \longrightarrow k$$

taking orthogonal spaces, since $KA^{\perp} = Z(A)$, $T_n(A)^{\perp}$ is an ideal of $Z(A)$.

Given an equivalence of triangulated categories of standard type (cf Section 1.2.2 below) $D^b(A) \longrightarrow D^b(B)$ between two symmetric k-algebras A and B, Rickard showed in [20] that this equivalence induces an isomorphism $Z(A) \simeq Z(B)$ and if k is a perfect field, it is shown in [22] that $T_n(A)^{\perp}$ is mapped by this isomorphism to $T_n(B)^{\perp}$.

Moreover A is symmetric if and only if $A \simeq A^* := Hom_k(A, k)$ as $A \otimes_k A^{op}$-modules. Since we supposed k to be perfect the Frobenius map is invertible and let $k \ni \lambda \mapsto \lambda^{p^{-1}} \in k$ be its inverse. Denote the n-th iterate by $p^{-n} := \left(p^{-1}\right)^n$. Since $A/KA \ni b \mapsto b^p \in A/KA$ is a well-defined additive mapping, semilinear with respect to the Frobenius mapping, one defines an element in $Hom_k(A, k)$ by

$$a \mapsto \left(b \mapsto < a, b^{p^n} >^{p^{-n}}\right)$$

and so for every $a \in Z(A)$ there is a unique element $\zeta_n(a) \in A$ so that

$$< a, b^{p^n} > = < \zeta_n(a), b >^{p^n} \; \forall \, b \in A.$$

Similarly, using that $Z(A)^{\perp} = KA$, one gets a mapping $\kappa_n : A/KA \longrightarrow A/KA$ satisfying

$$< a^{p^n}, b > = < a, \kappa_n(b) >^{p^n} \; \forall \, a \in Z(A), b \in A/KA.$$

In [23] this construction was generalised to a structure on Hochschild homology which is invariant under equivalences of derived categories. More precisely, let A be a *symmetric* finite dimensional k-algebra for k a perfect field of characteristic $p > 0$. Then, denoting by $\mathbb{B}A$ the bar resolution of A and $A^e = A \otimes_k A^{op}$,

$$Hom_k(\mathbb{B}A \otimes_{A^e} A, k) \simeq Hom_{A^e}(\mathbb{B}A, Hom_k(A, k)) \simeq Hom_{A^e}(\mathbb{B}A, A)$$

and so

$$Hom_k(HH_m(A, A), k) \simeq HH^m(A, A)$$

which induces a non degenerate pairing

$$< \, , \, >_m: HH^m(A, A) \times HH_m(A, A) \longrightarrow k \, .$$

By a construction analogous to the one for the pairing

$$< \, , \, >: Z(A) \times A/KA \longrightarrow k$$

one gets a mapping

$$\kappa_n^{(m);A} : HH_{p^n m}(A,A) \longrightarrow HH_m(A,A)$$

denoted simply by $\kappa_n^{(m)}$ if no confusion may occur, satisfying

$$< z^{p^n}, x >_{p^n m} = \left(< z, \kappa_n^{(m)}(x) >_m \right)^{p^n}$$

for all $x \in HH_{p^n m}$ and $z \in HH^m(A,A)$.

1.2.2. *On derived equivalences and Hochschild (co-)homology.* In [23] we studied invariance properties of these mappings with respect to equivalences between derived categories. Rickard showed in [20] that whenever A and B are k-algebras over a field k, and whenever $D^b(A)$ and $D^b(B)$ are equivalent as triangulated categories, then there is a complex $X \in D^b(B \otimes_k A^{op})$ so that $X \otimes_A^{\mathbb{L}} - : D^b(A) \longrightarrow D^b(B)$ is an equivalence. Such an equivalence is called of standard type and X is called a two-sided tilting complex. It is shown in [20] that every equivalence of triangulated categories $D^b(A) \longrightarrow D^b(B)$ coincides with a suitable standard one on objects.

A tilting complex T is a complex in $K^b(A)$, the homotopy category of bounded complexes of projective A-modules, satisfying $Hom_{D^b(A)}(T, T[i]) = 0$ for all $i \neq 0$, and $add(T)$ generates $K^b(A)$ as triangulated category. Two k-algebras A and B have equivalent derived categories $D^b(A)$ and $D^b(B)$ if and only if there is a tilting complex T in $D^b(A)$ with endomorphism ring B. Given an equivalence $D^b(B) \longrightarrow D^b(A)$, the image of the regular B-module is a tilting complex. Hence, by the above, for every tilting complex T in $D^b(A)$ there is a two-sided tilting complex X in $D^b(A \otimes_k B^{op})$ so that $X \simeq T$ in $D^b(A)$.

Let $F_X = X \otimes_A^{\mathbb{L}} - : D^b(A) \longrightarrow D^b(B)$ be an equivalence of derived categories with quasi-inverse $Y \otimes^{\mathbb{L}} -$ (cf Rickard [20]). If A is symmetric, then B is symmetric as well (cf [21] for general base rings, and [20] for fields) and

$$X \otimes_A^{\mathbb{L}} - \otimes_A^{\mathbb{L}} Y : D^b(A \otimes_k A^{op}) \longrightarrow D^b(B \otimes_k B^{op})$$

is an equivalence again. This equivalence induces an isomorphism

$$HH_*(F_X) : HH_*(A) \longrightarrow HH_*(B)$$

made explicit in [23]. Recall the precise mapping, defined on the level of complexes, which will be needed later.

$$\begin{array}{rcl} \mathbb{B}A \otimes_{A \otimes_k A^{op}} (X \otimes_B Y) & \longrightarrow & (Y \otimes_A \mathbb{B}A \otimes_A X) \otimes_{B \otimes_k B^{op}} B \\ u \otimes (x \otimes y) & \mapsto & (y \otimes u \otimes x) \otimes 1 \end{array}$$

Then $HH_*(F_X)$ is the mapping induced on the homology.

We recall one of the main results of [23]. This result will be one of our main ingredients in the proof of Theorem 2.

Theorem 1. [23] *Let A be a finite dimensional symmetric k-algebra over the perfect field k of characteristic $p > 0$. Let B be a second algebra such that $D^b(A) \simeq D^b(B)$ as triangulated categories, and let $m \in \mathbb{N}$. Then, there is a standard equivalence $F : D^b(A) \simeq D^b(B)$, and any such standard equivalence induces an isomorphism $HH_m(F) : HH_m(A,A) \longrightarrow HH_m(B,B)$ of all Hochschild homology groups, satisfying*

$$HH_m(F) \circ \kappa_n^{(m),A} = \kappa_n^{(m),B} \circ HH_{p^n m}(F) \ .$$

We take the opportunity to mention that in [23] in the statement of the theorem the condition on the field k to be perfect is unfortunately missing. However, this was a general assumption throughout [23].

1.2.3. *Trivial extension of an algebra; the Külshammer ideal theory in the general case.* Let k be a commutative ring and let A be a k-algebra. Then $Hom_k(A,k)$ is an $A\otimes_k A^{op}$-module by the action

$$((a,b)\cdot f)(x) := (a\cdot f\cdot b)(x) := f(bxa)\ \ \forall a,x\in A; b\in A^{op}; f\in Hom_k(A,k).$$

Now, the k-vector space

$$\mathbb{T}A := A\oplus Hom_k(A,k)$$

becomes a k-algebra by putting

$$(a,f)\cdot(b,g) := (ab, ag+fb)\ \ \forall a,b\in A; f,g\in Hom_k(A,k)\ .$$

Then

$$\begin{array}{ccc} A & \xrightarrow{\iota_A} & \mathbb{T}A \\ a & \mapsto & (a,0) \end{array}$$

is a ring homomorphism. Moreover $\{0\}\oplus Hom_k(A,k)$ is a two-sided ideal of $\mathbb{T}A$ and the canonical projection

$$\begin{array}{ccc} \mathbb{T}A & \xrightarrow{\pi_A} & A \\ (a,f) & \mapsto & a \end{array}$$

is a splitting for ι_A; i.e. $\pi_A\circ\iota_A = id_A$. The algebra $\mathbb{T}A$ is always symmetric via the bilinear form

$$\mathbb{T}A\otimes_k\mathbb{T}A \ni <(a,f);(b,g)> \mapsto f(b)+g(a)\in k$$

which induces, for k a field and A finite dimensional over k, an isomorphism

$$\begin{array}{ccc} \mathbb{T}A & \longrightarrow & Hom_k(\mathbb{T}A,k) \\ (a,f) & \mapsto & ((b,g)\mapsto f(b)+g(a)) \end{array}$$

of $\mathbb{T}A\otimes_k\mathbb{T}A^{op}$-modules.

In the joint paper [2] with Bessenrodt and Holm it is shown that the Külshammer ideals satisfy

$$T_n(\mathbb{T}A)^{\perp} = \{0\}\oplus Ann_{Hom_k(A,k)}(T_n(A))$$

for all $n\geq 1$, and therefore for all $n\geq 1$ the $Z(A)$-modules $Ann_{Hom_k(A,k)}(T_n(A))$ are invariants under derived equivalences.

Remark 1.1. It should be noted that for algebras A which are already symmetric, via the symmetrising form $<\ ,\ >$ we get an isomorphism of vector spaces $\lambda : Z(A)\longrightarrow Hom_k(A/KA,k)$. Now, every linear form on A equals a form of the shape $<z,->$ for some $z\in Z(A)$ and so

$$\lambda^{-1}(Ann_{Hom_k(A,k)}(T_n(A))) = \{z\in Z(A)\mid\ <z,T_n(A)>=0\} = T_n(A)^{\perp}.$$

We find back our original result.

1.3. **Relating Hochschild homology of an algebra and of its trivial extension.** Let R and S be k-algebras and let $\alpha : R\longrightarrow S$ and $\beta : S\longrightarrow R$ be algebra homomorphisms. Then it is well-known (cf e.g. Loday [18, Chapter 1, Section 1.1.4]) that Hochschild homology is functorial, i.e. α and β induce mappings

$$HH_*(\alpha) : HH_*(R)\longrightarrow HH_*(S) \text{ and } HH_*(\beta) : HH_*(S)\longrightarrow HH_*(R)$$

so that

$$HH_*(id_R) = id_{HH_*(R)} \text{ as well as } HH_*(\beta)\circ HH_*(\alpha) = HH_*(\beta\circ\alpha).$$

In particular in case of the mappings $\iota_A : A\longrightarrow\mathbb{T}A$ and $\pi_A : \mathbb{T}A\longrightarrow A$ from an algebra A to its trivial extension $\mathbb{T}A$ we get induced mappings giving a split projection

$$HH_n(\pi_A) : HH_n(\mathbb{T}A)\longrightarrow HH_n(A)$$

and a split injection

$$HH_n(\iota_A) : HH_n(A)\longrightarrow HH_n(\mathbb{T}A).$$

This can be defined on the level of complexes. As seen in Section 1.1 the algebra homomorphism $\alpha : R \longrightarrow S$ induces a morphism of complexes

$$\begin{array}{rcl} \mathbb{B}\alpha : \mathbb{B}R & \longrightarrow & \mathbb{B}S \\ (x_0 \otimes \cdots \otimes x_n) & \mapsto & (\alpha(x_0) \otimes \cdots \otimes \alpha(x_n)) \end{array}$$

and likewise for $\beta : S \longrightarrow R$. Also on the complex computing Hochschild homology an analogous morphism of complexes is defined by

$$\begin{array}{rcl} \mathbb{B}R \otimes_{R^e} R & \longrightarrow & \mathbb{B}S \otimes_{S^e} S \\ (x_0 \otimes \cdots \otimes x_n) \otimes x_{n+1} & \mapsto & (\alpha(x_0) \otimes \cdots \otimes \alpha(x_n)) \otimes \alpha(x_{n+1}) \end{array}$$

and likewise for β. Its homology induces a mapping $H_*\mathbb{B}\alpha : HH_*(R) \longrightarrow HH_*(S)$ which is easily seen to be $HH_*(\alpha)$ obtained above.

In particular in our situation

$$H_*\mathbb{B}\pi_A \circ H_*\mathbb{B}\iota_A = id_{\mathbb{B}A \otimes_{A^e} A}$$

and it is clear that

$$H_*\mathbb{B}\pi_A = HH_*(\pi_A) \text{ as well as } H_*\mathbb{B}\iota_A = HH_*(\iota_A).$$

We reproved the following proposition which was obtained in a much larger context and generality, and much more sophisticated methods, by Cibils, Marcos, Redondo and Solotar in [4, Theorem 5.8].

Proposition 1. [4, Theorem 5.8] *Let k be a field and let A be a k-algebra. Then the canonical embedding $A \longrightarrow \mathbb{T}A$ induces a canonical embedding of $HH_*(A)$ as a direct factor of $HH_*(\mathbb{T}A)$.*

2. Külshammer-like Hochschild invariants for non symmetric algebras

Let k be a perfect field of characteristic $p > 0$ and let A be a k-algebra. Recall from Section 1.2.1 that in case A is symmetric, we defined in [23] mappings

$$\kappa_n^{(m)} : HH_{p^n m}(A) \longrightarrow HH_m(A)$$

satisfying

$$\left(< x, \kappa_n^{(m)}(y) >_m\right)^{p^n} = < x^{p^m}, y >_{p^n m} \ \ \forall\, x \in HH^m(A); y \in HH_{p^n m}(A).$$

We should mention that the pairing $< \ , \ >_m$ is defined only on the (co-)homology of finite dimensional symmetric algebras by the isomorphism

$$\mathbb{B}A \otimes_{A^e} A \simeq Hom_k(Hom_k(\mathbb{B}A \otimes_{A^e} A, k), k) \simeq Hom_k(Hom_{A^e}(\mathbb{B}A, A), k)$$

whose homology gives an isomorphism

$$HH_*(A) \simeq Hom_k(HH^*(A), k)\ .$$

Observe that the double dual is the identity for finite dimensional vector spaces only, and hence here we use the fact that finite dimensional algebras have projective bimodule resolutions with finite dimensional homogeneous components.

Definition 1. Let A be a finite dimensional (not necessarily symmetric) k-algebra. Then put

$$\hat{\kappa}_n^{(m);A} := HH_m(\pi_A) \circ \kappa_n^{(m);\mathbb{T}A} \circ HH_{p^n m}(\iota_A)$$

We shall prove now that the invariant $\hat{\kappa}_n^{(m)}$ is an invariant of the derived category of A in the same sense as it was proved for symmetric algebras (cf Theorem 1).

Rickard showed in [19] that an equivalence between the derived categories of two k-algebras induces an equivalence between the derived categories of their trivial extension algebras. We need the following improvement of his result, which seems to be of interest in its own right.

Proposition 2. *Let A and B be finite dimensional k algebras and let T be a tilting complex in $D^b(A)$ with endomorphism ring B.*

(1) *(Rickard [19]) Then $\mathbb{T}A \otimes_A T$ is a tilting complex in $D^b(\mathbb{T}A)$ with endomorphism ring $\mathbb{T}B$.*
(2) *Let X be a two-sided tilting complex in $D^b(\mathbb{T}A \otimes_k \mathbb{T}B)$ so that $\mathbb{T}A \otimes_A T \simeq X$ in $D^b(\mathbb{T}A)$. Then $A \otimes_{\mathbb{T}A} X$ and $X \otimes_{\mathbb{T}B} B$ are two-sided tilting complexes in $D^b(A \otimes_k B)$. Moreover, $T \simeq A \otimes_{\mathbb{T}A} X$.*

Proof. Let T be a tilting complex in $D^b(A)$ with endomorphism ring B. Then by Rickard's theorem [19, Corollary 5.4] the complex $\mathbb{T}A \otimes_A T$ is a tilting complex in $D^b(\mathbb{T}A)$ with endomorphism ring $\mathbb{T}B$. By Keller's theorem [12] there is a two-sided tilting complex X in $D^b(\mathbb{T}A \otimes_k \mathbb{T}B^{op})$ so that ${}_{\mathbb{T}A}|X \simeq \mathbb{T}A \otimes_A T$.

Now,

$$\begin{aligned} Hom_{D^b(A)}(A \otimes_{\mathbb{T}A} X, A \otimes_{\mathbb{T}A} X) &= Hom_{D^b(A)}(A \otimes_{\mathbb{T}A} {}_{\mathbb{T}A}|X, A \otimes_{\mathbb{T}A} {}_{\mathbb{T}A}|X) \\ &\simeq Hom_{D^b(A)}(A \otimes_{\mathbb{T}A} \mathbb{T}A \otimes_A T, A \otimes_{\mathbb{T}A} \mathbb{T}A \otimes_A T) \\ &\simeq Hom_{D^b(A)}(T, T) \\ &\simeq B \end{aligned}$$

as rings. But, we know that $\mathbb{T}B$ acts on $A \otimes_{\mathbb{T}A} X$ by multiplication on the right. Hence, we get a ring homomorphism

$$\mathbb{T}B \longrightarrow Hom_{D^b(A)}(A \otimes_{\mathbb{T}A} X, A \otimes_{\mathbb{T}A} X).$$

Since we have seen that the endomorphism ring of $A \otimes_{\mathbb{T}A} X$ is isomorphic to B, the mapping

$$\mathbb{T}B \longrightarrow Hom_{D^b(A)}(A \otimes_{\mathbb{T}A} X, A \otimes_{\mathbb{T}A} X)$$

factorises through the canonical projection $\mathbb{T}B \longrightarrow B$. Hence, the action of $\mathbb{T}B$ on $A \otimes_{\mathbb{T}A} X$ has B^* in the kernel.

Now, the $\mathbb{T}B \otimes_k \mathbb{T}B^{op}$-module structure of $Hom_{D^b(A)}(A \otimes_{\mathbb{T}A} X, A \otimes_{\mathbb{T}A} X)$ is the following. The action of $\mathbb{T}B$ on the first argument gives the $\mathbb{T}B$-action from the left on $Hom_{D^b(A)}(A \otimes_{\mathbb{T}A} X, A \otimes_{\mathbb{T}A} X)$ and the action of $\mathbb{T}B$ on the right comes from the action of $\mathbb{T}B$ on the second argument. Both have the degree 2 nilpotent ideal B^* in the kernel and so, the natural action of $\mathbb{T}B \otimes_k \mathbb{T}B^{op}$ is actually an action of $B \otimes_k B^{op}$.

Therefore,

$$Hom_{D^b(A)}(A \otimes_{\mathbb{T}A} X, A \otimes_{\mathbb{T}A} X) \simeq B$$

as $B \otimes_k B^{op}$-modules.

Hence $A \otimes_{\mathbb{T}A} X$ is invertible from the left in the sense that

$$Hom_{D^b(A)}(A \otimes_{\mathbb{T}A} X, A \otimes_{\mathbb{T}A} X) \simeq Hom_A(A \otimes_{\mathbb{T}A} X, A) \otimes_A (A \otimes_{\mathbb{T}A} X) \simeq B$$

as $B \otimes_k B^{op}$-modules.

We still need to show that $A \otimes_{\mathbb{T}A} X$ is invertible from the right as well. Since $\mathbb{T}A$ and $\mathbb{T}B$ are both symmetric, and since X is a two-sided tilting complex in $D^b(\mathbb{T}A \otimes_k \mathbb{T}B^{op})$, its inverse complex is given by its k-linear dual $Hom_k(X, k) =: \check{X}$. We claim that $\check{X} \otimes_{\mathbb{T}A} A$ is a right inverse of $A \otimes_{\mathbb{T}A} X$. Indeed, by the previous paragraph, we see that $\mathbb{T}B$ acts on the right of $A \otimes_{\mathbb{T}A} X$ via the projection $\mathbb{T}B \longrightarrow B$, i.e. the nilpotent ideal B^* is in the kernel of this action. Analogously, the same holds for $\check{X} \otimes_{\mathbb{T}A} A$. Hence, the tensor product over $\mathbb{T}B$ equals the tensor product over B only.

$$\begin{aligned} (A \otimes_{\mathbb{T}A} X) \otimes_B (\check{X} \otimes_{\mathbb{T}A} A) &\simeq (A \otimes_{\mathbb{T}A} X) \otimes_{\mathbb{T}B} (\check{X} \otimes_{\mathbb{T}A} A) \\ &\simeq A \otimes_{\mathbb{T}A} (X \otimes_{\mathbb{T}B} \check{X}) \otimes_{\mathbb{T}A} A \\ &\simeq A \otimes_{\mathbb{T}A} \mathbb{T}A \otimes_{\mathbb{T}A} A \\ &\simeq A \end{aligned}$$

as $A \otimes_k A^{op}$-modules.

As a consequence $A \otimes_{\mathbb{T}A} X$ is a two-sided tilting complex with restriction to the left being T. Moreover, it is clear that π_A will map X to $A \otimes_{\mathbb{T}A} X$. ∎

Theorem 2. *Let k be a perfect field of characteristic $p > 0$, let A and B be finite dimensional k-algebras and suppose that $D^b(A) \simeq D^b(B)$ as triangulated categories. Let F be an explicit standard equivalence between $D^b(A)$ and $D^b(B)$. Then, F induces a sequence of isomorphisms $HH_m(F) : HH_m(A) \longrightarrow HH_m(B)$ so that*

$$HH_m(F) \circ \hat{\kappa}_n^{(m);A} = \hat{\kappa}_n^{(m);B} \circ HH_{p^n m}(F).$$

Proof. Let T be a tilting complex in $D^b(A)$ with endomorphism ring B and let X be a two-sided tilting complex in $D^b(\mathbb{T}A \otimes_k \mathbb{T}B^{op})$ with $\mathbb{T}A \otimes_A T \simeq X$ in $D^b(\mathbb{T}A)$. By Proposition 2 we get $A \otimes_{\mathbb{T}A} X$ is a two-sided tilting complex in $D^b(A \otimes_k B)$.

We now use Theorem 1:

$$HH_m(F_X) \circ \kappa_n^{(m);\mathbb{T}A} = \kappa_n^{(m);\mathbb{T}B} \circ HH_{p^n m}(F_X).$$

Multiplying with $HH_{p^n m}(\iota_A)$ from the right and with $HH_m(\pi_B)$ from the left gives then

$$HH_m(\pi_B) \circ HH_m(F_X) \circ \kappa_n^{(m);\mathbb{T}A} \circ HH_{p^n m}(\iota_A) = HH_m(\pi_B) \circ \kappa_n^{(m);\mathbb{T}B} \circ HH_{p^n m}(F_X) \circ HH_{p^n m}(\iota_A).$$

We now claim that

$$HH_{p^n m}(F_X) \circ HH_{p^n m}(\iota_A) = HH_{p^n m}(\iota_B) \circ HH_{p^n m}(F_{A \otimes_{\mathbb{T}A} X})$$

and

$$HH_m(\pi_B) \circ HH_m(F_X) = HH_m(F_{A \otimes_{\mathbb{T}A} X}) \circ HH_m(\pi_A).$$

Recall from Section 1.2.2 how $HH_m(F_X)$ is defined. Then, the commutativity relation will be proven as soon as we have that the diagrams

$$\begin{array}{ccc} \left(\check{X} \otimes_{\mathbb{T}A} \mathbb{B}(\mathbb{T}A) \otimes_{\mathbb{T}A} X\right) \otimes_{(\mathbb{T}B)^e} \mathbb{T}B & \longleftarrow & \mathbb{B}\mathbb{T}A \otimes_{(\mathbb{T}A)^e} (X \otimes_{\mathbb{T}B} \check{X}) \\ \uparrow HH(\iota_B) & & HH(\iota_A) \uparrow \\ \left((\check{X} \otimes_{\mathbb{T}A} A) \otimes_A \mathbb{B}A \otimes_A (A \otimes_{\mathbb{T}A} X)\right) \otimes_{B^e} B & \longleftarrow & \mathbb{B}A \otimes_{A^e} ((A \otimes_{\mathbb{T}A} X) \otimes_B (\check{X} \otimes_{\mathbb{T}A} A)) \end{array}$$

as well as

$$\begin{array}{ccc} \left(\check{X} \otimes_{\mathbb{T}A} \mathbb{B}(\mathbb{T}A) \otimes_{\mathbb{T}A} X\right) \otimes_{(\mathbb{T}B)^e} \mathbb{T}B & \longleftarrow & \mathbb{B}\mathbb{T}A \otimes_{(\mathbb{T}A)^e} (X \otimes_{\mathbb{T}B} \check{X}) \\ \downarrow HH(\pi_B) & & HH(\pi_A) \downarrow \\ \left((\check{X} \otimes_{\mathbb{T}A} A) \otimes_A \mathbb{B}A \otimes_A (A \otimes_{\mathbb{T}A} X)\right) \otimes_{B^e} B & \longleftarrow & \mathbb{B}A \otimes_{A^e} ((A \otimes_{\mathbb{T}A} X) \otimes_B (\check{X} \otimes_{\mathbb{T}A} A)) \end{array}$$

are commutative.

We use the canonical isomorphisms

$$(\check{X} \otimes_{\mathbb{T}A} A) \otimes_A \mathbb{B}A \otimes_A (A \otimes_{\mathbb{T}A} X) \simeq \check{X} \otimes_{\mathbb{T}A} \mathbb{B}A \otimes_{\mathbb{T}A} X$$

and

$$\mathbb{B}A \otimes_{A^e} (A \otimes_{\mathbb{T}A} X \otimes_B \check{X} \otimes_{\mathbb{T}A} A) \simeq \mathbb{B}A \otimes_{(\mathbb{T}A)^e} (X \otimes_{\mathbb{T}B} \check{X}),$$

which proves that we only need to show that the diagrams

$$\begin{array}{ccc} \left(\check{X} \otimes_{\mathbb{T}A} \mathbb{B}(\mathbb{T}A) \otimes_{\mathbb{T}A} X\right) \otimes_{(\mathbb{T}B)^e} \mathbb{T}B & \longleftarrow & \mathbb{B}\mathbb{T}A \otimes_{(\mathbb{T}A)^e} (X \otimes_{\mathbb{T}B} \check{X}) \\ \uparrow HH(\iota_B) & & HH(\iota_A) \uparrow \\ \left(\check{X} \otimes_{\mathbb{T}A} \mathbb{B}A \otimes_{\mathbb{T}A} X\right) \otimes_{B^e} B & \longleftarrow & \mathbb{B}A \otimes_{(\mathbb{T}A)^e} (X \otimes_{\mathbb{T}B} \check{X}) \end{array}$$

and

$$\begin{array}{ccc} \left(\check{X} \otimes_{\mathbb{T}A} \mathbb{B}(\mathbb{T}A) \otimes_{\mathbb{T}A} X\right) \otimes_{(\mathbb{T}B)^e} \mathbb{T}B & \longleftarrow & \mathbb{B}\mathbb{T}A \otimes_{(\mathbb{T}A)^e} (X \otimes_{\mathbb{T}B} \check{X}) \\ \downarrow HH(\pi_B) & & HH(\pi_A) \downarrow \\ \left(\check{X} \otimes_{\mathbb{T}A} \mathbb{B}A \otimes_{\mathbb{T}A} X\right) \otimes_{B^e} B & \longleftarrow & \mathbb{B}A \otimes_{(\mathbb{T}A)^e} (X \otimes_{\mathbb{T}B} \check{X}) \end{array}$$

are commutative. But this is clear since the unit element of $\mathbb{T}B$ (and of $\mathbb{T}A$ resp.) is the image of the unit element of B (and of A resp.) under ι.

Hence the claim is proven.

This implies now

$$\begin{aligned} HH_m(F_{A\otimes_{\mathbb{T}A}X})\circ\hat{\kappa}_n^{(m);A} &= HH_m(F_{A\otimes_{\mathbb{T}A}X})\circ HH_m(\pi_A)\circ\kappa_n^{(m);\mathbb{T}A}\circ HH_{p^nm}(\iota_A)\\ &= HH_m(\pi_B)\circ HH_m(F_X)\circ\kappa_n^{(m);\mathbb{T}A}\circ HH_{p^nm}(\iota_A)\\ &= HH_m(\pi_B)\circ\kappa_n^{(m);\mathbb{T}B}\circ HH_{p^nm}(F_X)\circ HH_{p^nm}(\iota_A)\\ &= HH_m(\pi_B)\circ\kappa_n^{(m);\mathbb{T}B}\circ HH_{p^nm}(\iota_B)\circ HH_{p^nm}(F_{A\otimes_{\mathbb{T}A}X})\\ &= \hat{\kappa}_n^{(m);B}\circ HH_{p^nm}(F_{A\otimes_{\mathbb{T}A}X})\end{aligned}$$

which proves Theorem 2. ■

3. Dual numbers as an example

It should be noted that the mappings $\kappa_n^{(m)}$ are not zero in general. The dual numbers provide an example. In this section we shall show this fact and prove moreover that $\kappa_m^{(n)} = \hat{\kappa}_m^{(n)}$ in this case.

Holm computed the cohomology ring of rings $k[X]/(f(X))$ for all polynomials $f(X)$ and all fields k. In particular for $f(X) = X^2$ one obtains a symmetric algebra, the algebra of dual numbers, whose Hochschild homology and cohomology are isomorphic. For fields of characteristic zero Lindenstrauss [16] computed in general the Hochschild homology of algebras $k[X_1,\dots,X_n]/\mathfrak{m}^m$ for $\mathfrak{m}$ being the maximal ideal corresponding to the point $(0,0,\dots,0)$ by exhibiting an explicit projective resolution.

We shall reprove parts of these statements since we will need quite detailed information about the (co-)cycles that represent each element in the Hochschild structure.

Let $A = k[\epsilon]/(\epsilon^2)$ throughout this section and let k be a field of characteristic $p > 0$.

Lemma 3. $\mathbb{T}A \simeq A\otimes_k A$.

Proof. Clearly, since A is commutative, also $\mathbb{T}A$ is commutative. Moreover, there is a k-basis $A = k\cdot 1\oplus k\cdot\epsilon$ and since A is symmetric, the symmetrising form being $\langle x+y\epsilon, x'+y'\epsilon\rangle := xy' + x'y$, we get a K-basis of $\mathbb{T}A = A^*\rtimes A$ by

$$\mathbb{T}A = k\cdot(\langle 1,-\rangle,0)\oplus k\cdot(\langle\epsilon,-\rangle,0)\oplus k\cdot(\langle 0,-\rangle,1)\oplus k\cdot(\langle 0,-\rangle,\epsilon).$$

Let $\delta := (\langle 1,-\rangle,0)$, $\sigma := (\langle\epsilon,-\rangle,0)$, $\varepsilon := (\langle 0,-\rangle,\epsilon)$ and $1 := (\langle 0,-\rangle,1)$ (remarking that this is still the unit element of the trivial extension algebra). We verify immediately that $\varepsilon^2 = \delta^2 = 0$. Furthermore, $\varepsilon\cdot\delta = \delta\cdot\varepsilon = \sigma$. Hence, $\mathbb{T}A$ is the quotient of the quiver algebra with one vertex and two loops by the relations saying that the two loops are nilpotent of order 2 and that they commute. This describes exactly the algebra $A\otimes_k A$. ■

Remark 3.1. If $p = 2$, then A is isomorphic to the group algebra of the cyclic group of order 2. Then $A\otimes_k A^{op}$ is isomorphic to the group algebra of the Klein four group and the Hochschild cohomology ring is known. Holm showed [8, Theorem 3.2.1] that in this case

$$HH^*(A\otimes_k A^{op})\simeq A[X,Y]$$

where X and Y are algebraically independent of degree 1.

For any $p > 0$ we know how to determine the Hochschild homology in this more general case as well by the Künneth formula. Indeed,

$$HH_*(A\otimes_k A)\simeq HH_*(A)\otimes_k HH_*(A)$$

by the Künneth formula.

We have to study the injection

$$HH_*(A)\longrightarrow HH_*(\mathbb{T}A)\simeq HH_*(A\otimes_K A)\simeq HH_*(A)\otimes_K HH_*(A)$$

given by the isomorphism $\mathbb{T}A \simeq A \otimes_K A$, the injection $A \longrightarrow \mathbb{T}A$ and the Künneth formula, as well as the projection

$$HH_*(A) \otimes_k HH_*(A) \simeq HH_*(A \otimes_k A) \simeq HH_*(\mathbb{T}A) \longrightarrow HH_*(A).$$

The algebra A is symmetric and hence

$$Hom_k(HH_m(A), k) \simeq HH^m(A)$$

by an isomorphism induced by the symmetrising bilinear form. Therefore there is a non degenerate pairing

$$HH^m(A) \times HH_m(A) \longrightarrow K$$

as usual.

Remark 3.2. The algebra $A \otimes A$ is the quotient of the quiver algebra with one vertex and two loops by the relations saying that the two loops are nilpotent of order 2 and that they commute. Hence, we may replace any loop by a non trivial linear combination of these two loops, completing by another linear combination of the two loops so that the determinant of the coefficient matrix is non zero. Hence, we may suppose that the inclusion $A \longrightarrow \mathbb{T}A$ is given by the inclusion $A \longrightarrow A \otimes A$ defined by $a \mapsto 1 \otimes a$. Indeed, the Hochschild homology computation by means of the corresponding double complex does not depend on the choice of a basis.

BACH [1] and Holm [7] computed an explicit resolution of bimodules for a monogenic algebra, such as the dual numbers. Abbreviate for simplicity $A = K[\epsilon]/\epsilon^2$ (as usual) and $A^2 := A \otimes_k A$. Then a free resolution C of A is periodic of period 2 and is given by

$$C : \quad (A \longleftarrow) A^2 \xleftarrow{d_1} A^2 \xleftarrow{d_2} A^2 \xleftarrow{d_1} A^2 \xleftarrow{d_2} A^2 \longleftarrow \cdots$$

where d_1 is multiplication by $1 \otimes \epsilon - \epsilon \otimes 1$ and d_2 is multiplication by $1 \otimes \epsilon + \epsilon \otimes 1$.

Applying the functor $- \otimes_{A^2} A$ gives a complex

$$A \xleftarrow{(d_1)_\otimes} A \xleftarrow{(d_2)_\otimes} A \xleftarrow{(d_1)_\otimes} A \xleftarrow{(d_2)_\otimes} A \longleftarrow \dots$$

with $(d_1)_\otimes = 0$ and $(d_2)_\otimes = 2\epsilon$.

Now, applying the functor $Hom_{A^2}(-, A)$ gives a complex $Hom_{A^2}(C, A)$

$$\dots \longrightarrow A \xrightarrow{(d_1)_h} A \xrightarrow{(d_2)_h} A \xrightarrow{(d_1)_h} A \xrightarrow{(d_2)_h} A$$

where again $(d_1)_h = 0$ and $(d_2)_h = 2\epsilon$.

The Künneth formula gives that the tensor product of the cohomology is the cohomology of the tensor product

$$H\left(Hom_{A^2}(C, A) \otimes Hom_{A^2}(C, A)\right) = H\left(Hom_{A^2}(C, A)\right) \otimes H\left(Hom_{A^2}(C, A)\right).$$

We observe that

$$Hom_{A^2}(C, A) \otimes Hom_{A^2}(C, A) \simeq Hom_{A^2 \otimes A^2}(C \otimes C, A \otimes A)$$

and this is the complex computing the Hochschild cohomology of $A \otimes_k A$, and whence for $\mathbb{T}A$.

As usual the Hochschild (co-)homology depends on weather $p = 2$ or $p > 2$. Our arguments use the explicit structure of the Hochschild (co-)homology and therefore we shall need to treat these two cases separately.

3.1. **The case $p > 2$.** Holm [7] shows that

$$HH^*(A) \simeq A[U, Z]/(Z\epsilon, U\epsilon, U^2)$$

for an element U in degree 1 and an element Z in degree 2. Hence $HH^m(A)$ is one-dimensional, generated by Z^n if $m = 2n$ or UZ^n if $m = 2n + 1$, for $m > 0$ and isomorphic to A in degree 0.

We may therefore choose for each $n \in \mathbb{N} \setminus \{0\}$ an element z_n in $HH_{2n}(A)$ which corresponds to Z^n in $HH^*(A)$ under the above isomorphism and we get $HH_{2n}(A) = kz_n$. Moreover, choose an element $uz_n \in HH_{2n+1}(A)$ which corresponds to UZ^n under the above isomorphism. Hence, $HH_{2n+1}(A) = kuz_n$.

Observe that

$$(HH_*(A) \otimes HH_*(A))^* \simeq HH_*(A)^* \otimes HH_*(A)^* \simeq HH^*(A) \otimes HH^*(A)$$

where the first isomorphism is canonical and the second one is induced by the isomorphism $HH_*(A)^* \simeq HH^*(A)$. Hence *using this chain of isomorphisms* we choose the k-basis

$$\begin{gathered}
x_i y_j \text{ in bidegree } (2i, 2j) \text{ of } HH_{2i+2j}(A \otimes A),\\
x_i v y_j \text{ in bidegree } (2i+1, 2j) \text{ of } HH_{2i+1+2j}(A \otimes A),\\
x_i y_j w \text{ in bidegree } (2i, 2j+1) \text{ of } HH_{2i+2j+1}(A \otimes A),\\
x_i v y_j w \text{ in bidegree } (2i+1, 2j+1) \text{ of } HH_{2i+1+2j+1}(A \otimes A),
\end{gathered}$$

which is the dual basis element to the corresponding monomial in X^iY^j, the monomial X^iVY^j, the monomial X^iY^jW or the monomial X^iVY^jW of $HH^{*+*}(A \otimes A)$ under that isomorphism. The p-th power of all these basis elements is zero, except the element X^iY^j, whose p-th power is $X^{pi}Y^{pj}$.

The (minimal) projective resolution of A^2 used above can most easily be expressed as a double complex $C \otimes C$, as was shown above.

$$\begin{array}{ccccccccccccc}
 & & A^2 & \longleftarrow & A^2 & \longleftarrow & A^2 & \longleftarrow & A^2 & \longleftarrow & A^2 & \longleftarrow \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
A^2 & \longleftarrow & A^2 \otimes A^2 & \longleftarrow & A^2 \otimes A^2 & \longleftarrow & A^2 \otimes A^2 & \longleftarrow & A^2 \otimes A^2 & \longleftarrow & A^2 \otimes A^2 & \longleftarrow \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
A^2 & \longleftarrow & A^2 \otimes A^2 & \longleftarrow & A^2 \otimes A^2 & \longleftarrow & A^2 \otimes A^2 & \longleftarrow & A^2 \otimes A^2 & \longleftarrow & A^2 \otimes A^2 & \longleftarrow \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
A^2 & \longleftarrow & A^2 \otimes A^2 & \longleftarrow & A^2 \otimes A^2 & \longleftarrow & A^2 \otimes A^2 & \longleftarrow & A^2 \otimes A^2 & \longleftarrow & A^2 \otimes A^2 & \longleftarrow \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
A^2 & \longleftarrow & A^2 \otimes A^2 & \longleftarrow & A^2 \otimes A^2 & \longleftarrow & A^2 \otimes A^2 & \longleftarrow & A^2 \otimes A^2 & \longleftarrow & A^2 \otimes A^2 & \longleftarrow \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow &
\end{array}$$

In order to view the image of $z_n \in HH_{2n}(A)$ in $HH_{2n}(\mathbb{T}A)$ we need to first establish the multiplicative structure of $HH^*(A \otimes A)$ in terms of maps in the double complex.

The element X is the degree $(2, 0)$ mapping consisting of the identity on all homogeneous components except on the borders of this bi-complex, and the element Y is the degree $(0, 2)$ mapping identity on all homogeneous components except on the borders of this bi-complex.

The element V is the degree $(1, 0)$ mapping consisting of $\epsilon \otimes 1$ in all degrees, except on the borders of the bi-complex, where it is 0. Likewise the element W is the degree $(0, 1)$ mapping consisting of $\epsilon \otimes 1$ in all degrees, except on the borders of the bi-complex, where it is 0.

We claim that the element X^iY^j is represented by the degree $(2i, 2j)$ mapping consisting of the identity on all homogeneous components except on the borders of this bi-complex. The proof of this statement is an easy induction on i and j. Actually, the composition of a morphism of bi-complexes of degree $(2i, 2j)$ of the given shape by a morphism of complexes of degree $(2, 0)$, or $(0, 2)$ respectively, corresponds to a morphism of bi-complexes of degree $(2(i + 1), 2j)$, or $(2i, 2(j + 1))$ respectively. This corresponds to the cup product of X^iY^j with X, or Y respectively.

The element X^iVY^j is represented by the degree $(2i+1,2j)$ mapping consisting of the mapping $(\epsilon\otimes id)\otimes(id\otimes id)$ on all homogeneous components except on the borders of this bi-complex. Likewise the element X^iY^jW is represented by the degree $(2i,2j+1)$ mapping consisting of the mapping $(id\otimes id)\otimes(\epsilon\otimes id)$ and the element X^iVY^jW is represented by the degree $(2i+1,2j+1)$ mapping consisting of the mapping $(\epsilon\otimes id)\otimes(\epsilon\otimes id)$.

The cup product on $HH^*(A)$ is similar: Z corresponds to the degree 2 mapping being the identity on all homogeneous of C, except the degree 0 and degree 1 component, where the mapping clearly is 0. Now, again by an analogous induction as in the bi-complex case, the element Z^n corresponds to the degree $2n$ mapping being the identity on all homogeneous of C, except the degrees up to $2n-1$ component, where the mapping clearly is 0. The element U corresponds to the degree 1 mapping $\epsilon\otimes 1$ in all degrees except the degree 0, where it is 0. The cup product with Z^n is just an additional shift in degree, the cup product of U with U is 0, since $(\epsilon\otimes 1)^2=0$.

We now need to compare this mapping to the dual of the Hochschild homology side. Hence, we dualise the bi-complex and the cocycle representing X^iY^j. This just reverses the arrows, identifying again A^* with A. The dual of the cocycle X^iY^j can also be obtained by first applying $Hom_{A^2\otimes A^2}(-,A^2)$ to the double complex and then K-dualising the result. We obtain the double complex $D_{HH(A^2)}$.

$$\begin{array}{ccccccccccc}
 & & \cdot & & \cdot & & \cdot & & \cdot & & \cdot & \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
\cdot & \longleftarrow & A^2 & \stackrel{0}{\longleftarrow} & A^2 & \longleftarrow & A^2 & \stackrel{0}{\longleftarrow} & A^2 & \longleftarrow & A^2 & \longleftarrow \\
 & & \uparrow 0 & & \uparrow 0 & & \uparrow 0 & & \uparrow 0 & & \uparrow 0 & \\
\cdot & \longleftarrow & A^2 & \stackrel{0}{\longleftarrow} & A^2 & \longleftarrow & A^2 & \stackrel{0}{\longleftarrow} & A^2 & \longleftarrow & A^2 & \longleftarrow \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
\cdot & \longleftarrow & A^2 & \stackrel{0}{\longleftarrow} & A^2 & \longleftarrow & A^2 & \stackrel{0}{\longleftarrow} & A^2 & \longleftarrow & A^2 & \longleftarrow \\
 & & \uparrow 0 & & \uparrow 0 & & \uparrow 0 & & \uparrow 0 & & \uparrow 0 & \\
\cdot & \longleftarrow & A^2 & \stackrel{0}{\longleftarrow} & A^2 & \longleftarrow & A^2 & \stackrel{0}{\longleftarrow} & A^2 & \longleftarrow & A^2 & \longleftarrow \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow &
\end{array}$$

which now represents exactly the complex computing Hochschild homology of $A\otimes A$. Half of the arrows represent the 0-morphism, the rest, without a side- or superscript in the diagram, represent the mapping 2ϵ. The cycle representing X^n is the $2n$-th mapping identity in the upper line. The complex computing the Hochschild homology of A is simpler:

$$C_{HH(A)}:\ \ A\stackrel{0}{\longleftarrow}A\longleftarrow A\stackrel{0}{\longleftarrow}A\longleftarrow A\stackrel{0}{\longleftarrow}\cdots$$

with the same convention on non superscribed arrows.

The injection $A\longrightarrow A\otimes A\simeq \mathbb{T}A$ maps $a\mapsto 1\otimes a$ and the projection $A\otimes A\longrightarrow A$ maps $a\otimes b\mapsto ab$. Hence, the projective resolution $P_\bullet$ of A as A^2-modules maps to the projective resolution $P_\bullet\otimes P_\bullet$ of A^2 as $A^2\otimes A^2$-modules as $x\mapsto 1\otimes x$. The second degree component is 0, except in degree 0, where it is constantly $1\otimes 1$. Hence, the Hochschild homology complex of A injects by the identity into the first line of the Hochschild homology (double-) complex of A^2. In other words, the injection produces the mapping of complexes from $C_{HH(A)}$ to the first line of the bi-complex $D_{HH(A^2)}$ by the mapping $A\ni a\mapsto 1\otimes a\in A\otimes A$ on the level of each homogeneous component.

Likewise, the projection produces a mapping of complexes in the inverse order, which compose to the identity on $C_{HH(A)}$. Hence z_n is mapped to x_n for all $n>0$ and z_nu is mapped to x_nv for all $n>0$.

Finally, since p is odd, p^nm is odd if and only if m is odd. We proved the following

Proposition 4. *Let k be a perfect field of characteristic $p>2$. Then $\hat{\kappa}_n^{(m),k[\epsilon]/\epsilon^2}=\kappa_n^{(m),k[\epsilon]/\epsilon^2}$ for all n and m. Moreover, $\kappa_n^{(m)}(z_{p^nm})=z_m$ and $\kappa_n^{(m)}(z_{p^nm-1}u)=0$.*

3.2. **The case** $p = 2$. The case of even characteristic is completely analogous, with a few exceptions. The generator Z of $HH^*(A)$ is in degree 1, Z is not annihilated by ϵ, and the differentials for the homology and the cohomology complex are all 0. The Hochschild cohomology ring does only contain nilpotency coming from the centre. The rest is immediate.

Proposition 5. *Let k be a perfect field of characteristic 2. Then $\hat{\kappa}_n^{(m),k[\epsilon]/\epsilon^2} = \kappa_n^{(m),k[\epsilon]/\epsilon^2}$ for all n and m. Moreover, $\kappa_n^{(m)}(z_{p^n m}) = z_m$ for all m and n.*

Proof. The proof is a straightforward analogue of the proof in the $p > 2$-case. ∎

References

[1] Buenos Aires Hochschild Homology Group, *Cyclic homology of algebras with one generator*, K-theory **5** (1991) 51-69.

[2] Christine Bessenrodt, Thorsten Holm, Alexander Zimmermann; *Generalized Reynolds ideals for non-symmetric algebras*, Journal of Algebra **312** (2007) 985-994; arXiv: math/0603189

[3] Jerzy Białkowski, Karin Erdmann and Andrzej Skowroński, *Deformed preprojective algebras of generalised Dynkin type*, Transactions of the American Mathematical Society, **359** (2007) 2625-2650.

[4] Claude Cibils, Eduardo Marcos, Maria-Julia Redondo and Andrea Solotar, *Cohomology of split algebras and of trivial extensions*, Glasgow Mathematical Journal **45** (2003) 21-40.

[5] Karin Erdmann and Andrzej Skowroński, *Periodic algebras*, in Trends in Representation Theory of Algebras and Related Topics, Skowroński (ed.), Series of Congress Reports, European Mathematical Society Publishing House Zürich 2008.

[6] László Héthelyi, Erszébet Horváth, Burkhard Külshammer and John Murray, *Central ideals and Cartan invariants of symmetric algebras*, Journal of Algebra **293** (2005) 243-260.

[7] Thorsten Holm, *Hochschild cohomology rings of algebras $k[X]/(f)$*, Beiträge zur Algebra und Geometrie **41** (2000), No.1, 291-301

[8] Thorsten Holm, *Blocks of tame representation type and related algebras: derived equivalences and Hochschild cohomology*, Habilitationsschrift (2001) Universität Magdeburg.

[9] Thorsten Holm and Andrzej Skowroński, *Derived equivalence classification of symmetric algebras of domestic type*, Journal of the Mathematical Society of Japan **58** (2006), no.4, 1133-1149. arXiv: math/0511610

[10] Thorsten Holm and Alexander Zimmermann, *Generalized Reynolds ideals and the scalar problem for derived equivalences of tame blocks of finite groups*, Journal of Algebra **320** (2008) 3425-3437. arXiv: math.RA/0807.0688

[11] Thorsten Holm and Alexander Zimmermann, *Generalized Reynolds ideals of deformed preprojective algebras of generalized Dynkin type L.* preprint (2008), 18 pages

[12] Bernhard Keller, *Bimodule complexes via strong homotopy actions*, Algebras and Representation Theory **3** (2000) 357-376.

[13] Steffen König, Yuming Liu and Guodong Zhou, *Transfer maps in Hochschild (co-)homology and their applications,* to appear in Transactions of the American Mathematical Society.

[14] Burkhard Külshammer, *Bemerkungen über die Gruppenalgebra als symmetrische Algebra I, II, III and IV*, Journal of Algebra **72** (1981) 1-7; **75** (1982) 59-69; **88** (1984) 279-291; **93** (1985) 310-323

[15] Burkhard Külshammer, *Group-theoretical descriptions of ring theoretical invariants of group algebras*, Progress in Mathematics **95** (1991) 425-441.

[16] Ayelet Lindenstrauss, *Hochschild Homology of Fatpoints*, Israel Journal of Mathematics **133** (2003) 177-188.

[17] Yuming Liu, Guodong Zhou and Alexander Zimmermann, *Higman ideal, stable Hochschild homology and Auslander-Reiten conjecture*, preprint (2008).

[18] Jean-Louis Loday, Cyclic homology, third edition, Springer Verlag, Berlin-Heidelberg 1998

[19] Jeremy Rickard, *Derived categories and stable equivalences*, Journal of Pure and Applied Algebra **61** (1989) 303–317.

[20] Jeremy Rickard, *Derived equivalences as derived functors*, Journal of the London Mathematical Society **43** (1991) 37-48.

[21] Alexander Zimmermann, *Tilted symmetric orders are symmetric orders*, Archiv der Mathematik **73** (1999) 15-17

[22] Alexander Zimmermann, *Invariance of generalized Reynolds ideals under derived equivalences.*, Mathematical Proceedings of the Royal Irish Academy **107**A (1) (2007) 1-9; arXiv: math/0509580

[23] Alexander Zimmermann, *Fine Hochschild invariants of derived categories for symmetric algebras*, Journal of Algebra **308** (2007) 350-367; arXiv: math/0608712

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